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Kolmogorov (1962)理論の拡張

電通大　細川　巌

Refined Similarity Hypothesis and Velocity Structure Functions in Fluid Turbulence

Iwao Hosokawa
University of Electro-Communications
Chofu, Tokyo 182, japan

The refined similarity hypothesis is generalized so that the statistics of similarity variable may have a slight scale-ratio dependence. This is more natural because the scale ratio r/L (L: macroscale) is another independent non-dimensional parameter than Re_r in the first hypothesis of Kolmogorov (1962). A possible detailed form of the dependence is proposed based on a recent multifractal model of intermittent energy dissipation and some theoretical and experimental knowledges. Thus the modified hypothesis makes it possible to predict a plausible value of Kolmogorov prefactor and reasonable scalings of longitudinal as well as transverse velocity structure functions in isotropic turbulence.

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Recently, people have been directing their interest to fundamental problems in isotropic fluid turbulence such as an asymptotic tendency of Kolmogorov constant for high Reynolds numbers [1-4] and a non-trivial difference between longitudinal and transeverse velocity structure functions [5-8]. All these matters are closely related to the refined similarity hypothesis (RSH) for isotropic turbulence that Kolmogorov [9] established in 1962. Here the author stresses the importance of prescribing the statistics of similarity variable in RHS in a reasonable form, instead of keeping it untouched as an unknown universal thing, in order to settle these fundamental problems in a consistent way. Since we have an appreciable knowledge of the statistics of similarity variable obtained from direct numerical simulation (DNS) [10], we are now in a good position to do it.

By RSH [9], longitudinal velocity increment Δu_r across distance r is related with energy dissipation rate ε_r averaged over a domain of scale r in isotropic turbulence, as follows;

$$\Delta u_r = v(r\varepsilon_r)^{1/3}. \qquad (1)$$

Here the similarity variable v was assumed to be independent of r, ε_r, and $Re_r \equiv r(r\varepsilon_r)^{1/3}/\nu$ in the inertial range of scale r by Kolmogorov (ν: kinematic viscodity). According to our investigation [10], this assumption is partly unrealistic; the statistics of v slightly depends on r while it does hardly on ε_r.

This fact is rationalized by reconsidering the RSH and recognizing that another independent parameter r/L (L: macroscale) should be introduced to govern the statistics of v. Since Kolmogorov acknowledged in 1962 the non-dimensional parameter r/L to come into RSH for the purpose of explaining the intermittency of ε_r in the form: $<\varepsilon_r/\varepsilon_L>^q = (r/L)^{\mu(q)}$ where the angular bracket denotes ensemble average and $\mu(q)$ is called intermittency exponents, it must be the most general that the statistics of v depends on Re_r as well as r/L. While the statistics of v is assumed to be indifferent to Re_r in the inertial range of r, it may always remain dependent on r/L as it is kept finite with the intermittency exponent $\mu(q)$. If so, our conventional view for the scaling of velocity structure function (such as described in [11]) can be basically changed because the moments of v may bring forth non-trivial additional scalings with the parameter r/L. We call this ansatz the generalized RSH (GRSH). How the probability density function (PDF) of v, P(v), depends on r/L is the next important problem.

We construct the PDF of Δu_r on the basis of (1) as

$$p_3(\Delta u_r) = \int P(\Delta u_r/x)/x\, p_2(x;\, r)dx, \qquad (4)$$

where $p_2(x;\, r)$ is the PDF of $x \equiv (r\varepsilon_r)^{1/3}$ [12]. Here and hereafter r is normalized by L, ε_r by ε_L and Δu_r by $(L\varepsilon_L)^{1/3}$, unless stated otherwise. $p_2(x;$

r) is the PDF of x which should be associated with a particular intermittency model for ε_r. For the case of the 3D binomial Cantor set model [13] it is explicitly given as

$$p_2(x; r) = 3x^2/r\ 2^{-\Omega} \sum_{k=0}^{\Omega} {}_{\Omega}C_k\ \delta(x^3/r - B^{\Omega-k}C^k), \tag{5}$$

where $\Omega = -\ln r/\ln A$, $A = 2^{1/3}$, $B = 1.2175$ and $C = 0.7825$. Other values of A, B and C may provide other binomial Cantor sets, including the p model [14]. Let us assume P(v) is a modified Gaussian with variance $\sigma(r)^2$, skewness S(r) and kurtosis K(r) (in the Gram-Charlier form). Then we have straightforwardly

