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Constructions of bounded weak approximate identities for Segal algebras on R^n.

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In this paper we study bounded weak approximate identities for Segal algebras on R^n. In particular we show that, if a Segal algebra A on R^n belongs to some familiar class, we can construct bounded weak approximate identities for A of norm 1. This results imply at once that Segal algebras in this class are BSE-algebras. Examples of Segal algebras which have no bounded weak approximate identities are also given.

1. Introduction. In the paper of Lahr [2], the bounded weak approximate identities for commutative Banach algebras were defined and used successfully to characterize the existence of the unit element in the structure space of a commutative convolution measure algebra. Further, an example of a convolution measure algebra which has bounded weak approximate identities without no bounded (norm) approximate identities was presented by Jones and Lahr [1].

On the other hand, it was discovered by the second author and Hatori [9] that the bounded weak approximate identities were also important to decide whether a commutative Banach algebra is a **BSE**-algebra or not. In particular, a Segal algebra A on G is **BSE** if and only if A has bounded weak approximate identities.

In this paper we investigate bounded weak approximate identities for Segal algebras A on R^n, and get the following results.

Theorem 3 *For each* $1 < p \leq \infty$, $S^p(R^n)$ *has bounded weak approximate identitities of norm 1.*

Theorem 5 *For each* $1 \leq p < \infty$, $A_p(R^n)$ *has bounded weak approximate identities of norm 1.*

Theorem 6 *Suppose that* G *is a non-disicrete LCA group,* $1 \leq p < \infty$, *and* ν *is a positive Radon measure on* $\hat{G}$ *which has un unbounded discrete part;* $\sum_{\gamma \in \hat{G}} \nu(\{\gamma\}) = \infty$. *Then* $A_{p,\nu}(G)$ *has no bounded weak approximate identities.*

2. Preliminaries. Let A be a commutative semisimple Banach algebra with the maximal ideal space Δ_A. A net $\{e_\lambda\}_{\lambda \in \Lambda}$ in A is called a bounded weak approximate identity for A of norm C if (i) $\sup\{\|e_\lambda\|_A : \lambda \in \Lambda\} = C < \infty$ and (ii) $\lim_\lambda \mid \phi(ae_\lambda) - \phi(a) \mid = 0 \ \ (\phi \in \Delta_A)$.

It is easy to see that a net $\{e_\lambda\}_{\lambda \in \Lambda}$ in A is an bounded weak approximate identity for A if and only if the relation $\lim_\lambda \phi(e_\lambda) = 1 \ \ (\phi \in \Delta_A)$ holds.

Let G be a locally compact abelian group with the dual gourp $\hat{G}$, and let $L^1(G)$ and $M(G)$ denote the group algebra and the measure algebra on G, respectively.

A subspace $S(G)$ of $L^1(G)$ is called a Segal algebra if (i) $S(G)$ is dense in $L^1(G)$, (ii) $S(G)$ is a Banach space under norm $\|\cdot\|_{S(G)}$ satisfying $\|f\|_{S(G)} \geq \|f\|_1 \ \ (f \in S(G))$, (iii) $S(G)$ is invariant under the translation; $f_a \in S(G) \ \ (f \in S(G), a \in G)$, where $f_a(x) = f(x-a)$. (iv) $\|f_a\|_{S(G)} = \|f\|_{S(G)} \ \ (f \in S(G), a \in G)$, (v) for each $f \in S(G)$, the map $a \to f_a$ is continuous from G into $S(G)$.

Further, it is known that a Segal algebra $S(G)$ satisfies the following additional properties: (vi) $\|f * \mu\|_{S(G)} \leq \|f\|_{S(G)}\|\mu\| \ \ (f \in S(G), \mu \in M(G))$, (vii) $S(G)$ is a

Bancach algebra whose maximal ideal space can be idenitified with $\hat{G}$, (viii) $S(G)$ has a bounded (norm) approximate identity if and only if $S(G)$ equals to the group algebra $L^1(G)$.

