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ON $-P \cdot P$ OF SURFACE SINGULARITIES

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1. Introduction

Let (X, x) be a normal surface singularity over the complex number field $\mathbb{C}$ and $f\colon (M, A) \to (X, x)$ a resolution of the singularity (X, x). Let K be the canonical divisor on M. Let $A = \bigcup_{i=1}^{k} A_i$ be the decomposition of the exceptional set A into irreducible components. Assume that f is the minimal good resolution, i.e., f is the smallest resolution for which A consists of non-singular curves intersecting among themselves transversally, with no three through one point. It is well known that there exists a unique minimal good resolution.

Definition 1.1. By [12, Theorem A.1], $K + A$ admits a unique Zariski-decomposition $P + N$, $P, N \in \sum_{i=1}^{k} \mathbb{Q}A_i$, where

(1) $(K + A) \cdot A_i = (P + N) \cdot A_i$ for all i.
(2) P is f-nef, i.e., $P \cdot A_i \geq 0$ for all i.
(3) N is effective.
(4) $P \cdot N = 0$.

Then we define the invariant P^2 by $P^2 := P \cdot P$.

The $P \cdot P$ is a topological invariant and its fundamental properties are stated in [15]. It is expected that P^2 has many of nice properties of the invariant $K \cdot K$ studied by Laufer [8]. The upper semicontinuity of $-P^2$ in a family of surface singularities follows from that of the L^2-plurigenera δ_m (cf. [2]), since the following equality holds (see [15, Introduction]):

$$-P \cdot P/2 = \limsup_{m \to \infty} \delta_m / m^2.$$

In this note, we prove the following.

Theorem . *Let $\pi\colon X \to T$ be a deformation of a normal Gorenstein surface singularity such that T is a neighborhood of the origin of $\mathbb{C}$. Let P_t^2 be the invariant of the fiber $X_t, t \in T$. Then the following conditions are equivalent:*

(1) *π admits the simultaneous log-canonical model.*
(2) *P_t^2 is constant.*

2. Preliminaries

Let X be a normal variety over $\mathbb{C}$ of dimension $d \geq 2$, and X_{sing} the singular locus of X. Let $f: Y \to X$ be a birational morphism of normal varieties and $E = f^{-1}(X_{sing})_{red}$ the largest reduced exceptional divisor on Y. For a $\mathbb{Q}$-Cartier divisor D on X, we denote by $f^{\dagger}D$ the sum of the divisors E and the strict transform of D under the morphism f. The morphism $f: Y \to X$ is called a good resolution of the pair (X, D), if Y is nonsingular and the support of $f^{\dagger}D$ is a divisor with only simple normal crossings.

Definition 2.1 (cf. [7], [13]). Let B be a reduced divisor on X. The divisor $K_X + B$ is said to be log-canonical if the following conditions are satisfied:

(1) $K_X + B$ is a $\mathbb{Q}$-Cartier divisor.
(2) There exists a good resolution $f: Y \to X$ of (X, B) such that

$$K_Y + f^{\dagger}B = f^*(K_X + B) + \sum a_i E_i$$

for $a_i \in \mathbb{Q}$ with the condition that $a_i \geq 0$, where the E_i are the exceptional prime divisors.

Definition 2.2 (cf. [7], [13]). Let $f: Y \to X$ be a partial resolution with the exceptional divisor $E = f^{-1}(X_{sing})_{red}$. Then the morphism $f: Y \to X$ is called a log-canonical model of X, if the divisor $K_Y + E$ is log-canonical and $K_Y + E$ is f-ample.

Theorem 2.3 (cf. [6], [13]). *Let X be a normal variety of dimension $d \leq 3$. Then there exists the log-canonical model $f: Y \to X$ of X. In fact, the following morphism gives the log-canonical model:*

$$\operatorname{Proj}\left(\bigoplus_{n \geq 0} f_* \mathcal{O}_Y(n(K_Y + E))\right) \to X,$$

where $f: Y \to X$ is a partial resolution with $E = f^{-1}(X_{sing})_{red}$ such that the divisor $K_Y + E$ is log-canonical.

3. The plurigenera

In this section, we describe basic facts concerning plurigenera of normal isolated singularities needed later.

