数理解析研究所講究録
1048巻 1998年 26-32

The Limiting Behavior of Fuzzy States in Dynamic Fuzzy Systems

北九州大学経済学部　吉田祐治　(Yuji YOSHIDA)

1. Introduction and notations

The limiting behavior of fuzzy states in dynamic fuzzy systems has been studied by Kurano et al. [4] and Yoshida et al. [7]. Under a contractive condition for the fuzzy relation, [4] showed that the limiting fuzzy state is a unique solution of a fuzzy relational equation. Also, [7] discussed the limit theorem in a monotone case. In this paper, we consider the limit theorem when fuzzy relations satisfy the transitive property. We show that, in this case, the limiting fuzzy state is a solution of the fuzzy relational equation. But the equation does not necessarily have a unique solution similarly to the monotone case, therefore we need to investigate the space of the solutions of the equation.

The existence and the uniqueness of the solutions of the fuzzy relational equation has been studied by Kurano et al. [5] under some assumptions. In the transitive case, this paper makes clear the structure of the space of the solutions of the fuzzy relational equation, and we give a simple characterization of the limiting fuzzy state by the fundamental solutions for the numerical calculation of the limiting fuzzy state.

We use some notations in [5]. Let E be a compact metric space. Let $\mathcal{C}(E)$ be the collection of all non-empty closed subsets of E, and let ρ be the Hausdorff metric on $\mathcal{C}(E)$. Then it is well-known ([3]) that $(\mathcal{C}(E),\rho)$ is a compact metric space. Let $\mathcal{F}(E)$ be the set of all fuzzy sets $\tilde{s} : E \to [0,1]$ which are upper semi-continuous and satisfy $\sup_{x\in E}\tilde{s}(x) = 1$. For $\tilde{s} \in \mathcal{F}(E)$, the α-cut $\tilde{s}_\alpha$, $\alpha \in [0,1]$, is defined by

$$\tilde{s}_\alpha := \{x \in E \mid \tilde{s}(x) \geq \alpha\}\ (\alpha \neq 0) \quad \text{and} \quad \tilde{s}_0 := \mathrm{cl}\{x \in E \mid \tilde{s}(x) > 0\},$$

where cl means the closure of a set. Let $\tilde{q} : E \times E \to [0,1]$ be a fuzzy relation on E satisfying $\tilde{q}(x,\cdot) \in \mathcal{F}(E)$ for $x \in E$. A fuzzy relation $\tilde{q}$ is called "transitive" (see Klir and Yuan [2]) if it satisfies

$$\tilde{q}(x,y) \geq \sup_{z\in E}\{\tilde{q}(x,z) \wedge \tilde{q}(z,y)\}, \quad x,y \in E.$$

Throughout this paper, we assume $\tilde{q}$ is transitive. In Sections 1 and 2, we consider the space of solutions $\tilde{p} \in \mathcal{F}(E)$ of the following fuzzy relational equation (see [5]) :

$$\tilde{p}(y) = \sup_{x\in E}\{\tilde{p}(x) \wedge \tilde{q}(x,y)\}, \quad y \in E, \tag{1}$$

where $a \wedge b := \min\{a,b\}$ for real numbers a and b. By the solutions of (1) (see [4, 7] for contractive/monotone fuzzy relations), in Section 3 we discuss the limiting behavior of the sequence of fuzzy states $\{\tilde{s}_n\}_{n=0}^{\infty} \subset \mathcal{F}(E)$ with an initial fuzzy state $\tilde{s} \in \mathcal{F}(E)$ which is defined by

$$\tilde{s}_0 := \tilde{s}, \quad \text{and} \quad \tilde{s}_{n+1}(y) := \sup_{x\in E}\{\tilde{s}_n(x) \wedge \tilde{q}(x,y)\}, \quad y \in E \quad \text{for } n = 0,1,2,\cdots. \tag{2}$$

Crisp sets $\tilde{q}_\alpha(x)$ $(x \in E,\ \alpha \in [0,1])$ are defined by

$$\tilde{q}_\alpha(x) := \begin{cases} \{y \in E \mid \tilde{q}(x,y) \geq \alpha\} & \text{for } \alpha \neq 0 \\ \mathrm{cl}\{y \in E \mid \tilde{q}(x,y) > 0\} & \text{for } \alpha = 0. \end{cases}$$