$$<\Delta u_r^3>/<\Delta u_r^2>^{3/2} = S(r)\ r^{(3/2)\mu(2/3)} = S(r)\ r^{-0.0346} \tag{6}$$

$$<\Delta u_r^4>/<\Delta u_r^2>^2 = K(r)\ r^{-\mu(4/3)+2\mu(2/3)} = K(r)\ r^{-0.0917} \tag{7}$$

where an angular bracket denotes the ensemble average with the PDF given by (4), and $\mu(q)$ given in this case as

$$\mu(q) = \log_A[(B^q + C^q)/2]. \tag{8}$$

To determine the forms of S(r) and K(r), we need some theoretical and experimental knowledges. If we assume that the kurtosis of longitudinal velocity gradient is equal to $<\Delta u_\eta^4>/<\Delta u_\eta^2>^2$ (where η is the normalized (by L) Kolmogorov length) and take into account Pullin and Saffman's [15] theoretical prediction based on Lundgren's dynamical model for turbulence [16] that the kurtosis is in proportion to $R_\lambda^{1/4}$ (R_λ: the Taylor-scale Reynolds number), we may set up the following formula:

$$\log_{10}(<\Delta u_\eta^4>/<\Delta u_\eta^2>^2) = a + 0.25 \log_{10} R_\lambda . \tag{9}$$

Here we take a = 0.25 which is reasonable in comparison with Van Atta and Antonia's [17] experimental plots of the kurtosis of longitudinal velocity gradient against R_λ. Since the data points in the figure are much scattered particularly for high R_λ, we avoid here 'best fitting' in the figure. Thus (9) is rewritten by the relation: $\eta = 15^{3/4} R_\lambda^{-3/2}$ (For example, see [18].) as

$$<\Delta u_\eta^4>/<\Delta u_\eta^2>^2 = 2.515\eta^{-0.1667}. \tag{10}$$

Taking account of the 3/8 rule: $|S| \sim K^{3/8}$ (For example, see [19].) and again matching the constant factor with their plots of the skewness against R_λ, we have

$$<\Delta u_\eta^3>/<\Delta u_\eta^2>^{3/2} = -0.321\eta^{-0.0625}. \tag{11}$$

If we assume the inertial range extends down to Kolmogorov scale, (6) and (7) are the conditions to decide the asymptotic behaviors of K and S. On the other hand, we observe that P(v) becomes Gaussian for r larger than some scale, r_c, which belongs to the pre-inertial range. $r_c = 1/2^3$ looks reasonable from observation of energy spectrum [20,21]. Therefore, we propose:

$$K(r) = 2.515\ r^{-0.0750}\ (1 + 0.1647\ r) \tag{12}$$

$$S(r) = -0.321\ r^{-0.0279}\ (1 - 2^3\ r) \tag{13}$$

for $r \le r_c$ so that they continuously reduce to K(r) = 3 and S(r) = 0 for $r \ge r_c$.

Reality may be more complex in detail but this assumption is the simplest and useful enough to give an essential sketch of the v statistics, as is known from the fact that $p_3(\Delta u_r)$ hence calculated in (4) for many r are very well comparable with the corresponding experiments [12].

It is known from [22] that predicting the main body of $p_3(\Delta u_r)$ except for rarely happening events is not so sensitive to the choice of a model for $p_2(x; r)$ (among the binomial Cantor set, the p and the lognormal model), but crucially affected by the form of P(v); particularly the proper r-dependent value of skewness is essential.