Definition 3.1 (cf. [9], [16]). Let (X, x) be a normal isolated singularity and $f:$ $(M, A) \to (X, x)$ a good resolution of the singularity (X, x). We define the log-plurigenera

$\{\lambda_m(X,x)\}_{m\in\mathbb{N}}$ and the L^2-plurigenera $\{\delta_m(X,x)\}_{m\in\mathbb{N}}$ by

$$\lambda_m(X,x) = \dim_{\mathbb{C}} \mathcal{O}_X(mK_X)/f_*\mathcal{O}_M(m(K_M+A)) \text{ and}$$
$$\delta_m(X,x) = \dim_{\mathbb{C}} \mathcal{O}_X(mK_X)/f_*\mathcal{O}_M(m(K_M+A)-A), \text{ respectively.}$$

The definition does not depend on the choice of the good resolution.

Lemma 3.2. *Let X be a normal variety and B a reduced divisor on X such that K_X+B is log-canonical. Let $f\colon Y\to X$ be a good resolution of the pair (X,B) with $B_Y := f^{\dagger}B$. Then we have $f_*\mathcal{O}_Y(m(K_Y+B_Y)) = \mathcal{O}_X(m(K_X+B))$.*

Proof. It is clear that $f_*\mathcal{O}_Y(m(K_Y+B_Y)) \subset \mathcal{O}_X(m(K_X+B))$. We assume that X is affine, and we show that $f_*\mathcal{O}_Y(m(K_Y+B_Y)) \supset \mathcal{O}_X(m(K_X+B))$.

Let r be the index of the divisor K_X+B and m a positive integer which divides by r. By assumption, we have that $m(K_Y+B_Y)\geq f^*(m(K_X+B))$. Hence we obtain that

$$H^0(\mathcal{O}_Y(m(K_Y+B_Y))) \supset H^0(f^*\mathcal{O}_X(m(K_X+B))) = H^0(\mathcal{O}_X(m(K_X+B))).$$

For any positive integer m and any element ω in $H^0(\mathcal{O}_X(m(K_X+B)))$, we obtain that $v_{E_i}(\omega^r)\geq -mr$ for all exceptional prime divisor E_i on Y, where v_{E_i} is the valuation associated to the prime divisor E_i. Hence ω belongs to $H^0(\mathcal{O}_Y(m(K_Y+B_Y)))$. □

Corollary 3.3. *Let (X,x) be a normal isolated singularity and $f\colon Y\to X$ a partial resolution with $E=f^{-1}(x)_{red}$ such that K_Y+E is log-canonical. Then we have*

$$\lambda_m(X,x) = \dim_{\mathbb{C}} \mathcal{O}_X(mK_X)/f_*\mathcal{O}_Y(m(K_Y+E)).$$

Let $\pi\colon X\to T$ be a deformation of a normal Gorenstein surface singularity $(X_0,x) = \pi^{-1}(0)$, where T is a neighborhood of the origin of $\mathbb{C}$. Put $X_t := \pi^{-1}(t)$. Then we define the m-th log-plurigenus and m-th L^2-plurigenus of X_t by

$$\lambda_m(X_t) := \sum_{p\in(X_t)_{sing}} \lambda_m(X_t,p) \quad\text{and}\quad \delta_m(X_t) := \sum_{p\in(X_t)_{sing}} \delta_m(X_t,p).$$

Let $\psi_t\colon M_t\to X_t$ be the minimal good resolution of the singularities and K_t the canonical divisor on M_t. Let $A_{t,p}$ be the connected component of the exceptional set A_t on M_t which blows down to $p\in(X_t)_{sing}$. Let $P_{t,p}+N_{t,p}$ be the Zariski decomposition of $K_t+A_{t,p}$. Here, $P_{t,p}$ and $N_{t,p}$ are $\mathbb{Q}$-divisor supported in $A_{t,p}$. We define the $\mathbb{Q}$-divisor P_t on M_t by $P_t := \sum_{p\in(X_t)_{sing}} P_{t,p}$. We put $P_t^2 := -P_t\cdot P_t$ and define the function $\mathcal{P}\colon T\to\mathbb{Q}$ by $\mathcal{P}(t) = -P_t^2$. From [15, Theorem 1.6], [11, Remark 2.7] and Introduction, we obtain the following.