In this paper, we assume the map $\tilde{q}_\alpha(\cdot) : E \mapsto \mathcal{C}(E)$ is continuous for all $\alpha \in [0,1]$. We also define $\tilde{q}_\alpha(D) := \bigcup_{x \in D} \tilde{q}_\alpha(x)$ for $D \in \mathcal{C}(E)$, and then we note that $\tilde{q}_\alpha : \mathcal{C}(E) \to \mathcal{C}(E)$. For $x \in E$ and $\alpha \in [0,1]$, a sequence $\{\tilde{q}_\alpha^n(x)\}_{n=0}^\infty \subset \mathcal{C}(E)$ is defined iteratively by

$$\tilde{q}_\alpha^0(x) := \{x\}, \quad \text{and} \quad \tilde{q}_\alpha^{n+1}(x) := \tilde{q}_\alpha(\tilde{q}_\alpha^n(x)), \quad n = 0,1,2,\cdots.$$

Then we have the following lemma for a sequence of fuzzy relations $\{\tilde{q}^n\}_{n=1}^\infty$ defined by

$$\tilde{q}^1 := \tilde{q}, \quad \text{and} \quad \tilde{q}^{n+1}(x,y) := \sup_{z \in E} \{\tilde{q}^n(x,z) \wedge \tilde{q}(z,y)\}, \quad x,y \in E \quad \text{for } n = 1,2,\cdots. \tag{3}$$

Lemma 1.1. *The following (i) and (ii) hold:*

(i) *For all* $n = 1,2,\cdots,$

$$\tilde{q}^n(x,y) \geq \tilde{q}^{n+1}(x,y), \quad x,y \in E, \tag{4}$$

and

$$\tilde{q}_\alpha^{n+1}(x) \subset \tilde{q}_\alpha^n(x), \quad x \in E,\ \alpha \in [0,1]. \tag{5}$$

(ii) $\tilde{q}^n(x,x) = \tilde{q}(x,x)$ *for all* $x \in E$ *and* $n = 1,2,\cdots.$

Let

$$R_\alpha := \left\{ x \in E \mid x \in \bigcup_{n=1}^\infty \tilde{q}_\alpha^n(x) \right\}, \quad \alpha \in [0,1].$$

Each state of R_α is called "α-recurrent" (see [8]). From Lemma 1.1(i), we have

$$R_\alpha = \{x \in E \mid \tilde{q}(x,x) \geq \alpha\}, \quad \alpha \in (0,1]. \tag{6}$$

Let $x \in E$. The following crisp sets are used in [5] to analyze the solutions of (1):

$$F_\alpha(x) := \bigcup_{n=0}^\infty \tilde{q}_\alpha^n(x),$$

and

$$\hat{F}_\alpha(x) := \bigcap_{\alpha' < \alpha} \mathrm{cl}\{F_{\alpha'}(x)\}\ (\alpha \neq 0) \quad \text{and} \quad \hat{F}_0(x) := \mathrm{cl}\{F_0(x)\}.$$

In the transitive case, by Lemma 1.1(i) they are reduced to the following (7):

$$\hat{F}_\alpha(x) = \{x\} \cup \tilde{q}_\alpha(x), \quad x \in E,\ \alpha \in [0,1]. \tag{7}$$

Especially we have

$$\hat{F}_\alpha(z) = \tilde{q}_\alpha(z), \quad z \in R_1,\ \alpha \in [0,1]. \tag{8}$$

Therefore, we obtain the following lemma.

Lemma 1.2 (Kurano et al. [5, Lemma 1.2(i) and Theorem 2.1(ii)]).

(i) *For $\tilde{p} \in \mathcal{F}(E)$, $\tilde{p}$ satisfies (1) if and only if*

$$\tilde{q}_\alpha(\tilde{p}_\alpha) = \tilde{p}_\alpha, \quad \alpha \in [0,1]. \tag{9}$$

(ii) *Let $z \in R_1$. Define a fuzzy state*

$$\tilde{p}^z(x) := \sup_{\alpha \in [0,1]} \left\{ \alpha \wedge 1_{\hat{F}_\alpha(z)}(x) \right\} = \tilde{q}(z,x), \quad x \in E. \tag{10}$$

Then $\tilde{p}^z \in \mathcal{F}(E)$ satisfies (1).

