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SINGULARITIES OF $\mathbb{R}P^2$-VALUED GAUSS MAPS OF SURFACES IN MINKOWSKI 3-SPACE

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1. INTRODUCTION

In [1], D.Bleecker and L.Wilson studied the classification of singularities and the stability of the Gauss map of a closed surface in Euclidean 3-space. In this paper, we study the same theme as in [1] for a closed surface in Minkowski 3-space. Classically, for an oriented surface in Euclidean 3-space, the Gauss map sends each point on the surface to the unit normal, so the value of Gauss map is in the unit sphere S^2. In Minkowski 3-space, there are three kinds of vectors named space-like, time-like and light-like. In particular, the norm of a light-like vector is zero.

On the other hand, we can always determine the pseudo-normal vector of the surface associated with Minkowski metric. When the pseudo-normal vector of the surface is light-like, we can not consider the unit vector along it. Because of this reason, the notion which is analogous to the Euclidean Gauss map can only be defined at the point where the pseudo-normal direction is not light-like. In order to avoid the above difficulty, we consider $\mathbb{R}P^2$-valued Gauss maps. We now formulate as follows:

Let $\mathbb{R}^3 = \{(x_1, x_2, x_3) | x_1, x_2, x_3 \in \mathbb{R}\}$ be a 3-dimensional vector space, $\boldsymbol{x} = (x_1, x_2, x_3)$ and $\boldsymbol{y} = (y_1, y_2, y_3)$ be two vectors in $\mathbb{R}^3$, the *pseudo scalar product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined by $< \boldsymbol{x}, \boldsymbol{y} >= -x_1y_1 + x_2y_2 + x_3y_3$. $(\mathbb{R}^3, <,>)$ is called a 3-dimensional *pseudo Euclidean space*, or *Minkowski 3-space*. We denote ${\mathbb{R}_1}^3$ as $(\mathbb{R}^3, <,>)$. For any $\boldsymbol{x} = (x_1, x_2, x_3)$, $\boldsymbol{y} = (y_1, y_2, y_3) \in {\mathbb{R}_1}^3$, the *pseudo vector product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined by

$$\boldsymbol{x} \wedge \boldsymbol{y} = \begin{vmatrix} -e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = (-(x_2y_3 - x_3y_2), x_3y_1 - x_1y_3, x_1y_2 - x_2y_1).$$

We say that $\boldsymbol{x}$ is *pseudo perpendicular* to $\boldsymbol{y}$ if $< \boldsymbol{x}, \boldsymbol{y} >= 0$. Clearly, we get $< \boldsymbol{x} \wedge \boldsymbol{y}, \boldsymbol{x} >=< \boldsymbol{x} \wedge \boldsymbol{y}, \boldsymbol{y} >= 0$, so that $\boldsymbol{x} \wedge \boldsymbol{y}$ is pseudo perpendicular to both of $\boldsymbol{x}$ and $\boldsymbol{y}$. Moreover, $\boldsymbol{x}$ in ${\mathbb{R}_1}^3$ is called a *space-like vector*, a *light-like vector* or a *time-like vector* if $< \boldsymbol{x}, \boldsymbol{x} > > 0$, $< \boldsymbol{x}, \boldsymbol{x} > = 0$ or $< \boldsymbol{x}, \boldsymbol{x} > < 0$ respectively. Let $\boldsymbol{a} = (a_1, a_2, a_3)$ be a point and $\boldsymbol{n} = (n_1, n_2, n_3)$ a vector in ${\mathbb{R}_1}^3$. Then the equation $< \boldsymbol{n}, \boldsymbol{x} - \boldsymbol{a} >= 0$ (i.e. $-n_1(x_1 - a_1) + n_2(x_2 - a_2) + n_3(x_3 - a_3) = 0$) which passes through the point $\boldsymbol{a}$ and is pseudo perpendicular to the vector $\mathbf{n}$ is called an *equation of the plane*, where $\boldsymbol{x} = (x_1, x_2, x_3) \in {\mathbb{R}_1}^3$, and $\mathbf{n}$ is called a *pseudo normal vector* of the plane. We also say

*On leave from Department of Mathematics, North East Normal University, Chang Chun 130024, P.R.China that the plane is *time-like, light-like* or *space-like* if the pseudo normal vector $\mathbf{n}$ is space-like, light-like or time-like respectively. Let M be a compact 2-dimensional manifold and $f : M \longrightarrow {\mathbb{R}_1}^3$ be an immersion. We now define a map $N(f) : M \longrightarrow \mathbb{R}P^2$ by

$$M \ni x \longmapsto \langle X_u(x) \wedge X_v(x) \rangle_{\mathbb{R}}.$$

We call $N(f)$ the *$\mathbb{R}P^2$-valued Gauss map* associated with the immersion f. Here, $X = X(u,v)$ is a local parametrization of $f(M)$. By the previous argument, $X_u(x) \wedge X_v(x)$ is the pseudo normal vector of the tangent plane $T_{f(x)}f(M)$. We can separate M into three parts as follows:

$$\begin{aligned} {M_s}^f &= \{x \in M \mid X_u(x) \wedge X_v(x) : \text{time-like}\}; \\ {M_l}^f &= \{x \in M \mid X_u(x) \wedge X_v(x) : \text{light-like}\}; \\ {M_t}^f &= \{x \in M \mid X_u(x) \wedge X_v(x) : \text{space-like}\}. \end{aligned}$$

We respectively call ${M_s}^f$, ${M_l}^f$ or ${M_t}^f$ a *space-like part*, a *light-like part* or a *time-like part*. It is clear that ${M_s}^f$,${M_t}^f$ are open submanifolds. We now formulate the main result in this paper as follows:

Let M be a compact 2-dimensional manifold and $I(M, {\mathbb{R}_1}^3)$ the space of C^∞ immersions $f : M \longrightarrow {\mathbb{R}_1}^3$ equipped with the Whitney C^∞-topology. For any $f \in I(M, {\mathbb{R}_1}^3)$, the singular set of $\mathbb{R}P^2$-valued Gauss map $N(f)$ is called a *parabolic set* of f. Moreover, when $g : N \longrightarrow P$ is a C^∞ map between two 2-dimensional manifolds, a point $x \in N$ is called a *fold point* of g if there exist local coordinates (x_1, x_2) and (y_1, y_2) in neighbourhoods of x and $g(x)$ respectively, such that $y_1 \circ g = x_1$ and $y_2 \circ g = {x_2}^2$. A point $x \in N$ is called a *cusp point* of g if there exist local coordinates (x_1, x_2) and (y_1, y_2) such that $y_1 \circ g = x_1$ and $y_2 \circ g = {x_2}^3 + x_1 x_2$. Our main theorem is as follows.

Theorem A. *There exists a dense set $\mathcal{O} \subset I(M, {\mathbb{R}_1}^3)$ such that the following conditions hold for any $f \in \mathcal{O}$.*

(1) *The parabolic set of f consists of regular curves (called a parabolic locus in M).*
(2) *The set of cusp points on parabolic locus of f is a finite set and other points are fold points.*
(3) *The light-like part ${M_l}^f$ is a union of regular curves (called a light-like locus in M).*
(4) *The light-like locus and the parabolic locus in M intersect transversally, the intersections consist of fold points of $N(f)$.*
(5) *The set of points in ${M_l}^f$ consisting of the points at where the tangent line of ${M_l}^f$ is light-like is a set of isolated points.*
(6) *The set of points in the parabolic locus consisting of the points at where the tangent line of the parabolic locus is light-like is a set of isolated points.*

Remark. We can show that there exists an open dense set $\mathcal{O} \subset I(M, {\mathbb{R}_1}^3)$ such that $N(f)$ is stable for any $f \in \mathcal{O}$. Nevertheless, we omit the proof.