From the well-known Kolmogorov relation:

$$<\Delta u_r^3> = -4/5\, r\varepsilon_L \qquad (14)$$

and the third-order moment of (1), we have the relation

$$<v^3> = -4/5 = S(r)\sigma(r)^3, \qquad (15)$$

which gives variance $\sigma(r)^2$ explicitly in the inertial range from the knowledge of S(r). Thus we can formulate the unnormalized second-order structure function as

$$<\Delta u_r^2> = A(r/L)\ (r/L)^{\beta}\ r^{2/3}\varepsilon_L^{2/3}, \qquad (16)$$

where

$$A(r/L) = [(4/5)/0.321]^{2/3}/(1 - 8r/L)^{2/3} = 1.838/(1 - 8r/L)^{2/3}, \qquad (17)$$

$$\beta = 0.0279\times(2/3) - \mu(2/3) = 0.0417, \qquad (18)$$

with $\mu(2/3) = -0.02310$. Hence we know the Kolmogorov prefactor depends basically on scale-ratio r/L rather than Reynolds number, even if a proper average of A(r/L) over the inertial range may bring forth a systematic dependence on R_λ. The only absolute constant prefactor in (16) is A(0) =1.838, that is somewhat lower than the value (4.02 × 0.52 =) 2.09 recommended by Sreenivasan [2] but looks still good as the Kolmogorov constant (to be expected for $R_\lambda = \infty$). But we rather surprise that our simple phenomenological approach gives such a plausible result. The enhanced scaling index in (16), $\beta + 2/3 \equiv \zeta_L(2)$, is very close to 0.71 of the DNS [8] and 0.70 of the experiment [9]. [Further we note that (15) breaks down before r/L reaches 1/8 because the neglected term, $6\nu d/dr<\Delta u_r^2>$, becomes non-negligible there, so that (17) is valid for r/L much less than 1/8. (16) is restricted also by the condition: $<\Delta u_L^2> = 2U^2$ where $U \equiv (L\varepsilon_L)^{1/3}$, so that the real A(r/L) cannot overpass the expected prefactor value for r/L = 1 which can not be very far from 2.]

Now we are interested in the scaling indices $\zeta_L(n)$ of longitudinal velocity structure functions for n > 3. To calculate those, however, the Gram-Charlier form of P(v) is not suitable because S and K make a trouble as r (normailzed again) → 0. The better form to keep the same quality as the

Gram-Charlier form for small values of S and K, is given by the cumulant expansion form of characteristic function:

$$\phi(y) = \exp[-(\sigma y)^2/2 - i(\sigma y)^3 S/3! + (\sigma y)^3 (K - 3)/4! - c(\sigma y)^6/6!] \qquad (19)$$

where $c = 5(K - 3)^2/8$. The last term is the minimum necessary to guarantee that $|\phi(y)| \leq 1$. It was verified [12] that this approach is practically useful for predicting $p_3(\Delta u_r)$ for various r, however small r may be. The true cumulant expansion may need other higher-order terms but here we avoid getting in such a complication, expecting the present truncation to be efficient enough to embody, at least, the main character of the v statistics. Hence the characteristic function of $p_3(\Delta u_r)$ is easily written in terms of $\phi(y)$ and all moments of Δu_r are calculated by differentiation of it. The scaling indices $\zeta_L(n)$ are determined by those of the most dominant terms in the derivatives as $r \to 0$, which are in proportion to $\sigma^n K^{\kappa(n)}$ for n even and $\sigma^{n-3} K^{\kappa(n-3)}$ for n (≥ 3) odd where

$$\kappa(n) = 2[n/6] + [(n - 6[n/6])/4]\,. \qquad (20)$$

(Note $\sigma^3 S$ has no scaling as (11) dictates.) Here the square brackets imply Gauss brackets. Thus we obtain

$$\zeta_L(n) = 0.0093\, n - 0.0750\, \kappa(n) + n/3 - \mu(n/3) \qquad (21)$$

for n even and

$$= 0.0093\, (n-3) - 0.0750\, \kappa(n-3) + n/3 - \mu(n/3) \qquad (22)$$

for n odd.

The first two terms in (21) and (22) correct the conventional formula originated with Kolmogorov: $\zeta_L(n) = n/3 - \mu(n/3)$ appreciably. Fig. 1 shows the corrected $\zeta_L(n)$ (upper dots) in comparison with the conventional ones by the 3D binomial Cantor set model (solid line) and by She-Leveque's model (dotted line) [23]. Our values are $\zeta_L(2) = 0.708$, $\zeta_L(4) = 1.250$, $\zeta_L(6) = 1.706$, and $\zeta_L(8) = 2.159$, which are comparable with the corresponding values of Chen et al. [7,24], 0.695 ± 0.003, 1.279 ± 0.004, 1.772 ± 0.015, and 2.188 ± 0.027, respectively. The experimental values of Dhruva et al. [8] are not far from those. But $\zeta_L(n)$ for odd n cannot be compared with their data, since both Chen et al. and Dhruva et al. treated the structure functions of the absolute magnitude of velocity increment, instead of velocity increment itself. But it is a well-known property that the $\zeta_L(n)$ of the latter for odd n is slightly larger than that of the former, as is seen in Fig. 1.