Examples ([8]). (i) For each $1 < p \leq \infty$, we put $S^p(G) = L^1(G) \cap L^p(G)$. Define norm by

$$\|f\|_{S^p(G)} := \max\{\|f\|_1, \|f\|_p\} \quad (f \in S^p(G)).$$

Then $Sp(G)$ is a Segal algebra.

(ii) Let G be a non-discrete LCA group, and let ν be an unbounded positive Radon measure on $\hat{G}$ and $1 \leq p < \infty$. We put

$$A_{p,\nu}(G) = \{f \in L^1(G) : \|\hat{f}\|_{p,\nu} := \left(\int_{\hat{G}} | \hat{f}(\gamma) |^p \, d\nu(\gamma)\right)^{1/p} < \infty\},$$

where $\hat{f}$ is the Fourier transform of f. Then $A_{p,\nu}(G)$ is a Segal algebra with norm

$$\|f\|_{A_{p,\nu}(G)} = \max\{\|f\|_1, \|\hat{f}\|_{p,\nu}\} \quad (f \in A_{p,\nu}(G)).$$

In particular, when ν is the Haar measure $m_{\hat{G}}$ on $\hat{G}$, we simply denote $A_{p,m_{\hat{G}}}(G)$ by $A_p(G)$.

$S^p(G)$ and $A_p(G)$ are typical and well known example of Segal algebras, which were studied by several authors ([3], [4], [5], [6], [7], [8], [10]).

3. Bounded weak approximate identities for Segal algebras

Lemma 1 (Jones and Lahr [1]). *Let G be an infinite discrete abelian group. Then there exists a net $\{g_\lambda\} \subseteq G \setminus \{0\}$, such that* $\lim_\lambda (g_\lambda, \gamma) = 1$ $(\gamma \in \hat{G})$.

Lemma 2 *Let $\{x_1, ..., x_m\}$ be a finite subset of R, and let $\varepsilon > 0$ be arbitrary. Then there exists a sequence $\{n_1, n_2, ...\}$ of natural numbers such that $n_1 < n_2 < ...$ and $| e^{in_k x_j} - 1 | < \varepsilon$ $(1 \leq j \leq m,\ k = 1, 2, ...)$.*

Proof. By Lemma 1, there exists a net of non-zero integers $\{n_\lambda\}_{\lambda\in\Lambda}$, such that $e^{in_\lambda x} \to 1$ for each $x \in R$. Since $e^{-in_\lambda x} \to 1$ for each $x \in R$, we can assume that all n_λ are positive integers. In this case, we have $n_\lambda \to \infty$. In fact if not, then there exists a subnet $\{n_{\lambda'}\}$ of $\{n_\lambda\}$ which converges to some integer $n_0 \neq 0$, hence $e^{in_0 x} = 1$ for each $x \in R$, a contradiction. Then the elementary convergence argument implies the desired result. Q.E.D.

Theorem 3 For each $1 < p \leq \infty$, $S^p(R^n)$ has weak approximate identitities of norm 1.

Proof. First we prove the case $n = 1$. Let $\varepsilon > 0$ and a finite subset $F = \{\xi_1, \xi_2, ..., \xi_{m_0}\}$ of R be arbitrary. Then we can fined $M > 0$ and a natural number n_0 such that

$$F \subseteq [-M, M], \tag{1}$$

$$\left| n_0 \hat{\mathcal{X}}_E(\xi) - 1 \right| \leq \varepsilon/2 \quad (\xi \in [-M, M]), \tag{2}$$

where $\hat{\mathcal{X}}_E$ is Fourier transform of the characteristic function $\mathcal{X}_E$ of the interval $E = [\frac{-1}{2n_0}, \frac{1}{2n_0}]$. Further, by Lemma 2, we can choose positive integers $N_1, N_2, ..., N_{n_0}$ such that

$$(E - N_k) \cap (E - N_\ell) = \emptyset \;\; (1 \leq k < \ell \leq n_0), \tag{3}$$

$$\mid e^{iN_k \xi_j} - 1 \mid \leq \frac{\varepsilon}{\varepsilon + 2} \quad j = 1, ...m_0, \;\; k = 1, ..., n_0. \tag{4}$$