In §2 we give the proof of theorem A. The geometric meanings and properties of the $\mathbb{R}P^2$-valued Gauss map will be discussed in §3. Especially, Theorem A will be interpreted geometrically (cf., Theorem 3.5). Some examples will be given in §4.

All the manifolds and maps we consider in this paper are of class C^∞ unless otherwise specified.

2. Proof of Theorem A

In this section we give the proof of Theorem A. The idea of the proof for the assertions (1),(2),(3) is analogous to that of Theorem 1.1 in Bleecker and Wilson [1].

Let M be a compact 2-dimensional manifold. For any $f \in I(M, {\mathbb{R}_1}^3)$, we have the $\mathbb{R}P^2$-valued Gauss map $N(f) : M \longrightarrow \mathbb{R}P^2$. This correspondence induces a map $N : I(M, {\mathbb{R}_1}^3) \longrightarrow C^\infty(M, \mathbb{R}P^2)$. Then we have the following lemma.

Lemma 2.1. *The map $N : I(M, {\mathbb{R}_1}^3) \longrightarrow C^\infty(M, \mathbb{R}P^2)$ is continuous, where we also consider the Whitney C^∞-topology on $C^\infty(M, \mathbb{R}P^2)$.*

Proof. Define $I^1(2,3) = \{j^1 f(0) \in J^1(2,3) : \mathrm{rank} J_f|_0 = 2\}$. For an open set $U \subset M$, we also define $I^1(U, {\mathbb{R}_1}^3) = \{j^1 f(x) \in J(U, {\mathbb{R}_1}^3) : \mathrm{rank} J_{f(x)} = 2\}$. Let u_x denote the partial derivative of a function $u : U \longrightarrow \mathbb{R}$ with respect to a coordinate x. We can choose $(f_x, f_y)(0) = (u_x, v_x, w_x, u_y, v_y, w_y)(0)$ as coordinates of $j^1 f(0) \in J^1(2,3)$, where $f = (u, v, w)$. If $j^1 f(0) \in I^1(2,3)$, then

$$\gamma = (u_x, v_x, w_x) \wedge (u_y, v_y, w_y) \neq 0$$

and γ is pseudo normal to the image of f.

We now define a map $\rho : I^1(2,3) \longrightarrow \mathbb{R}P^2$ by

$$\rho(j^1 f(0)) = \langle \gamma \rangle_{\mathbb{R}}.$$

Then we can extend the map to the C^∞ map on $I^1(M, {\mathbb{R}_1}^3)$. In fact

$$I^1(M, {\mathbb{R}_1}^3; p, q) = I^1(U, V; p, q)$$

$$= I^1(\varphi(U), \psi(V); 0, 0) = I^1({\mathbb{R}_1}^2, {\mathbb{R}_1}^3; 0, 0).$$

i.e.

$$\Phi : I^1(U, V; p, q = f(p)) \longrightarrow I^1(\varphi(U), \psi(V); 0, 0)$$

$$\Phi(j^1 f(p)) = j^1(\psi \circ f \circ \varphi^{-1})(0)$$

is an isomorphism, where (U, φ) is a coordinate neighbourhood of M and (V, ψ) a coordinate neighbourhood of ${\mathbb{R}_1}^3$. The map

$$j^1 : I(M, {\mathbb{R}_1}^3) \longrightarrow C^\infty(M, I^1(M, {\mathbb{R}_1}^3))$$

is continuous by II 3.4 of [3], ρ_* is continuous by II 3.5 of [3]. Thus $\rho_* \circ j^1(f) = N(f)$ is also continuous. Therefore $N(f)$ is continuous. □

Since $f : M \longrightarrow {\mathbb{R}_1}^3$ is an immersion, $f(M)$ can be at least locally written as the graph of a function on a neighbourhood of each point. We can distinguish three cases for the local representation as the graph of functions.

Case 1). When $f(M)=\{(x,y,F(x,y))|(x,y)\in\mathbb{R}^2\}$, we may write $f(x,y)=(x,y,F(x,y))$. Let $[\chi;\eta;\zeta]$ denote homogeneous coordinates on $\mathbb{R}P^2$, then $N(f)(x,y)=[F_x;-F_y;1]$. Hence $N(f)(x,y)=(F_x,-F_y)$ in the affine coordinate neighbourhood $(U_\zeta,(X,Y))$, where $U_\zeta=\{[\chi;\eta;\zeta]|\zeta\neq 0\}$, $X=\frac{\chi}{\zeta}$ and $Y=\frac{\eta}{\zeta}$. If we consider the linear transformation $(X,Y)\xrightarrow{A}(X,-Y)$, then $A\circ N(f)(x,y)=(F_x,F_y)=\mathrm{grad}F(x,y)$.

Case 2). When $f(M)=\{(x,F(x,z),z)|(x,z)\in\mathbb{R}^2\}$, we may also write $f(x,z)=(x,F(x,z),z)$, so we have $N(f)(x,z)=[-F_x;-1;F_z]$. By the same arguments as that of in the case 1), we have $N(f)(x,z)=(F_x,F_z)=\mathrm{grad}F(x,z)$ in the affine coordinate neighbourhood $(U_\eta,(X,Z))$. Hence $N(f)(x,z)=(F_x,F_z)=\mathrm{grad}F(x,z)$ by the linear transformation $(X,Z)\xrightarrow{A}(X,-Z)$.

Case 3). When $f(M)=\{(F(y,z),y,z)|(y,z)\in\mathbb{R}^2\}$, we may also write $f(y,z)=(F(y,z),y,z)$, then $N(f)(y,z)=[-1;-F_y;-F_z]$. Hence $N(f)(y,z)=(F_y,F_z)$ $=\mathrm{grad}F(y,z)$ in the affine coordinate neighbourhood $(U_\chi,(Y,Z))$.

For each pair of manifolds M,N and nonincreasing, finite sequence $\omega=(i_1,i_2,\ldots,i_k)$ of nonnegative integers there is a fiber subbundle S^ω of $J^k(M,N)$ called a *Thom-Boardman singularity*. Let $S^{i_1}(f)=\{x\in M:\dim(\ker T_xf)=i_1\}$, $S^{i_1,i_2}(f)=\{x\in M:\dim(\ker T_xf|_{S^{i_1}(f)})=i_2\}$ $(S^\omega(f)=\{x\in M:j^kf(x)\in S^\omega\})$, etc. then $J^3(\mathbb{R}^3,\mathbb{R}^2)=S^0\bigcup S^1\bigcup S^2$. Here, $S^1=S^{1,0}\bigcup S^{1,1}; S^{1,1}=S^{1,1,0}\bigcup S^{1,1,1}$. Let I_k denote $(1,1,\ldots,1)$ k-times, then we have $\mathrm{codim}S^2=4$; $\mathrm{codim}S^{I_k}=k$ (c.f., [3], II.5.4).