Theorem 3.4. *For any $m \in \mathbb{N}$,*

$$\lambda_m(X_t) = \mathcal{P}(t)m^2/2 + P_t \cdot K_t m/2 + b_t(m) \quad \text{and} \tag{3.1}$$

$$\delta_m(X_t) = \mathcal{P}(t)(m-1)^2/2 - P_t \cdot K_t(m-1)/2 + b'_t(m), \tag{3.2}$$

where b_t and b'_t are bounded functions. Furthermore, the function $\mathcal{P}$ is upper semicontinuous.

4. Some invariants which depend on a deformation

In this section, we fix the following notation. Let $\pi \colon X \to T$ be a deformation of a normal Gorenstein surface singularity $(X_0, x) = \pi^{-1}(0)$, where T is a neighborhood of the origin of $\mathbb{C}$. Then X is a three-dimensional Gorenstein variety. Therefore, for any $t \in T$, we have the isomorphism $\mathcal{O}_{X_t}(mK_X) \cong \mathcal{O}_{X_t}(mK_{X_t})$. We denote by Y_t the fiber $f^{-1}(t)$ and put $f_t := f|_{Y_t}$. Let $f \colon Y \to X$ be the log-canonical model of X with $E = f^{-1}(X_{sing})_{red}$. We define the sheaves by $\mathcal{I}_m := f_*\mathcal{O}_Y(m(K_Y + E))$ and $\mathcal{Q}_m := \mathcal{O}_X(mK_X)/\mathcal{I}_m$ for any $m \in \mathbb{N}$. We put $T^* := T \setminus \{0\}$. We assume that T is sufficiently small.

Let $\mathbb{C}(t)$ be the residue field of $t \in T$, i.e., $\mathbb{C}(t) = \mathcal{O}_{T,t}/\mathcal{M}_t$, where $\mathcal{M}_t$ is the maximal ideal. We use the symbol $\otimes\mathbb{C}(t)$ instead of $\otimes_{\mathcal{O}_T}\mathbb{C}(t)$. By Nakayama's Lemma, we obtain that

$$\dim_{\mathbb{C}} \mathcal{Q}_m \otimes \mathbb{C}(t) \le \dim_{\mathbb{C}} \mathcal{Q}_m \otimes \mathbb{C}(0), \tag{4.1}$$

where the equality holds if and only if $\mathcal{Q}_m$ is a torsion free $\mathcal{O}_T$-module. Let $\mathcal{I}_{m,0}$ be the image of the homomorphism $\mathcal{I}_m \otimes \mathbb{C}(0) \to \mathcal{O}_{X_0}(mK_{X_0})$.

The following Lemmas are proved by an argument similar to that in [4, §1].

Lemma 4.1. *The following conditions are equivalent.*

(1) *The equality in* (4.1) *holds.*
(2) *$\mathcal{Q}_m$ is a torsion free $\mathcal{O}_T$-module.*
(3) *$\mathcal{I}_m \otimes \mathbb{C}(0) = \mathcal{I}_{m,0}$.*

Lemma 4.2. *For any $t \in T^*$, the restriction $f_t \colon Y_t \to X_t$ is the log-canonical model of X_t. Moreover, for each $m \in \mathbb{N}$, there exists a closed analytic subset S_m of T containing the origin such that $\lambda_m(X_t) = \dim_{\mathbb{C}} \mathcal{Q}_m \otimes \mathbb{C}(t)$, for all $t \in T \setminus S_m$.*

Let $\psi \colon (M, A) \to (X_0, x)$ be a good resolution. For every $m \in \mathbb{N}$, we put $\mathcal{A}_m := \psi_*\mathcal{O}_M(m(K_M + A))$ and define the invariant ϵ_m and θ_m by

$$\epsilon_m := \dim_{\mathbb{C}} \mathcal{A}_m/(\mathcal{I}_{m,0} \cap \mathcal{A}_m)$$
$$\theta_m := \dim_{\mathbb{C}} \mathcal{I}_{m,0}/(\mathcal{A}_m \cap \mathcal{I}_{m,0}).$$

Then we have the diagram

$$\begin{array}{ccc} \mathcal{A}_m \cap \mathcal{I}_{m,0} & \longrightarrow & \mathcal{I}_{m,0} \\ \downarrow & & \downarrow \\ \mathcal{A}_m & \longrightarrow & \mathcal{O}_{X_0}(mK_{X_0}). \end{array}$$

From (4.1) and Lemma 4.2, we have the following inequality for every $m \in \mathbb{N}$:

$$\lambda_m(X_t) \leq \lambda_m(X_0) + \epsilon_m - \theta_m. \tag{4.2}$$

Lemma 4.3. *There exist $a, b \in \mathbb{Q}$ such that $\epsilon_m \leq am + b$.*