Put $e = e(F, \varepsilon) := \sum_{j=1}^{n_0} \mathcal{X}_{E-N_j}$, then it follows from (2) and (3) that

$$\left| \hat{\mathcal{X}}_E(\xi_j) \right| \leq \frac{2+\varepsilon}{2n_0} \;\; (j = 1, 2, ..., m_0), \tag{5}$$

$$\|e\|_{S^p(R)} = 1. \tag{6}$$

Further, by (2), (4) and (5), we have

$$\mid \hat{e}(\xi_j) - 1 \mid \;\; = \;\; \left| \sum_{k=1}^{n_0} \hat{\mathcal{X}}_{E-N_k}(\xi_j) - 1 \right|$$

$$
\begin{aligned}
&= \left|\sum_{k=1}^{n_0} e^{i\xi_j N_k}\hat{\mathcal{X}}_E(\xi_j) - n_0\hat{\mathcal{X}}_E(\xi_j) + n_0\hat{\mathcal{X}}(\xi_j) - 1\right| \\
&\le \sum_{k=1}^{n_0}\left|e^{i\xi_j N_k} - 1\right|\left|\hat{\mathcal{X}}_E(\xi_j)\right| + \left|n_0\hat{\mathcal{X}}_E(\xi_j) - 1\right| \\
&\le n_0\frac{\varepsilon}{2+\varepsilon}\frac{2+\varepsilon}{2n_0} + \frac{\varepsilon}{2} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad (j = 1, 2, ..., m_0). \qquad (7)
\end{aligned}
$$

Let $\Lambda := \{e(F, \varepsilon) : F \subseteq R \text{ (finite set)}, \ \varepsilon > 0\}$, and define partial order in Λ by

$$
(F_1, \varepsilon_1) \le (F_2, \varepsilon_2) \text{ if and only if } F_1 \subseteq F_2, \text{ and } \varepsilon_2 \le \varepsilon_1.
$$

Then it is obvious by (6) and (7) that $\{e(F, \varepsilon)\}_{(F,\varepsilon)\in\Lambda}$ is a bounded weak approximate identity for $S^p(R)$ of norm 1.

Now, let us consider the general case; $n \ge 1$. Let $\varepsilon > 0$ and a finite subset $F = \{\xi_1, ..., \xi_{m_0}\}$ of R^n be arbitrary, where $\xi_j = (\xi_{j,1}, ..., \xi_{j,n})$ for $j = 1, ..., m_0$. Set $F_k = \{\xi_{1,k}, ..., \xi_{m_0,k}\}$ for $k = 1, ..., n$. As we have proved above, we can choose $e_k \in L^1(R)$ $k = 1, 2, ..., n$ such that

$$
\max\{\|e_k\|_1, \|e_k\|_p\} \le 1, \quad | \widehat{e_k}(\xi_{j,k}) - 1 | \le \varepsilon/n \ (j = 1, ..., m_0).
$$

We define $e_{(F,\varepsilon)} \in L^1(R^n)$ by

$$
e_{(F,\varepsilon)}((x_1, ..., x_n)) := \prod_{k=1}^{n} e_k(x_k) \ \ ((x_1, ..., x_n) \in R^n).
$$