We define a map $\Gamma:J^4(\mathbb{R}^2,\mathbb{R})\longrightarrow J^3(\mathbb{R}^2,\mathbb{R}^2)$ by $\Gamma(j^4F(x))=j^3(\text{grad F})(x)$. Let $T^\omega=\Gamma^{-1}S^\omega$ for each ω. Then we have the following lemma.

Lemma 2.2. *(Bleecker-Wilson [1], the proof of Proposition 2.2)*

(1) T^0,T^{I_k},T^2 *are submanifolds of* $J^4(\mathbb{R}^2,\mathbb{R})$ *with codim* $T^0=0$, *codim* $T^{I_k}=k$ *and codim*$T^2=4$.

(2) j^4F *is transversal to* T^{I_k} *if and only if* j^3*(grad F) is transversal to* S^{I_k}.

We say that a map $g\in C^\infty(\mathbb{R}^2,\mathbb{R}^2)$ is *excellent* (respectively, *good*) if $j^3g\pitchfork S^2$ (respectively, $j^1g\pitchfork S^2$), and $j^3g\pitchfork S^{I_k}$ (respectively, $j^1g\pitchfork S^{I_k}$). Where $\pitchfork$ denote the transversal intersection. When g is excellent, it is well-known that $S^{1,0}$ is the fold points set, $S^{1,1,0}$ is the cusp points set (c.f., [3]). Since $\mathrm{codim}S^{1,1,1}>2$ and $\mathrm{codim}S^2>2$, $S^{1,1,1}(f)=S^2(f)=\phi$.

Proposition 2.3. *Let M be a compact 2-dimensional manifold. We denote that*

$$\mathcal{Q}_e=\{f\in I(M,{\mathbb{R}_1}^3)|N(f):\textit{excellent}\},$$

then $\mathcal{Q}_e$ is an open and dense subset of $I(M,{\mathbb{R}_1}^3)$.

Proof. Since $S^{1,0}=(S^1-S^{1,1})$ is the set of fold points and $S^{1,1,0}=(S^{1,1}-S^{1,1,1})$ is the set of cusp points, $\mathcal{Q}_e$ is the set of $f\in I(M,{\mathbb{R}_1}^3)$ which satisfies $j^3N(f)\cap(S^2\cup S^{1,1,1})=\phi$. Since $S^2,S^{1,1,1}$ are closed sets and N is continuous by Lemma 2.1, $\mathcal{Q}_e$ is an open set. Define

$$I(M,{\mathbb{R}_1}^3)=\{j^4f(x)\in J^4(M,{\mathbb{R}_1}^3)|\mathrm{rank}df(x)=2\},$$

then it is an open subset of $J^4(M, {\mathbb{R}_1}^3)$. We also define

$$O_1 = \{z = j^4(f_1, f_2, f_3)(x) | H_1 = (f_2, f_3) : \text{nonsingular}, at\ x\},$$

then O_1 is also an open subset of $I^4(M, {\mathbb{R}_1}^3)$, and O_2, O_3 are defined analogously. In this case, the map $\pi_1 : O_1 \longrightarrow J^4(\mathbb{R}^2, \mathbb{R}^1)$ defined by

$$\pi_1(z) = j^4(f_1 \circ {H_1}^{-1})(y)$$

is a submersion, where $z \in O_1$ and $y = H_1(x)$. We define a map

$$\widetilde{H_4} : J^4(U, \mathbb{R}^1) \longrightarrow J^4(U, \mathbb{R}^1);$$

by

$$\widetilde{H_4}(j^4 g(x)) = j^4 g \circ {H_1}^{-1}(y)$$

(U is an open subset of $\mathbb{R}^2$), then the differential map

$$d\widetilde{H^4} : T_x J^4(\mathbb{R}^2, \mathbb{R}^1) \longrightarrow T_x J^4(\mathbb{R}^2, \mathbb{R}^1)$$

is an isomorphism. And the map $P : O_1 \longrightarrow J^4(\mathbb{R}^2, \mathbb{R}^1)$ defined by

$$P(z) = j^4 f_1(x),$$

Then the differential map

$$dP : T_z O_1 \longrightarrow T_x J^4(\mathbb{R}^2, \mathbb{R}^1)$$

is onto. Thus $d\pi_1$ is surjective by the following commutative diagram, so π_1 is a submersion.

$$\begin{tikzcd}
T_z O_1 \arrow[r, "dP"] \arrow[d, "d\pi_1"'] & T_x J^4(\mathbb{R}^2, \mathbb{R}^1) \arrow[d, "d\widetilde{H^4}"] \\
T_x J^4(U, \mathbb{R}^1) \arrow[r, equal] & T_x J^4(U, \mathbb{R}^1)
\end{tikzcd}$$

Similarly

$$\pi_i : O_i \longrightarrow J^4(\mathbb{R}^2, \mathbb{R}^1) \ \ (i = 2, 3)$$

is also a submersion. Moreover, for each ω,

$$(\pi_i|_{O_i \cap O_j})^{-1}(T^\omega) = (\pi_j|_{O_i \cap O_j})^{-1}(T^\omega) \ (i, j = 1, 2, 3)$$

holds. In fact, without the loss of generality, we consider the case that $i = 2, j = 3$, For any $j^4 f(x) \in (\pi_2|_{O_2 \cap O_3})^{-1} T^\omega$, we denote that

$$\begin{cases} f = (f_1, f_2, f_3) \\ g = (f_1, f_3, f_2) = (g_1, g_2, g_3) \\ G_2 = (f_1, f_2) = H_3; \ G_3 = (f_1, f_3) = H_2\ . \end{cases}$$

Then we have

$$\pi_2(j^4 f(x)) = j^4(f_2 \circ H_2{}^{-1})(y) \subset \pi_2(\pi_2{}^{-1}T^\omega) \subset T^\omega$$

for $x \in M, y = H_2(x)$. Since $j^4 g_2(x) = j^4 f_3(x) \in O_2 \cap O_3$, we have

$$\pi_3(j^4 f(x)) = j^4(f_3 \circ H_3{}^{-1})(y) = j^4(g_2 \circ G_2{}^{-1})(y) \in T^\omega.$$

It follows that $j^4 f(x) \in (\pi_3|_{O_2 \cap O_3})^{-1}(T^\omega)$. Hence, we have

$$(\pi_2|_{O_2 \cap O_3})^{-1}(T^\omega) \subset (\pi_3|_{O_2 \cap O_3})^{-1}(T^\omega).$$

Similarly, we have

$$(\pi_2|_{O_2 \cap O_3})^{-1}(T^\omega) \supset (\pi_3|_{O_2 \cap O_3})^{-1}(T^\omega).$$