Proof. First, we show that $\psi_* \mathcal{O}_M(mK_M + (m-1)A) \subset \mathcal{I}_{m,0} \cap \mathcal{A}_m$. Let ω be a section of $\psi_* \mathcal{O}_M(mK_M + (m-1)A)$. By [2, Theorem 2.1], there exists a section ω' of $f_* \mathcal{O}_Y(mK_Y + (m-1)E)$ of which the image in $\mathcal{O}_{X_0}(mK_{X_0})$ is ω. Since $f_* \mathcal{O}_Y(mK_Y + (m-1)E) \subset \mathcal{I}_m$, we see that ω belongs to $\mathcal{I}_{m,0}$. Hence we obtain the inclusion. Then the inclusion implies that

$$\epsilon_m \leq \dim_{\mathbb{C}} \mathcal{A}_m / \psi_* \mathcal{O}_M(mK_M + (m-1)A)) = \delta_m(X_0, x) - \lambda_m(X_0, x).$$

From Theorem 3.4, we obtain the assertion. □

In [14], Tomari and Watanabe proved their main theorem by using Izumi's results on the analytic orders [5]. We need their useful arguments. The following lemma is the version due to Ishii.

Lemma 4.4 (Ishii [3, Lemma 1.5]). *Let (W, w) be a d-dimensional normal isolated singularity and $h: W_1 \to W$ a resolution of the singularity which factors through the blowing up by the maximal ideal of the singular point. Let $F = \bigcup_{i=1}^k F_i$ be the exceptional divisor on W_1, where the F_i are irreducible components. Then there exist positive numbers $\beta \in \mathbb{R}$ and $b \in \mathbb{N}$ such that:*

For an $\mathcal{O}_W$-ideal $J = h_* \mathcal{O}_{W_1}(-\sum_{i=1}^k a_i F_i)$ with $a_i > b$ for some i, the inequalities $\dim_{\mathbb{C}} \mathcal{O}_W / J \geq \beta (a_i)^d$ *$(i = 1, \ldots, k)$ hold.*

Lemma 4.5. *If $\theta_r \neq 0$ for some $r \in \mathbb{N}$, then there exists a positive integer $c \in \mathbb{R}$ such that $\theta_{mr} \geq cm^2$ for all $m \in \mathbb{N}$.*

Proof. Assume $\theta_r \neq 0$. By Lemma 3.2, we may assume that $\psi: (M, A) \to (X_0, x)$ is a good resolution of the singularity which factors through the blowing up by the maximal ideal of the singular point. Let ω be a section of $\mathcal{I}_{r,0}$ which does not belong to $\mathcal{A}_r$. We define a homomorphism $\varphi_m: \mathcal{O}_{X_0} \to \mathcal{I}_{mr,0}$ by $\varphi_m(s) = s\omega^m$ for every $m \in \mathbb{N}$. We denote by J_m the inverse image $\varphi_m^{-1}(\mathcal{A}_{mr} \cap \mathcal{I}_{mr,0})$. Then we have the injection

$$\mathcal{O}_{X_0} / J_m \to \mathcal{I}_{mr,0} / \mathcal{A}_{mr} \cap \mathcal{I}_{mr,0}.$$

We put $a_i := \min\{ v_i(\omega) + r, 0 \}$, where v_i is the valuation at an irreducible component A_i of A. Then $J_m = \psi_* \mathcal{O}_M(\sum m a_i A_i)$. By the choice of ω, there exists a component A_i such that $a_i < 0$. By Lemma 4.4, there exists $c \in \mathbb{R}$ such that $\theta_{mr} \geq cm^2$ for any $m \in \mathbb{N}$. □

Corollary 4.6. *If $\mathcal{P}(t)$ is constant, then $\theta_m = 0$ for all $m \in \mathbb{N}$.*

Proof. It follows from Theorem 3.4, (4.2) and lemmas above. □

5. Main theorem

In this section, we prove the main theorem. We use the same notation as in the preceding section.

Definition 5.1. Let $f\colon Y \to X$ be the log-canonical model of X with the exceptional divisor E. We call f the simultaneous log-canonical model, SLC model for short, if the restriction $f_t\colon Y_t \to X_t$ is the log-canonical model of X_t and $K_{Y_t} + E_t$ is log-canonical for any $t \in T$.