Then we have

$$
\max\{\|e_{(F,\varepsilon)}\|_1, \|e_{(F,\varepsilon)}\|_p\} = \max\{\prod_{k=1}^{n} \|e_k\|_1, \prod_{k=1}^{n} \|e_k\|_p\} \le 1,
$$

and

$$
\begin{aligned}
| \widehat{e_{(F,\varepsilon)}}(\xi_j) - 1 | &\le \ | \sum_{k=2}^{n}\prod_{\ell=1}^{k} \widehat{e_\ell}(\xi_{j,\ell}) - \prod_{\ell=1}^{k-1} \widehat{e_\ell}(\xi_{j,\ell}) | + | \widehat{e_1}(\xi_{j,1}) - 1 | \\
&\le \sum_{k=1}^{n} | \widehat{e_k}(\xi_{j,k}) - 1 | \le \varepsilon \ \ (j = 1, ..., m_0).
\end{aligned}
$$

Therefore we can constract a bouded weak approximate identity $\{e_{(F,\varepsilon)}\}_{(F,\varepsilon)\in\Lambda}$ for $S^p(R^n)$ of norm 1, where $\Lambda = \{(F, \varepsilon) : F \subset R^n(\text{finite set}), \varepsilon > 0\}$ is a directed set with the partial order defined in the same way as in the case $n = 1$. Q.E.D.

Lemma 4 *For each* $1 \leq p < \infty$, *we have*

$$A_1(R^n) \subseteq A_p(R^n) \text{ and } \|f\|_{A_p(R^n)} \leq \|f\|_{A_1(R^n)} \quad (f \in A_1(R^n)).$$

Proof. Let $f \in A_1(R^n)$ and $1 \leq p < \infty$. We consider the two cases:

(i) $\|\hat{f}\|_1 \leq \|f\|_1$. Set $g = \frac{f}{\|f\|_1}$. Then $\|\hat{g}\|_\infty \leq \|g\|_1 = 1$ and hence $\|\hat{g}\|_p^p \leq \|\hat{g}\|_1 \leq 1$. Therefore $\|g\|_{A_p(R^n)} = \|g\|_{A_1(R^n)} (= 1)$ and so $\|f\|_{A_p(R^n)} = \|f\|_{A_1(R^n)}$.

(ii) $\|\hat{f}\|_1 > \|f\|_1$. Set $g = \frac{f}{\|\hat{f}\|_1}$. Then $1 = \|\hat{g}\|_1 > \|g\|_1$ and hence $\|g\|_{A_1(R^n)} = 1$. Also since $\|\hat{g}\|_\infty \leq \|g\|_1 < 1$, it follows that $\|\hat{g}\|_p^p \leq \|\hat{g}\|_1 = 1$ and so $\|\hat{g}\|_{A_p(R^n)} \leq 1$. Thus we obtain $\|g\|_{A_p(R^n)} \leq \|g\|_{A_1(R^n)}$, which implies that $\|f\|_{A_p(R^n)} \leq \|f\|_{A_1(R^n)}$. Q.E.D.

Theorem 5 *For each* $1 \leq p < \infty$, $A_p(R^n)$ *has bounded weak approximate identities of norm 1.*

Proof. By Lemma 4, it suffices to prove only for $p = 1$.

First we prove the case $n = 1$. Consider any finite set $F = \{\xi_1, ..., \xi_{m_0}\}$ of R and any $\varepsilon > 0$. Set $u = \mathcal{X}_{[-1/2,1/2]} \star \mathcal{X}_{[-1/2,1/2]}$ and let ϕ be the Fourier inverse transform of u. Then $\hat{\phi} = u$ and $\|\phi\|_1 = \|u\|_1 = 1$. Choose $\delta > 0$ such that $1 - u(\xi) < \frac{\varepsilon}{2}$ for all $\xi \in [-\delta, \delta]$, and take a natural number n_0 such that $|\frac{\xi_j}{n_0}| < \delta$ $(j = 1, ..., m_0)$. Furthermore, take a sufficiently large number $L_0 > 0$ such that $|\widehat{\mathcal{X}}_{[-n_0,n_0]}(\xi)| < \varepsilon$ for all $| \xi | \geq L_0$. Also, by Lemma 2, we can choose a finite set $\{N_1, ..., N_{n_0}\}$ of natural numbers such that $L_0 \leq N_1$, $L_0 \leq N_{j+1} - N_j$ $(j = 1, ..., n_0 - 1)$ and $| e^{i\xi_j N_k} - 1 | < \frac{\varepsilon}{4}$ $(j = 1, ..., m_0; k = 1, ..., n_0)$. Set