By the same arguments as the above, we also have the inclusion of the converse direction. Then we have $(\pi_2|_{O_2 \cap O_3})^{-1}(T^\omega) = (\pi_3|_{O_2 \cap O_3})^{-1}(T^\omega)$. Therefore we have a submanifold

$$W^\omega = \bigcup_{i=1}^{3} \pi_i{}^{-1} T^\omega$$

for each ω. Since $\pi_i \pitchfork T^\omega$, then codim$W^\omega$ =codimT^ω. For $i = 1$, the following diagram is commutative:

$$\begin{array}{ccccc}
W^\omega \subset O_1 & \xrightarrow{\pi_1} & J^4(\mathbb{R}^2, \mathbb{R}^1) & \xrightarrow{\Gamma} & J^3(\mathbb{R}_1{}^2, \mathbb{R}_1{}^2) \supset S^\omega \\
\uparrow{\scriptstyle j^4 f} & & \uparrow{\scriptstyle j^4 \widetilde{f}_1} & & \uparrow{\scriptstyle j^3 \mathrm{grad}\, \widetilde{f}_1} \\
M & = & M & = & M,
\end{array}$$

where $j^4 \widetilde{f}_1(x) = j^4(f_1 \circ H_1{}^{-1})(y)$ and Γ is the mapping defined by Lemma 2.2. Since

$$\Gamma^{-1}(S^\omega) = T^\omega,\ W^\omega|_{O_1} = \pi_1{}^{-1} T^\omega,$$

$j^4 f \pitchfork W^\omega$ if and only if $j^4 \widetilde{f}_1 \pitchfork T^\omega$. When $\omega = I_k$, $j^4 \widetilde{f}_1 \pitchfork T^\omega$ if and only if $j^3(\mathrm{grad}(f_1 \circ H_1{}^{-1})) \pitchfork S^\omega$ by Lemma 2.2. For $i = 2, 3$ the same assertion as in case $i = 1$ holds. By Thom's Transversality theorem, the set of the immersion f such that $j^4 f \pitchfork W^\omega$ is dense in $I(M, \mathbb{R}_1{}^3)$. If we choose coordinate neighbourhood at every point of M and $\mathbb{R}P^2$, $N(f)$ can be written in the form grad$(f_i \circ H_i{}^{-1})$ with respect to $i = 1, 2, 3$. This means that $N(f)$ is excellent for such f. $\Box$

We consider the light-like part as follows.

Proposition 2.4. *Let $I(M, \mathbb{R}_1{}^3) \supset Q_l = \{f | M_l{}^f$: regular curve$\}$, then Q_l is a residual set .*

proof. We define an open subset $O_1 \subset I^2(M, \mathbb{R}_1{}^3)$ exactly the same way as O_1 in Proposition 2.3. For any $p \in M_f{}^l$,we consider the local parametrization $X(u, v) = (X_1(u, v), X_2(u, v), X_3(u, v))$ of $f(M)$ around $f(p) \in f(M)$.

Since $< X_u(p) \wedge X_v(p), X_u(p) \wedge X_v(p) >= 0$, we have

$$\begin{vmatrix} X_{2u}(p) & X_{3u}(p) \\ X_{2v}(p) & X_{3v}(p) \end{vmatrix} \neq 0.$$

It follows that $j^2 f({M_f}^l) \subset O_1$. We also have the submersion $\pi_1 : O_1 \longrightarrow J^2(\mathbb{R}^2, \mathbb{R}^1)$.

On the other hand, we denote $\alpha = (y, z, w, a_1, a_2, a_{11}, a_{12}, a_{22})$ the coordinates of $J^2(\mathbb{R}^2, \mathbb{R}^1)$. (where, $w = f(y,z), a_1 = f_y, a_2 = f_z, a_{11} = f_{yy}, a_{12} = f_{yz}, a_{22} = f_{zz}$). We now define maps

$$\rho_i : J^2(\mathbb{R}^2, \mathbb{R}^1) \longrightarrow \mathbb{R} \ (i = 1, 2, 3)$$

by

$$\begin{cases} \rho_1(\alpha) = {a_1}^2 + {a_2}^2 - 1 \\ \rho_2(\alpha) = a_1 \cdot a_{11} + a_2 \cdot a_{12} \\ \rho_3(\alpha) = a_1 \cdot a_{12} + a_2 \cdot a_{22} \ . \end{cases}$$

The Jacobian matrix of the map (ρ_1, ρ_2, ρ_3) is calculated as follows:

$$J(\rho_1, \rho_2, \rho_3) = \begin{pmatrix} 2a_1 & 2a_2 & 0 & 0 & 0 \\ a_{11} & a_{12} & a_1 & a_2 & 0 \\ a_{12} & a_{22} & 0 & a_1 & a_2 \end{pmatrix}.$$

Since $(a_1, a_2) \neq (0,0)$ on ${\rho_i}^{-1}(0)$, $\mathrm{rank} J(\rho_1, \rho_2, \rho_3) = 3$.
Therefore, ${\rho_1}^{-1}(0) \cap {\rho_2}^{-1}(0) \cap {\rho_3}^{-1}(0)$ is a submanifold with codimension 3.
On the graph $\{(g(y,z), y, z) | (y,z) \in \mathbb{R}^2\}$ of function $g(y,z)$, the light-like part is the set satisfying ${g_y}^2 + {g_z}^2 = 1$. Thus we have

$$(j^2 f)^{-1}({\pi_1}^{-1}({\rho_1}^{-1}(0))) = {M_l}^f.$$

Since π_1 is a submersion, ${\pi_1}^{-1}({\rho_1}^{-1}(0))$ is an algebraic set of O_1, and singular set of ${\pi_1}^{-1}({\rho_1}^{-1}(0))$ is the submanifold ${\pi_1}^{-1}({\rho_1}^{-1}(0) \cap {\rho_2}^{-1}(0) \cap {\rho_3}^{-1}(0))$ with codimension 3. Hence, $\mathcal{Q}_l$ is residual set by Thom's Transversality theorem. □

Moreover, we have the following proposition.

Proposition 2.5. *There exists a residual subset $\mathcal{Q}_l' \subset I(M, {\mathbb{R}_1}^3)$ such that the condition (5) in Theorem A holds for any $f \in \mathcal{Q}_l'$.*

proof. Here, we use the same notion as those of the proof of Proposition 2.4.
Since $j^2 f({M_l}^f) \subset O_1$, we may consider that $f(M)$ is the graph of a function. If $f(M)$ is the graph $\{(g(y,z), y, z) | (y,z) \in \mathbb{R}^2\}$ and ${M_l}^f$ is a regular curve, then the tangent line of the light-like locus $T_{x_0}{M_l}^f$ is the set of vectors of the form $\begin{pmatrix} \zeta \\ \xi \\ \eta \end{pmatrix} \in T_{x_0}\mathbb{R}^3$ such that $\zeta = g_y \cdot \xi + g_z \cdot \eta$ and

$$(g_y \cdot g_{yy} + g_z \cdot g_{zy})\xi + (g_y \cdot g_{yz} + g_z \cdot g_{zz})\eta = 0.$$

If the direction of the line $T_{x_0}M_l{}^f$ is light-like, then we have

$$(g_y \cdot \xi + g_z \cdot \eta)^2 = \xi^2 + \eta^2,$$

so we have

$$\begin{aligned}&\{g_y(g_y \cdot g_{yz} + g_z \cdot g_{zz}) - g_z(g_y \cdot g_{yy} + g_z \cdot g_{zy})\}^2\\ &= (g_y \cdot g_{yz} + g_z \cdot g_{zz})^2 + (g_y \cdot g_{yy} + g_z \cdot g_{zy})^2.\end{aligned}$$