2. The space of the solutions

We put $\mathcal{P} := \{\tilde{p} \in \mathcal{F}(E) \mid \tilde{p} \text{ is a solution of (1)}\}$. Then the space $\mathcal{P}$ has the following property:

Lemma 2.1 (Kurano et al. [5, Theorem 2.2(ii)]). *Let $\tilde{p}^k \in \mathcal{P}$ $(k = 1, 2, \cdots, l)$, and let $\{\alpha^k \in [0,1] \mid k = 1, 2, \cdots, l\}$ satisfy $\sup_{k=1,2,\cdots,l} \alpha^k = 1$. Put*

$$\tilde{p}(x) := \max_{k=1,2,\cdots,l} \{\alpha^k \wedge \tilde{p}^k(x)\}, \quad x \in E. \tag{11}$$

Then $\tilde{p} \in \mathcal{P}$.

The purpose of this section is to prove an inverse of the statement of Lemma 2.1. Namely, we represent general solutions $\tilde{p} \in \mathcal{P}$ of (1) by the fundamental solutions $\tilde{p}^z$ of (10). From now on, we assume $R_1 \neq \emptyset$. We identify the states of R_1 with respect to the following equivalent relation $\sim$ on R_1 (see [5] and (8)): For $z_1, z_2 \in R_1$,

$$z_1 \sim z_2 \quad \text{means that} \quad z_1 \in \tilde{q}_1(z_2) \text{ and } z_2 \in \tilde{q}_1(z_1).$$

Then we put $R_1^{\sim} := R_1 / \sim$.

Assumption A. Let $\alpha \neq 0$ and $A \in \mathcal{C}(E)$. If $\tilde{q}_\alpha(A) = A$ holds, then

$$R_\alpha \cap A \subset \bigcup_{z \in R_1^{\sim} \cap A} \tilde{q}_\alpha(z).$$

From now on, we suppose that Assumption A holds (c.f. [5, Assumption A3]).

Theorem 2.1. *Let $\tilde{p}$ be a solution of (1). Then, there exists a family of coefficients $\{\alpha^z \in [0,1] \mid z \in R_1^{\sim}\}$ satisfying $\sup_{z \in R_1^{\sim}} \alpha^z = 1$ and*

$$\tilde{p}(x) = \sup_{z \in R_1^{\sim}} \{\alpha^z \wedge \tilde{p}^z(x)\} = \sup_{z \in R_1^{\sim}} \{\alpha^z \wedge \tilde{q}(z,x)\}, \quad x \in E. \tag{12}$$

3. A Limit Theorem

We uses the convergency of fuzzy states in following sense.

Definition (see [7]). Let $\tilde{s}_n, \tilde{p} \in \mathcal{F}(E)$. Then

$$\lim_{n\to\infty} \tilde{s}_n = \tilde{p} \quad \text{means} \quad \rho(\tilde{s}_{n,\alpha}, \tilde{p}_\alpha) \to 0 \quad (n \to \infty) \quad \text{for all } \alpha \in [0,1],$$

where $\tilde{s}_{n,\alpha}$ are the α-cuts of $\tilde{s}_n$ and ρ is the given Hausdorff metric.

Fix an initial fuzzy state $\tilde{s} \in \mathcal{F}(E)$. In this section, first we discuss the convergence of the sequence of fuzzy states $\{\tilde{s}_n\}_{n=0}^{\infty}$ defined by (2), and we prove the limiting fuzzy state is a solution of the fuzzy relational equation (1). Next, we give a representation of the limiting fuzzy state for the numerical calculation, by using the characterization (12).

Lemma 3.1. *Let $\alpha \in [0,1]$. Then*

$$\lim_{n\to\infty} \tilde{s}_{n,\alpha} = \bigcap_{n\geq 1} \bigcup_{x\in\tilde{s}_\alpha} \tilde{q}_\alpha^n(x) = \bigcup_{x\in\tilde{s}_\alpha} \bigcap_{n\geq 1} \tilde{q}_\alpha^n(x). \tag{13}$$

We use the following lemma to construct the limiting fuzzy state.