Definition 5.2. For any $m \in \mathbb{N}$, we define the function $\Lambda_m\colon T \to \mathbb{Z}$ by $\Lambda_m(t) := \lambda_m(X_t)$.

The following Lemma is proved by an argument similar to that in Lemma 4.5.

Lemma 5.3. *Let $g\colon (X', B) \to (X_0, x)$ be a partial resolution such that $K_{X'} + B$ is log-canonical. Let D be a reduced divisor on X' such that $0 \leq D \leq B$. For every $m \in \mathbb{N}$, we define the invariant $\nu_m(X'; B, D)$ by*

$$\nu_m(X'; B, D) = \dim_{\mathbb{C}} g_* \mathcal{O}_M(m(K_{X'} + B))/g_* \mathcal{O}_M(m(K_{X'} + D)).$$

If $\nu_r(X'; B, D) \neq 0$ for some $r \in \mathbb{N}$, then there exists a positive integer $c \in \mathbb{R}$ such that $\nu_{mr}(X'; B, D) \geq cm^2$ for all $m \in \mathbb{N}$.

Proposition 5.4. *Assume that there exists the SLC model of the deformation $\pi\colon X \to T$. Then the function Λ_m is constant for $m >> 0$.*

Proof. Let $f\colon Y \to X$ be the SLC model of the deformation π. Since $K_Y + E$ is f-ample, $R^1 f_* \mathcal{O}_Y(m(K_Y + E)) = 0$ for $m >> 0$. From the exact sequence (cf. [10])

$$0 \to f_* \mathcal{O}_Y(m(K_Y + E)) \to f_* \mathcal{O}_Y(m(K_Y + E)) \to f_* \mathcal{O}_{Y_0}(m(K_{Y_0} + E_0)) \\ \to R^1 f_* \mathcal{O}_Y(m(K_Y + E)),$$

we have $f_* \mathcal{O}_{Y_0}(m(K_{Y_0} + E_0)) = \mathcal{I}_m \otimes \mathbb{C}(0)$ for $m >> 0$. Since $f_* \mathcal{O}_{Y_0}(m(K_{Y_0} + E_0))$ is a submodule of $\mathcal{O}_{X_0}(mK_{X_0})$, we have the equality $\mathcal{I}_m \otimes \mathbb{C}(0) = \mathcal{I}_{m,0}$. Then Lemma 4.1 and Lemma 4.2 imply that

$$\lambda_m(X_t) = \dim_{\mathbb{C}} \mathcal{Q}_m \otimes \mathbb{C}(0).$$

We denote by B the exceptional set on Y_0. Since $E_0 \leq B$, we obtain the equality

$$\dim_{\mathbb{C}} \mathcal{Q}_m \otimes \mathbb{C}(0) = \lambda_m(X_0, x) + \nu_m(Y_0; B, E_0).$$

Since $\mathcal{P}(t)$ is upper semicontinuous, $\nu_m(Y_0; B, E_0) = 0$ by the lemma above. □

Lemma 5.5. *$\mathcal{Q}_m$ is a torsion free $\mathcal{O}_T$-module for any $m \in \mathbb{N}$, if $\mathcal{P}$ is constant.*

Proof. We assume that there exists a section $\omega \in \mathcal{O}_X(rK_X) \setminus \mathcal{I}_r$ of which the image in $\mathcal{Q}_r$ is a torsion element. Then there exists an exceptional prime divisor F on Y lying over X_0 such that $v_F(\omega) < -r$. We note that F is a projective surface. Let $\mathcal{I}_F$ be the $\mathcal{O}_Y$-ideal of the subvariety F, and let $L_m := m(K_Y + E)$. Since L_1 is f-ample, there exists an integer $n \in \mathbb{N}$ such that $\mathcal{O}_F(L_n)$ is a very ample invertible sheaf and the following sequence is exact for any $m \in \mathbb{N}$:

$$0 \to f_*(\mathcal{I}_F \mathcal{O}_Y(L_{mn} + F)) \to f_*\mathcal{O}_Y(L_{mn} + F) \to H^0(\mathcal{O}_F(L_{mn} + F)) \to 0.$$