We also denote $\alpha = (y, z, w, a_1, a_2, a_{11}, a_{12}, a_{22})$ the coordinates of $J^2(\mathbb{R}^2, \mathbb{R})$. Thus we have the following equations:

$$a_1{}^2 + a_2{}^2 - 1 = 0$$

and

$$\begin{aligned}&\{a_1(a_1 \cdot a_{12} + a_2 \cdot a_{22}) - a_2(a_1 \cdot a_{11} + a_2 \cdot a_{12})\}^2\\ &= (a_1 \cdot a_{12} + a_2 \cdot a_{22})^2 + (a_1 \cdot a_{11} + a_2 \cdot a_{12})^2.\end{aligned}$$

These equations give an algebraic subset V of $J^2(\mathbb{R}^2, \mathbb{R})$ and the codimension of V is two. By Thom's Transversality theorem, there exists a residual set $\mathcal{Q}' \subset I(M, \mathbb{R}_1{}^3)$ such that $(j^2 f)^{-1}(\pi_1{}^{-1}(V))$ is the set of isolated points. If we put $\mathcal{Q}_l{}' = \mathcal{Q}_l \cap \mathcal{Q}'$, it is also a residual set in $I(M, \mathbb{R}_1{}^3)$ and the condition (5) in Theorem A holds for any $f \in \mathcal{Q}_l{}'$. □

Similarly, we have the following proposition.

Proposition 2.6. *There exists a residual subset $\mathcal{Q}_e{}' \subset I(M, \mathbb{R}_1{}^3)$ such that the condition (6) in Theorem A holds for any $f \in \mathcal{Q}_e{}'$.*

proof. We adopt the residual set $\mathcal{Q}_e$ which is given in Proposition 2.3. For any $f \in \mathcal{Q}_e$, the parabolic set is a union of regular curves. Like as the previous arguments, we may only consider the case, when $f(M)$ is the graph $\{(g(y, z), y, z) | (y, z) \in \mathbb{R}^2\}$. In this case the parabolic locus P_f is given by the equation $g_{yy} \cdot g_{zz} - g_{yz}^2 = 0$. So the tangent line of the parabolic locus $T_{x_0}P_f$ is the set of vectors $\begin{pmatrix}\zeta\\ \xi\\ \eta\end{pmatrix} \in T_{x_0}\mathbb{R}^3$ such that $\zeta = g_y \cdot \xi + g_z \cdot \eta$ and

$$(g_{yyy} \cdot g_{zz} + g_{yy} \cdot g_{zzy} - 2g_{yz} \cdot g_{yzy})\xi + (g_{yyz} \cdot g_{zz} + g_{yy} \cdot g_{zzz} - 2g_{yz} \cdot g_{yzz})\eta = 0.$$

If the direction of the line $T_{x_0}P_f$ is light-like, then we have

$$(g_y \cdot \xi + g_z \cdot \eta)^2 = \xi^2 + \eta^2.$$

In this case we also denote $\alpha = (y, z, w, a_1, a_2, a_{11}, a_{12}, a_{22})$ the coordinates of $J^2(\mathbb{R}^2, \mathbb{R})$. It follows that the condition of the parabolic locus is light-like is given by the equations

$$a_{11} \cdot a_{22} - a_{12}{}^2 = 0$$

and

$$(a_1 \cdot (a_{112} \cdot a_{22} + a_{11} \cdot a_{222} - 2a_{12} \cdot a_{122}) - a_2 \cdot (a_{111} \cdot a_{22} + a_{11} \cdot a_{122} - 2a_{12} \cdot a_{112}))^2$$

$$= (a_{112} \cdot a_{22} + a_{11} \cdot a_{222} - 2a_{12} \cdot a_{122})^2 + (a_{222} \cdot a_{22} + a_{11} \cdot a_{122} - 2a_{12} \cdot a_{112})^2.$$

This condition gives an algebraic subset of $J^3(\mathbb{R}^2, \mathbb{R})$ with the codimension 2. It also follows from Thom's Transversality theorem that there exists a residual set $\mathcal{Q}_e{}'$ and the condition (6) is Theorem A holds for any $f \in \mathcal{Q}_e{}'$. □

Proof of Theorem A. By Propositions 2.5 and 2.6, $O_e{}'$ and $O_l{}'$ are residual sets, then the intersection $O_e{}' \cap O_l{}'$ is also a residual set. By definition of $O_e{}'$ and $O_l{}'$, $f \in O_e{}' \cap O_l{}'$ satisfies the condition (1),(2),(3) (5),(6) of Theorem A. Thus, we only need to prove that the immersion $f \in O_e{}' \cap O_l{}'$ has the property (4). Because have discussed on points of $M_f{}^l$, we can consider $I^2(M, \mathbb{R}_1{}^3) \supset O_1$ by the similar reason as that of Proposition 2.3. Since the Gauss map is locally given by $N(f)(y,z) = [-1; -g_y; -g_z]$ on graph$\{(g(y,z), y, z)|\ (y,z) \in \mathbb{R}^2$ of function $g(y,z)$, it's parabolic locus satisfies the equation $g_{yy} \cdot g_{zz} - g_{yz}{}^2 = 0$.

On the other hand, since the point in $M_f{}^l$ satisfies the equation $g_y{}^2 + g_z{}^2 = 1$, the intersection of $M_f{}^l$ and the parabolic locus is given by the equations

$$\begin{cases} g_{yy} \cdot g_{zz} - g_{yz}{}^2 = 0 \\ g_y{}^2 + g_z{}^2 = 1\ . \end{cases}$$

We define functions

$$\sigma_i : J^2(\mathbb{R}^2, \mathbb{R}) \longrightarrow \mathbb{R}\ (i = 1, 2)$$

by

$$\begin{cases} \sigma_1(\alpha) = a_{11} \cdot a_{22} - a_{12}{}^2 \\ \sigma_2(\alpha) = a_1{}^2 + a_2{}^2 - 1\ . \end{cases}$$

The Jacobian matrix of the map (σ_1, σ_2) is calculated as follows:

$$J(\sigma_1, \sigma_2) = \begin{pmatrix} 0 & 0 & a_{22} & -2a_{12} & a_{11} \\ 2a_1 & 2a_2 & 0 & 0 & 0 \end{pmatrix}.$$

Since $(a_1, a_2) \neq (0,0)$ on $a_1{}^2 + a_2{}^2 = 1$, rank $J(\sigma_1, \sigma_2) = 2$ if and only if $(a_{11}, a_{12}, a_{22}) \neq 0$. It follows that the singular set $\sum(\sigma_1, \sigma_2)$ of $\sigma_1{}^{-1}(0) \cap \sigma_2{}^{-1}(0)$ is given by the equations

$$\begin{cases} a_1{}^2 + a_2{}^2 = 1 \\ a_{11} = a_{12} = a_{22} = 0 \end{cases}$$

and codim $\sum(\sigma_1, \sigma_2) = 3$. Since submersion $\pi_1 : O_1 \longrightarrow J^2(\mathbb{R}^2, \mathbb{R})$ is a submersion, the pull-back $\pi_1{}^{-1}(\sigma_1{}^{-1}(0) \cap \sigma_2{}^{-1}(0))$ is a submanifold with codimension 2, expect the singular set $\pi_1{}^{-1}(\sum(\sigma_1, \sigma_2))$. And $\pi_1{}^{-1}(\sum(\sigma_1, \sigma_2))$ is a submanifold with codimension 3. If $j^2 f \pitchfork \pi_1{}^{-1}(\sigma_1{}^{-1}(0) \cap \sigma_2{}^{-1}(0))$, then $(j^2 f)^{-1}(\pi_1{}^{-1}(\sigma_1{}^{-1}(0) \cap \sigma_2{}^{-1}(0)))$ is a isolated point of M. Which is a both of parabolic point and light-like point of f.