$$\mu := \frac{1}{n_0} \sum_{k=1}^{n_0} \frac{\delta_{N_k} + \delta_{-N_k}}{2} \quad \text{and} \quad \eta_{(F,\varepsilon)} := \mu \star \mu \star \phi_{n_0},$$

where δ_{N_k} is an unit point mass at N_k, and $\phi_{n_0}(x) = n_0 \phi(n_0 x)$ $(x \in R)$. Obviously we have $\hat{\delta}_x(\xi) = e^{-i\xi x}$ $(x \in R)$, $\hat{\mu}(\xi) = \frac{1}{2n_0} \sum_{k=1}^{n_0} (e^{-i\xi N_k} + e^{i\xi N_k})$ and $\hat{\phi}_{n_0}(\xi) = u(\frac{\xi}{n_0})$ $(\xi \in R)$. We shall show that

$$\|\eta_{(F,\varepsilon)}\|_{A_1(R)} \leq 1 + \varepsilon \quad \text{and} \quad | \hat{e}(\xi_j) - 1 | < \varepsilon \ (j = 1, ..., m_0).$$

To do this, note that

$$\widehat{\eta_{(F,\varepsilon)}}(\xi) = \hat{\mu}(\xi)^2\hat{\phi}_{n_0}(\xi) = u\Big(\frac{\xi}{n_0}\Big)\frac{1}{4n_0^2}\sum_{k,l=1}^{n_0}(e^{-i\xi N_k} + e^{i\xi N_k})(e^{-i\xi N_l} + e^{i\xi N_l}),$$

for each $\xi \in R$. Hence we have

$$\begin{aligned}
\|\widehat{\eta_{(F,\varepsilon)}}\|_1 &\leq \frac{1}{4n_0^2}\sum_{k,l=1}^{n_0}\int_{-n_0}^{n_0}(e^{-i\xi N_k} + e^{i\xi N_k})(e^{-i\xi N_l} + e^{i\xi N_l})d\xi \\
&= \frac{1}{4n_0^2}\Big[\sum_{k\neq l}\sum_{\varepsilon_1,\varepsilon_2=\pm 1}\widehat{\mathcal{X}_{[-n_0,n_0]}}(\varepsilon_1 N_k + \varepsilon_2 N_l) \\
&\qquad + \sum_{k=1}^{n_0}\Big(\widehat{\mathcal{X}_{[-n_0,n_0]}}(-2N_k) + \widehat{\mathcal{X}_{[-n_0,n_0]}}(2N_k) + 4n_0\Big)\Big] \\
&\leq \frac{1}{4n_0^2}\Big(4(n_0^2 - n_0)\varepsilon + 2n_0\varepsilon + 4n_0^2\Big) < 1 + \varepsilon.
\end{aligned}$$

Also $\|\eta_{(F,\varepsilon)}\|_1 \leq \|\mu\|^2\|\phi_{n_0}\|_1 \leq \|\phi\|_1 = 1$, so we have $\|\eta_{(F,\varepsilon)}\|_{A_1(R)} \leq 1+\varepsilon$. Moreover,