Lemma 3.2 ([4, 6]). *Let a family of subsets $\{D_\alpha \mid \alpha \in [0,1]\} \subset \mathcal{C}(E)$ satisfies the following conditions (a) and (b):*

(a) $D_\alpha \subset D_{\alpha'}$ *for* $0 \leq \alpha' < \alpha \leq 1$.

(b) $\lim_{\alpha'\uparrow\alpha} D_{\alpha'} = D_\alpha$ *for* $\alpha \in (0,1]$.

Then $\tilde{s}(x) := \sup_{\alpha\in[0,1]}\{\alpha \wedge 1_{D_\alpha}(x)\}$, $x \in E$, satisfies $\tilde{s} \in \mathcal{F}(E)$ and $\tilde{s}_\alpha = D_\alpha$ for all $\alpha \in [0,1]$, where 1_D denotes the characteristic function of a set $D \in \mathcal{C}(E)$.

From Lemma 3.1, we define

$$\tilde{p}(x) := \sup_{\alpha\in[0,1]} \{\alpha \wedge 1_{D_\alpha}(x)\}, \quad x \in E, \tag{14}$$

where

$$D_\alpha := \lim_{n\to\infty} \tilde{s}_{n,\alpha} = \bigcap_{n\geq 1} \bigcup_{x\in\tilde{s}_\alpha} \tilde{q}_\alpha^n(x) = \bigcup_{x\in\tilde{s}_\alpha} \bigcap_{n\geq 1} \tilde{q}_\alpha^n(x), \quad \alpha \in [0,1]. \tag{15}$$

Theorem 3.1. *$\tilde{p}$ has the following property (i) and (ii):*

(i) $\tilde{p} = \lim_{n\to\infty} \tilde{s}_n \in \mathcal{F}(E)$.

(ii) *$\tilde{p}$ is a solution of (1).*

Finally, by using the limit theorem (Theorem 3.1) and the characterization of the solutions of the fuzzy relational equation (Theorem 2.1), we give a simple representation of the limiting fuzzy state $\tilde{p}$ for the numerical calculation (Section 4).

Theorem 3.2. *The coefficients in Theorem 2.1 are given by*

$$\alpha^z = \tilde{q}(\tilde{s})(z) = \tilde{p}(z), \quad z \in R_1^{\sim}, \tag{16}$$

where $\tilde{q}(\tilde{s})(z) = \sup_{x\in E}\{\tilde{s}(x) \wedge \tilde{q}(x,z)\}$. *Namely,*

$$\tilde{p}(x) = \sup_{z\in R_1^{\sim}}\{\tilde{q}(\tilde{s})(z) \wedge \tilde{q}(z,x)\} = \sup_{z\in R_1^{\sim}}\{\tilde{p}(z) \wedge \tilde{q}(z,x)\}, \quad x \in E. \tag{17}$$

4. A numerical example

We consider a one-dimensional numerical example to illustrate our results in Sections 2 and 3. Let $E = [-2,2]$. We give the transitive fuzzy relation $\tilde{q}$ (see Fig.1) by

$$\tilde{q}(x,y) = \begin{cases} \left(\frac{f(y)+y-2x}{y-f(y)} \vee 0\right) \wedge 1 & \text{if } xy > 0,\ y \neq -1, 1 \\ 1 & \text{if } y = 0 \\ 1 & \text{if } x \geq 1,\ y = 1 \\ 1 & \text{if } x \leq -1,\ y = -1 \\ 0 & \text{otherwise,} \end{cases} \tag{18}$$

where $f(y) := y^5 - 2y^3 + 2y$, and we put $a \vee 0 := \max\{a, 0\}$ and $a \wedge 1 := \min\{a, 1\}$ for real numbers a. The α-cut of fuzzy relation $\tilde{q}$ is written as follows (see Fig.2):

$$\tilde{q}_\alpha(x) := \begin{cases} [0, \tilde{q}_\alpha(x)^*] & \text{if } x \geq 0 \\ [\tilde{q}_\alpha(x)^*, 0] & \text{if } x < 0, \end{cases} \tag{19}$$

where

$$\tilde{q}_\alpha(x)^* := \begin{cases} \max \tilde{q}_\alpha(x) & \text{if } x \geq 0 \\ \min \tilde{q}_\alpha(x) & \text{if } x < 0. \end{cases}$$