On the other hand, under the above condition, $(\sigma_1, \sigma_2) \circ \pi_1 \circ j^2 f$ is submersion if and only if it is a local diffeomorphism. Hence, $\sigma_1 \circ \pi_1 \circ j^2 f$ and $\sigma_2 \circ \pi_1 \circ j^2 f$ are submersion. It follows that $(\sigma_1 \circ \pi_1 \circ j^2 f)^{-1}(0)$ is a parabolic locus and $(\sigma_2 \circ \pi_1 \circ j^2 f)^{-1}(0)$ is a light-like locus. If these curve does not intersect transversally, we have

$$T_{(y_0, z_0)}(\sigma_1 \circ \pi_1 \circ j^2 f)^{-1}(0) = T_{(y_0, z_0)}(\sigma_2 \circ \pi_1 \circ j^2 f)^{-1}(0).$$

Since

$$T_{(y_0,z_0)}(\sigma_1 \circ \pi_1 \circ j^2 f)^{-1}(0) = \mathrm{ker} d(\sigma_1 \circ \pi_1 \circ j^2 f)$$

and

$$T_{(y_0,z_0)}(\sigma_2 \circ \pi_1 \circ j^2 f)^{-1}(0) = \mathrm{ker} d(\sigma_2 \circ \pi_1 \circ j^2 f),$$

we have

$$\mathrm{ker} d(\sigma_1 \circ \pi_1 \circ j^2 f) = \mathrm{ker} d(\sigma_2 \circ \pi_1 \circ j^2 f).$$

It follows that the dimension of the space

$$\mathrm{ker} d(\sigma_1 \circ \pi_1 \circ j^2 f) \cap \mathrm{ker} d(\sigma_2 \circ \pi_1 \circ j^2 f) = \mathrm{ker} d((\sigma_1, \sigma_2) \circ \pi_1 \circ j^2 f)$$

is equal to one. However, $\sigma_2 \circ \pi_1 \circ j^2 f$ is local- diffeomorphism, so we have

$$\mathrm{ker} d(\sigma_2 \circ \pi_1 \circ j^2 f) = 0$$

This is a contradiction.

Moreover, we can show that the intersection consists of fold points of the Gauss map. In fact, if the intersection is a cusp point, then it satisfies $g_{yy} \cdot g_{zz} - {g_{yz}}^2 = 0$, and can be written an algebraic condition of 3rd-order partial derivative of g at (y, z). In this case, $S^{1,1,0}$ is a submanifold with codimension 2. Since the equations of $S^{1,1,0}$ is described in terms of 2rd and 3rd order derivatives of 3-jets, these equations and ${g_y}^2 + {g_z}^2 = 1$ are linearly independent except at the points which satisfy $g_{yy} = g_{zz} = g_{yz} = 0$. So the set of 3-jets which corresponds to cusp points of $N(f)$ on ${M_f}^l$ is an algebraic set in O_1 whose codimension is greater than three . Thus, the set of immersions which satisfies the condition (1)-(6) in Theorem A is a dense set by Thom's Transversality Theorem. $\square$

3. Gauss maps on non-light like surfaces.

In this section we consider the geometric meaning of singularities of the $\mathbb{R}P^2$-valued Gauss map restricted on the space-like part or the time-like part.

Define

$${H_1}^2 = \{p \in {\mathbb{R}_1}^3 \mid < p, p >= -1\};$$

$${S_1}^2 = \{p \in {\mathbb{R}_1}^3 \mid < p, p >= 1\}.$$

We respectively call ${H_1}^2$, ${S_1}^2$ a *hyperbolic-plane*, a *pseudo sphere*. And for $\boldsymbol{x} \in {\mathbb{R}_1}^3$, the *norm* of $\boldsymbol{x}$ is defined by $|\boldsymbol{x}| = \sqrt{\varepsilon(\boldsymbol{x}) < \boldsymbol{x}, \boldsymbol{x} >}$, and $\boldsymbol{x}$ is called *unit vector* if $|\boldsymbol{x}| = 1$, where $\varepsilon(\boldsymbol{x}) = \mathrm{sign}(\boldsymbol{x})$ denotes the *signature* of $\boldsymbol{x}$ which is given by

$$\mathrm{sign}(\boldsymbol{x}) = \begin{cases} 1 & \boldsymbol{x} : \text{space-like} \\ 0 & \boldsymbol{x} : \text{light-like} \\ -1 & \boldsymbol{x} : \text{time-like} . \end{cases}$$

So we can distinguish two cases for the local representation of the Gauss map at a nonlight-like point on the surface.

For convenience we identify (at least locally) M and $f(M)$ for any $f \in I(M, {\mathbb{R}_1}^3)$.

Case 1). When $p \in M_s{}^f$, since $< X_u(p) \wedge X_v(p), X_u(p) \wedge X_v(p) > < 0$, we have

$$\frac{X_u(p) \wedge X_v(p)}{|X_u(p) \wedge X_v(p)|} \in H_1{}^2.$$

Here, $X = X(u,v)$ $((u,v) \in U_s)$ is a local parametrization of $f(M)$ and U_s is an open neighbourhood of p in $M_s{}^f$, and the subscripts u and v indicate partial differentiation. So $N(f)|_{U_s}$ can be considered as a map from U_s to $H_1{}^2$. We call $N(f)|_{U_s}$ the *space-like Gauss map* or *S-Gauss map* associated with the immersion f, and denoted by $N^s{}_{U_s}(f)$. That is

$$N^s{}_{U_s}(f) : U_s \longrightarrow H_1{}^2;\ N^s(f)(p) = \frac{X_u(p) \wedge X_v(p)}{|X_u(p) \wedge X_v(p)|}.$$

In this case, the derivative of $N^s{}_{U_s}(f)$ is denoted by

$$dN^s(f)_p : T_p(M_t{}^f) \longrightarrow T_{N^s(f)(p)}(H_1{}^2).$$

Under the identification of $M_s{}^f = f(M_s{}^f)$, since $T_p(M_s{}^f)$ and $T_{N^s(f)(p)}(H_1{}^2)$ are parallel planes at p, the map $dN^s{}_{U_s}(f)_p$ can be looked upon as a linear map on $T_p(M_s{}^f)$. And $K_S := \det dN^s(f)_p$ is called a *space-like Gauss curvature* or *S-Gauss curvature* at $p \in M_s{}^f$ on the surface $M_s{}^f$.