By [1, III, Ex. 5.2], there exists a polynomial q' of degree 2 such that

$$\dim_{\mathbb{C}} f_*\mathcal{O}_Y(L_{mn} + F)/f_*(\mathcal{I}_F \mathcal{O}_Y(L_{mn} + F)) = q'(m)$$

for $m >> 0$. Since $\mathcal{I}_F \mathcal{O}_Y(L_{mn} + F)$ is isomorphic to $\mathcal{O}_Y(L_{mn})$ outside a one-dimensional subvariety in F, there exists a polynomial q of degree 2 such that $\dim_{\mathbb{C}} f_*\mathcal{O}_Y(L_{mn} + F)/\mathcal{I}_{mn} \geq q(m)$ for $m >> 0$. Since any section of the sheaf $f_*\mathcal{O}_Y(L_{mn} + F)/\mathcal{I}_{mn}$ is a torsion element of $\mathcal{Q}_{mn}$, we obtain the inequality (cf. (4.2))

$$\dim_{\mathbb{C}} \mathcal{Q}_{mn} \otimes \mathbb{C}(t) \leq \dim_{\mathbb{C}} \mathcal{Q}_{mn} \otimes \mathbb{C}(0) - q(m).$$

Since $\dim_{\mathbb{C}} \mathcal{Q}_{mn} \otimes \mathbb{C}(0) - \dim_{\mathbb{C}} \mathcal{Q}_{mn} \otimes \mathbb{C}(t)$ is bounded by a linear function, we are led to a contradiction. □

Remark 5.6. From the proof above, we see that Y_0 is irreducible. Thus any irreducible component of E dominates T. Since Y_0 is a principal divisor, for any irreducible component F of E, the intersection $F \cap Y_0$ is a one-dimensional variety.

Lemma 5.7. *$\mathcal{I}_{m,0} = \mathcal{A}_m$ for any $m \in \mathbb{N}$, if $\mathcal{P}$ is constant.*

Proof. The inclusion $\mathcal{I}_{m,0} \subset \mathcal{A}_m$ follows from Corollary 4.6. Let ω be a section of $\mathcal{A}_m$ and ω' a section of $\mathcal{O}_X(mK_X)$ of which the image in $\mathcal{O}_{X_0}(mK_{X_0})$ is ω. If $v_F(\omega') < -m$ for an irreducible component F of E, then there exists an irreducible component A_i of A lying over the variety $F \cap Y_0$ such that $v_{A_i}(\psi^*\omega) < -m$. It contradicts the definition of ω. Hence ω' belongs to $\mathcal{I}_m$, and ω also belongs to $\mathcal{I}_{m,0}$. □

Theorem 5.8. *The following conditions are equivalent.*

(1) $\pi: X \to T$ *admits the SLC model.*

(2) *The map* $\Lambda_m: T \to \mathbb{Z}$ *is constant for any* $m \in \mathbb{N}$.

(3) *The map* $\mathcal{P}: T \to \mathbb{Q}$ *is constant.*

Proof. We consider the following condition: (2)′ The map $\Lambda_m: T \to \mathbb{Z}$ is constant for $m >> 0$. By Proposition 5.4 (1) implies (2)′. It follows from Theorem 3.4 that (2)′ implies (3). We assume that $\mathcal{P}$ is constant. Then, from Lemma 4.1 and lemmas above, we obtain the following equalities for any $m \in \mathbb{N}$:

$$\mathcal{I}_m \otimes \mathbb{C}(0) = \mathcal{I}_{m,0} = \mathcal{A}_m, \quad \dim_{\mathbb{C}} \mathcal{Q}_m \otimes \mathbb{C}(t) = \dim_{\mathbb{C}} \mathcal{Q}_m \otimes \mathbb{C}(0).$$

Now it is clear that (2) holds, and that $Y_0 = \mathrm{Proj}(\bigoplus_{m\in\mathbb{N}} \mathcal{I}_m \otimes \mathbb{C}(0))$ is the log-canonical model of X_0. Since $\mathcal{A}_m = \mathcal{I}_m \otimes \mathbb{C}(0) = f_*\mathcal{O}_{Y_0}(m(K_{Y_0} + E_0))$ for $m >> 0$ (cf. proof of Proposition 5.4) and $K_{Y_0} + E_0$ is ample, $K_{Y_0} + E_0$ is log-canonical. On the other hand, $f_t: Y_t \to X_t$ is the log-canonical model for $t \in T^*$ by Lemma 4.2. Hence we obtain the condition (1). □

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