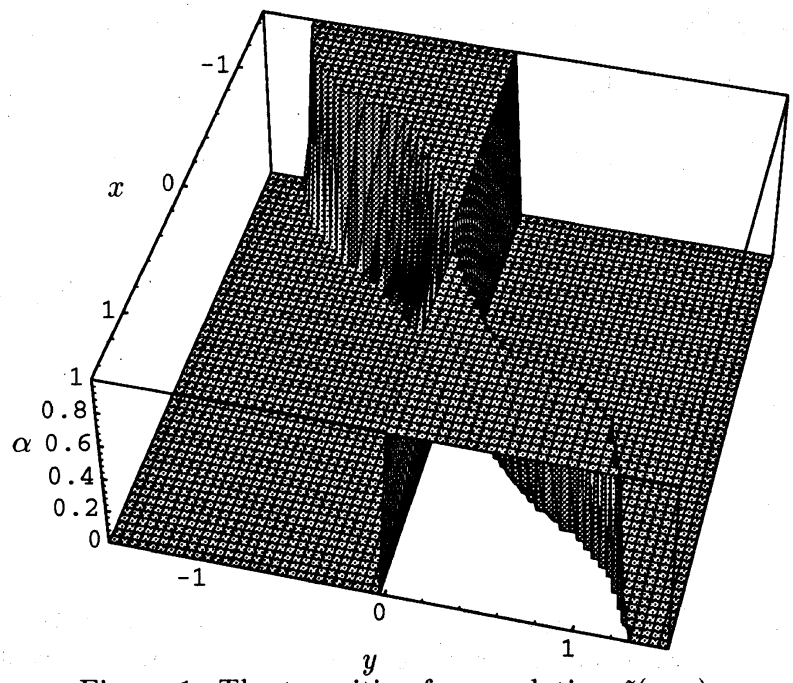

Figure 1. The transitive fuzzy relation $\tilde{q}(x,y)$.

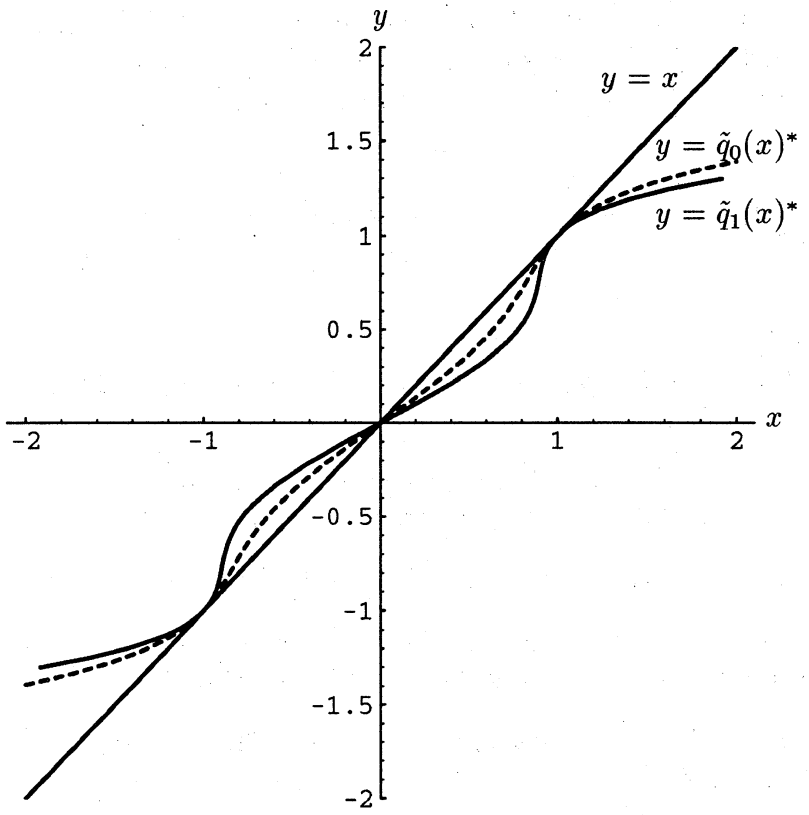

Figure 2. The α-cut of the fuzzy relation $\tilde{q}$.