Case 2). When $p \in M_t{}^f$, we also have

$$\frac{X_u(p) \wedge X_v(p)}{|X_u(p) \wedge X_v(p)|} \in S_1{}^2.$$

Here, $X = X(u,v)$ $((u,v) \in U_t)$ is a local parametrization of $f(M)$ and U_t is an open neighbourhood of p in $M_t{}^f$, and the subscripts u and v indicate partial differentiation. So $N(f)|_{U_t}$ can be considered as a map from U_t to $S_1{}^2$. We call $N(f)|_{U_t}$ the *time-like Gauss map* or *T-Gauss map* associated with the immersion f, and denoted by $N^t{}_{U_t}(f)$. That is

$$N^t{}_{U_t}(f) : U_t \longrightarrow S_1{}^2;\ N^t(f)(p) = \frac{X_u(p) \wedge X_v(p)}{|X_u(p) \wedge X_v(p)|}.$$

In this case, the derivative of $N^t{}_{U_t}(f)$ is denoted by

$$dN^t{}_{U_t}(f)_p : T_p(M_t{}^f) \longrightarrow T_{N^t(f)(p)}(S_1{}^2).$$

Under the identification of $M_t{}^f = f(M_t{}^f)$, since $T_p(M_t{}^f)$ and $T_{N^t(f)(p)}(S_1{}^2)$ are parallel planes at p, the map $dN^t(f)_p$ can also be looked upon as a linear map on $T_p(M_t{}^f)$. And $K_T := \det dN^t(f)_p$ is called a *time-like Gauss curvature* or *T-Gauss curvature* of the surface $M_t{}^f$ at $p \in M_t{}^f$.

By definition and the above local representation, a non-light like point p is the parabolic point if and only if the space-like (or time-like) Gauss curvature vanishes at p. Since the induced metric on the space-like part $M_s{}^f$ is positive definite, the space-like Gauss map has the almost same properties as those of Gauss maps of surfaces in Euclidean space. So we only discuss the properties on the time-like Gauss map in ${\mathbb{R}_1}^3$ as follows:

For $\forall v \in T_p({M_t}^f)$, the quadratic form II_p defined by

$$II_p(v) = - < dN^t(f)_p(v), v >$$

is called the *second fundamental form* of ${M_t}^f$ at p. Let $\alpha : I \longrightarrow {M_t}^f$ be a *regular curve* (i.e. $\alpha'(t) \neq 0,\ \forall t \in I$) which passes through the point $p \in {M_t}^f$, k a *curvature* and $\mathbf{n}$ a *unit normal vector* of the curve α at p, and N a *unit normal vector* of the surface ${M_t}^f$ at p. If $k \neq 0$ then we call $k_n = k < \boldsymbol{n}, N >$ the *normal curvature* of the curve $\alpha \subset {M_t}^f$ at p, where I is an open interval of $\mathbb{R}$. In this case, for the T-Gauss map $N^t(f)_p$ associated with $f \in I(M, {\mathbb{R}_1}^3)$ and $v \in T_p{M_t}^f$, we have $II_p(v) = k_n(p)$ by the Frenet-Serret type formula (cf., [4]).

In order to consider the principal curvature, we consider the eigenvector of $dN^t(f)_p$. Let $\mathbb{C}^2 = \{(u_1, u_2) | u_1, u_2 \in \mathbb{C} : \text{complex}\}$ be a 2-dimensional complex vector space, $u = (u_1, u_2)$ and $v = (v_1, v_2)$ be two vectors in $\mathbb{C}^2$, the *pseudo Hermitian-scalar product* of u and v is defined by $< u, v >= -u_1\bar{v}_1 + u_2\bar{v}_2$. $(\mathbb{C}^2, <,>)$ is called a 2-dimensional *complex Minkowski space* or 2-dimensional *pseudo complex Hermitian space*. We denote ${\mathbb{C}_1}^2$ as $(\mathbb{C}^2, <,>)$. Then we have the following simple lemma in linear algebra [6].

Lemma 3.1. *If $N^t : U_t \longrightarrow {S_1}^2$ is a T-Gauss map associated with $f \in I(M, {\mathbb{R}_1}^3)$ at $p \in {M_t}^f$, then the differential $dN^t(f)_p$ of $N^t(f)$ at p is a self-adjoint linear map. The eigenvalue and corresponding eigenvector are real.*

Proposition 3.2. *Let $N^t : U_t \longrightarrow {S_1}^2$ be a T-Gauss map associated with $f \in I(M, {\mathbb{R}_1}^3)$, the numbers λ_1 and λ_2 in $\mathbb{C}$ with $\lambda_1 \neq \lambda_2$ (in this case $\lambda_1, \lambda_2 \in \mathbb{R}$, by the Lemma 3.1). If the map $dN^t(f)_p : T_p({M_t}^f) \longrightarrow T_p({M_t}^f)$ satisfies $dN^t(f)_p(e_1) = -\lambda_1 e_1$ and $dN^t(f)_p(e_2) = -\lambda_2 e_2$, then e_1 and e_2 are pseudo-orthogonal.*

Proof. Since $dN^t(f)_p$ is self-adjoint, we have

$$< dN^t(f)_p(e_1), e_2 >=< e_1, dN^t(f)_p(e_2) > .$$

It follows that

$$< \lambda_1 \cdot e_1, e_2 >= \bar{\lambda}_2 < e_1, e_2 >= \lambda_2 < e_1, e_2 >,$$

thus we have

$$(\lambda_1 - \lambda_2) < e_1, e_2 >= 0\ (\lambda_1 \neq \lambda_2). \quad \square$$

The assertions of Proposition 3.2 implies that there exist nonlight-like pseudo orthonormal basis associated with pseudo scalar product on ${M_t}^f$ induces form ${\mathbb{R}_1}^3$.

Proposition 3.3. *If $p \in {M_t}^f$, and $\{e_1, e_2\}$ is a orthogonal basis of the tangent plane $T_p({M_t}^f)$, then the vectors e_1 and e_2 are nonlight-like.*

Proof. We may consider that $T_p{M_t}^f$ is $\mathbb{R}^2$ with the pseudo-inner product $< x, y >= -x_1 \cdot y_1 + x_2 \cdot y_2$. If one of the pseudo orthogonal basis is given by $e_1 = (1, 1)$ and $e_2 = (x, y)$ is another vector of the pseudo orthogonal basis in ${\mathbb{R}_1}^2$. Then we have $x = y$ by $< e_1, e_2 >= 0$. This means that e_1 and e_2 are linear dependent. $\square$

Theorem 3.4. *Let $\{e_1, e_2\}$ be a pseudo-orthonormal basis of the tangent plane $T_p(M_t{}^f)$ at $p \in M_t{}^f$, then for any $v \in T_p(M_t{}^f)$ which is given by $v = x \cdot e_1 + y \cdot e_2$,*

$$II_p(v) = k_n(p) = \lambda_1 \cdot \varepsilon(e_1) \cdot x^2 + \lambda_2 \cdot \varepsilon(e_2) \cdot y^2.$$

Here $dN^t(f)_p(e_i) = -\lambda_i \cdot e_i$ $(i = 1, 2;\ \lambda_1 \neq \lambda_2)$, and $\varepsilon(e_i) = sign\ (e_i)_{i=1,2}$.