$$\begin{aligned}
|\ \widehat{\eta_{(F,\varepsilon)}}(\xi_j) - 1\ | &= \Big|\hat{\mu}(\xi_j)^2 u(\frac{\xi_j}{n_0}) - 1\Big| \\
&\leq |\ \hat{\mu}(\xi_j)^2 - 1\ | + \Big|\hat{\mu}(\xi_j)^2\Big(u(\frac{\xi_j}{n_0}) - 1\Big)\Big| \\
&\leq 2\ |\ \hat{\mu}(\xi_j) - 1\ | + \Big|u(\frac{\xi_j}{n_0}) - 1\Big| \\
&\leq 2\frac{1}{2n_0}\sum_{k=1}^{n_0}\Big(|\ e^{-i\xi_j N_k} - 1\ | + |\ e^{i\xi_j N_k} - 1\ |\Big) + \frac{\varepsilon}{2} \\
&\leq 2\frac{1}{2n_0}n_0\Big(\frac{\varepsilon}{4} + \frac{\varepsilon}{4}\Big) + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

for all $j = 1, ..., m_0$.

Let $\Lambda = \{(F,\varepsilon) : F \subseteq R \text{ (finite set)},\ \varepsilon > 0\}$ be the directed set introduced in the proof of Theorem 3. For each $\Lambda \ni (F,\varepsilon)$, set $e_{(F,\varepsilon)} = \frac{1}{1+\varepsilon/2}\eta_{(F,\varepsilon/2)}$. Then we have $\|e_{(F,\varepsilon)}\|_{A_1(R)} \leq 1$ and

$$|\ \widehat{e_{(F,\varepsilon)}}(\xi) - 1\ | = \frac{1}{1+\varepsilon/2}\Big|\widehat{e_{(F,\varepsilon/2)}}(\xi) - 1 - \varepsilon/2\Big| \leq \frac{2\varepsilon/2}{1+\varepsilon/2} \leq \varepsilon \quad (\xi \in F).$$

Therefore we see that $\{e_{(F,\varepsilon)}\}_{(F,\varepsilon)\in\Lambda}$ is a bounded weak approximate identity for $A_1(R)$ of norm 1.

Now let us consider the general case; $n \geq 1$. In this case, we can constract bounded weak approximate identities for $A_1(R^n)$ of norm 1 from bounded weak approximate identities for $A_1(R)$ of norm 1 by the same way as in theorem 3 above, and the proof is complete. Q.E.D.

Although, as we have seen, $S^p(R^n)$ and $A_p(R^n)$ have weak approximate identities, it is not true that every Segal algebra on R^n has weak approximate identities, as the next theorem shows.

Theorem 6 *Suppose that G is a non-disicrete LCA group, $1 \leq p < \infty$, and ν is a positive Radon measure on $\hat{G}$ which has un unbounded discrete part; $\sum_{\gamma \in \hat{G}} \nu(\{\gamma\}) = \infty$. Then $A_{p,\nu}(G)$ has no bounded weak approximate identities.*

Proof. Suppose, on the contrary, that there exists a bounded weak approximate identity $\{e_\lambda\}_{\lambda \in \Lambda}$ for $A_{p,\nu}(G)$. Since ν has an unbounded discrete part, we can choose an infinite sequence $\{\gamma_n\}$ of elemtnts of $\hat{G}$ such that $\sum_{n=1}^{\infty} \nu(\{\gamma_n\}) = \infty$. Then by the definition of the norm of $\|e_\lambda\|_{A_{p,\nu}(G)}$, we get

$$\sup_\lambda \|e_\lambda\|_{A_{p,\nu}(G)} \geq \sup_\lambda \Big(\sum_{n=1}^{\infty} \mid \widehat{e_\lambda}(y_n) \mid^p \nu(\{\gamma_n\}) \Big)^{1/p}. \tag{8}$$

But the right side of the inequality (8) is infinite since $\sum_n \nu(\{\gamma_n\}) = \infty$ and $\lim_\lambda \widehat{e_\lambda}(\gamma) = 1$ $(\gamma \in \hat{G})$. This contradict to the assumption that $\{e_\lambda\}_{\lambda \in \Lambda}$ has a bounded norm. Q.E.D.

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