We can easily check $\tilde{q}$ is transitive (c.f. (5)) since the trnasitivity is equivalent to

$$\tilde{q}_\alpha^2(x) \subset \tilde{q}_\alpha(x), \quad x \in E, \ \alpha \in [0,1]. \tag{20}$$

It is trivial that the map $\tilde{q}_\alpha(\cdot) : E \mapsto \mathcal{C}(E)$ is continuous for all $\alpha \in [0,1]$. Then, we have $R_1^{\sim} = R_1 = \{-1,0,1\}$ and $R_\alpha = \{-1,0,1\}$ $(\alpha \in [0,1])$. If $A \in \mathcal{C}(E)$ satisfies $\tilde{q}_\alpha(A) = A$ for some $\alpha \in (0,1]$, then $A = \{0\}$ or $[-1,0]$ or $[0,1]$ or $[-1,1]$. Therefore, we can easily check that $\tilde{q}$ satisfies Assumption A. From Theorem 3.2, we have the coefficients

$$\alpha^z = \tilde{q}(\tilde{s})(z) = \sup_{x \in E}\{\tilde{s}(x) \wedge \tilde{q}(x,z)\} = \begin{cases} 1 & \text{if } z = 0 \\ \sup_{x \le -1} \tilde{s}(x) & \text{if } z = -1 \\ \sup_{x \ge 1} \tilde{s}(x) & \text{if } z = 1 \end{cases} \tag{21}$$

since we get $\tilde{q}(x,0) = 1$, $\tilde{q}(x,-1) = 1_{[-2,-1]}(x)$ and $\tilde{q}(x,1) = 1_{[1,2]}(x)$ from (18). From (18) and (10), we get the fundamental solutions of (1):

$$p^z(x) = \tilde{q}(z,x) = \begin{cases} 1_{\{0\}}(x) & \text{if } z = 0 \\ 1_{[-1,0]}(x) & \text{if } z = -1 \\ 1_{[0,1]}(x) & \text{if } z = 1. \end{cases} \tag{22}$$

For example, we put an initial fuzzy state $\tilde{s}$ by

$$\tilde{s}_0(x) = \tilde{s}(x) = \max\{1 - |x - 1/2|, 0\}, \quad x \in E.$$

Then, from (21), we have $\alpha^0 = 1$, $\alpha^{-1} = 0$ and $\alpha^1 = 1/2$. By Theorem 3.1, the sequence of fuzzy states $\{\tilde{s}_n\}_{n=0}^{\infty}$ converges to the the limiting fuzzy state $\tilde{p}$. Therefore, from (21), (22) and Theorem 3.2, we obtain the the limiting fuzzy state

$$\tilde{p}(x) = \max_{z=-1,0,1}\{\alpha^z \wedge p^z(x)\} = 1_{\{0\}}(x) \vee \{1/2 \wedge 1_{[0,1]}(x)\} = \begin{cases} 1 & \text{if } x = 0 \\ 1/2 & \text{if } 0 < x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

References

[1] R.E.Bellman and L.A.Zadeh, Decision-making in a fuzzy environment, *Management Sci. Ser B.* **17** (1970) 141-164.

[2] G.J.Klir and B.Yuan, *Fuzzy Sets and Fuzzy Logic: Theory and Applications* (Prentice-Hall, London, 1995). Interscience Publishers, New York, 1958).

[3] K.Kuratowski, *Topology I* (Academic Press, New York, 1966).

[4] M.Kurano, M.Yasuda, J.Nakagami and Y.Yoshida, A limit theorem in some dynamic fuzzy systems, *Fuzzy Sets and Systems* **51** (1992) 83-88.

[5] M.Kurano, M.Yasuda, J.Nakagami and Y.Yoshida, A fuzzy relational equation in dynamic fuzzy systems, to appear in *Fuzzy Sets and Systems.*

[6] V.Novák, *Fuzzy Sets and Their Applications* (Adam Hilder, Bristol-Boston, 1989).

[7] Y.Yoshida, M.Yasuda, J.Nakagami and M.Kurano, A limit theorem in some dynamic fuzzy systems with a monotone property, *Fuzzy Sets and Systems* **94** (1998) 109-119.

[8] Y.Yoshida, The recurrence of dynamic fuzzy systems, to appear in *Fuzzy Sets and Systems.*