Proof.

$$II_p(v) = - < dN^t(f)_p(v), v >= - < -\lambda_1 \cdot x \cdot e_1 - \lambda_2 \cdot y \cdot e_2, x \cdot e_1 + y \cdot e_2 >$$

$= \lambda_1 \cdot \varepsilon(e_1) \cdot x^2 + \lambda_2 \cdot \varepsilon(e_2) \cdot y^2$ □

Let

$$k_i = \lambda_i \cdot \varepsilon(e_i) = \lambda_i < e_i, e_i >,$$

then

$$k_n(p) = II_p(v) = k_1 \cdot x^2 + k_2 \cdot y^2.$$

We say that the numbers k_1, k_2 are *principal curvature* at $p \in M_t{}^f$. The corresponding directions that are given by the eigenvectors e_1, e_2 are called *principal directions* at $p \in M_t{}^f$. It follows that $K_T = k_1 \cdot k_2$ like as the Euclidean case.

On the other hand, we consider the case that $f \in I(M, \mathbb{R}_1{}^3)$ has properties in Theorem A. Let $p \in M_t{}^f$ be a parabolic point, $\{e_1, e_2\}$ be a pseudo orthonormal basis of the $T_p(M_t{}^f)$ and k_1 and k_2 be eigenvalues of $dN^t(f)_p$ with eigenvectors e_1 and e_2 respectively. Then e_1 and e_2 are nonlight-like by Proposition 3.3. Since $K_T = 0$ and $dK_T \neq 0$ at the parabolic point $p \in M_t{}^f$, we have $k_1 = 0$ and $k_2 \neq 0$. In this case, both of e_1 and e_2 are not light-like vectors. Moreover, the dimension of $\ker dN_p$ is one by Theorem A. The kernel of the derivative of $N^t(f)_p$ is a line corresponds to the *zero principal curvature direction*. This line is called a *zero principal curvature line*. So we have the following theorem which describe the generic geometric properties of the parabolic set on the nonlight-like part.

Theorem 3.5. *Let $f \in I(M, \mathbb{R}_1{}^3)$ be an immersion which has properties(1)-(6) of Theorem A. Then*

(1) *$p \in M_t{}^f$ (respectively, $p \in M_s{}^f$) is a fold point of the T-Gauss map $N^t(f)$ (respectively, S-Gauss map $N^s(f)$) if and only if a zero principal curvature line of f is transverse to the parabolic locus of f at p.*

(2) *$p \in M_t{}^f$ (respectively, $p \in M_s{}^f$) is a cusp point of the T-Gauss map $N^t(f)$ (respectively, S-Gauss map $N^s(f)$) if and only if a zero principal curvature line of f is tangent to the parabolic locus of f at p.*

proof. We only consider the case that $p \in M_t{}^f$. Locally, $f(M)$ can be written as the graph of a function $h \in C^\infty(\mathbb{R}^2, \mathbb{R}^1)$, and $N^t(f) = \mathrm{grad}(h|_U)$ by §2. Let $g = \mathrm{grad}(h|_U)$, so the smooth map $N^t(f) = g : U \longrightarrow \mathbb{R}^2$ is good by Theorem A, where U is open neighbourhood of p in $\mathbb{R}^2$. If p is a singular point of the good map g, then we have

$$\det J_g(p) = 0,\ \ \mathrm{grad}\det J_g(p) \neq 0.$$

In general, if g is a good map, the singular locus C of g is a regular curve in M. Moreover, it has been known that a singular point of g is a fold point if and only if the tangent line of the singular locus C of g is transverse to the direction of $\ker dg_p$ (cf., §3 in [1]).

On the other hand, if g is the T-Gauss map, $K_T = \det J_g(p)$. A singular point of g is a cusp point if and only if the zero principal direction line is tangent to the direction of $\ker dg_p$. This completes the proof. □

4. Example

We now give some examples which are illustrating the main results:

Example 1. The shoe surface:

$$X(x,y) = (x, y, f(x,y)) = (x, y, \frac{1}{3}x^3 - \frac{1}{2}y^2).$$

The local representation of the Gauss mapping is $N(f) = (f_x, f_y) = (x^2, -y)$, and the parabolic locus is obtained by solving $\Delta = f_{xx} \cdot f_{yy} - {f_{xy}}^2 = -2x = 0$. Since grad $\Delta = (-2, 0) \neq 0$ on the parabolic locus, N is good. The light-like locus is obtained by equation $-{f_x}^2 + {f_y}^2 + 1 = 0$, so the light-like locus is given by $-x^4 + y^2 - 1 = 0$. The parabolic locus can be parametrized by $x(t) = 0, y(t) = t$. So the Gauss mapping restricted to the parabolic locus is $N(t) = (0, -t)$, with $N'(t) = (0, -1) \neq 0$, hence N is excellent. Moreover, N has no cusp points.

Example 2. The Menn's surface:

$$X(y,z) = (f(y,z), y, z) = (-\frac{1}{2}y^4 + y^2 z - z^2, y, z).$$

The local representation of the Gauss mapping is
$N(f) = (f_y, f_z) = (-2y^3 + 2yz, y^2 - 2z)$, and the parabolic locus is $8y^2 - 4z = 0$. Since grad $\Delta = (16y, -4) \neq 0$ on the parabolic locus, N is good. The light-like locus is $(-2y^3 + 2yz)^2 + (y^2 - 2z)^2 - 1 = 0$. The parabolic locus can be parametrized by $y(t) = t, z(t) = 2t^2$, so the Gauss mapping restricted to the parabolic locus is $N(t) = (2t^3, -3t^2)$, $N'(t) = (6t^2, -6t)$, $N''(t) = (12t, -6)$, hence $N'(0) = (0,0)$, $N''(0) = (0, -6) \neq 0$. The Gauss map has a cusp point $(0,0)$, and N is excellent. Clearly $(0,0) \notin {M_l}^f$.

Example 3. The saddle surface:

$$X(y,z) = (f(y,z), y, z) = (\frac{1}{3}y^3 - yz^2 + \frac{1}{2}(y^2 + z^2)).$$

The local representation of the Gauss mapping is $N(f) = (y^2 - z^2 + y, -2yz + z)$, and the parabolic locus is $y^2 + z^2 = \frac{1}{4}$. So grad $\Delta = 4(-2y, -2z) \neq 0$ on the parabolic locus, N is good. The light-like locus is $(y^2 - z^2 + y)^2 + (-2yz + z)^2 - 1 = 0$. The parabolic locus can be parametrized by $y(t) = \frac{1}{2}\cos t, z(t) = \frac{1}{2}\sin t$, so the Gauss mapping restricted to the parabolic locus is

$$N(t) = (\frac{1}{4}\cos 2t + \frac{1}{2}\cos t, -\frac{1}{4}\sin 2t + \frac{1}{2}\sin t),$$

$$N'(t) = (-\frac{1}{2}\sin 2t - \frac{1}{2}\sin t, -\frac{1}{2}\cos 2t + \frac{1}{2}\cos t),$$

$$N''(t) = (-\cos 2t - \frac{1}{2}\cos t, \sin 2t - \frac{1}{2}\sin t).$$

Hence $t = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$ by $N'(t) = 0$. And $N'(t) = 0$ implies $N''(t) \neq 0$. We have cusp points $(\frac{1}{4}, 0)$, $(-\frac{1}{4}, \frac{\sqrt{3}}{4})$, $(-\frac{1}{4}, -\frac{\sqrt{3}}{4})$, and N is excellent. Clearly, cusp points $(\frac{1}{4}, 0), (-\frac{1}{4}, \frac{\sqrt{3}}{4})$, $(-\frac{1}{4}, -\frac{\sqrt{3}}{4}) \notin {M_l}^f$.

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