The above-described approach may be applicable to transverse velocity structure functions, if the same GRSH holds for this case but with a different PDF of similarity variable; which we denote as $P'(v')$ to avoid confusion. Then we may write

$$\Delta v_r = v'(r\varepsilon_r)^{1/3}. \qquad (23)$$

where Δv_r is transverse velocity increment. Obviously $P'(v')$ has no skewness (because of reflective symmetry) but would have different variance $\sigma'(r)^2$ and kurtosis $K'(r)$. $\sigma'(r)^2$ can be exactly related to $\sigma(r)^2$ by virtue of the kinematics of $<\Delta u_r^2>$ and $<\Delta v_r^2>$ [19]. In our case, the ratio of the latter to the former is $4/3 + 0.0209$ as $r \to 0$, and so is $\sigma'(r)^2/\sigma(r)^2$. Hence, (1) and (23) lead naturally to $\zeta_L(2) = \zeta_T(2)$, where $\zeta_T(n)$ is the scaling indices of transverse structure functions. On the other hand, we may estimate $K'(r)/K(r) \sim r^{-0.07}$ from the experimental data for $R_\lambda = 10000\text{-}15000$ by Dhruva et al. [8] with (1) and (23) in mind. If we assume $K'(r)$ becomes Gaussian for $r \geq r_c$, we may have

$$K'(r)/K(r) = 0.8645\, r^{-0.07}. \qquad (24)$$

Therefore, by substitution of (8), we obtain for $r \leq r_c$

$$K'(r) = 2.174\, r^{-0.1450}\,(1 + 0.1647\, r), \qquad (25)$$

which means that Δv_r must be more intermittent than Δu_r, as $r \to 0$. [Hosokawa et al. [25] proposed another ratio $K'(r)/K(r) = 0.7935\, r^{-0.1112}$ from the DNS data for a lower R_λ. But (24), using (4), (5) and (19), gives a slightly better prediction of $p_3(\Delta v_r)$ for $r = 0.00415$ as compared with the corresponding PDF of Vincent and Meneguzzi [26] for $R_\lambda \approx 150$.] In this case, it is easy to formulate

$$\zeta_T(n) = 0.0093\, n - 0.1450\, \kappa(n) + n/3 - \mu(n/3) \qquad (26)$$

for n even and $\zeta_T(n) = 0$ for n odd, in place of (21) and (22).

This result is plotted in Fig. 1 by lower dots. It is likely to be a result of low-order truncation in the cumulant expansion of the PDF that the dots for n even does not step up smoothly with n. [This actually stems from Gauss brackets in (20). A smoother step-up would be made up by considering proper scale-ratio dependences of higher-order cumulant terms. The same situation exists more or less for $\zeta_L(n)$, too.] But the main feature of the scaling indices looks well grasped by the present approach. In fact, our results of $\zeta_T(n)$ are well comparable with those of the recent DNS [7] and experiement [8]. That is, $\zeta_T(2) = 0.708$, $\zeta_T(4) = 1.180$, $\zeta_T(6) = 1.566$, and $\zeta_T(8) = 2.019$ in our case, while 0.71 ± 0.04, 1.25 ± 0.067, 1.63 ± 0.079, and 1.87 ± 0.078, respectively, in Chen et al.'s case [7] and the corresponding experimental values [8] are not far from those. Particularly we note that our approach corrects the consideration of Chen et al. [8] (based on the 1962 Kolmogorov concept) on the scaling of both structure functions by $0.0093\, n - 0.0750\, \kappa(n)$ and $0.0093\, n - 0.1450\, \kappa(n)$, respectively, as a result of the scaling behaviors of similarity variables. The latter values compensate the deficency of $\zeta_T(n) - \mu(n/3)$ from $n/3$ considerably well, while the former values tend to overcorrect it only slightly. See Fig. 4 in [7]. These facts seem to sign "no problem" in the use of ERSH to longitudinal as well as transverse velocity increments, if similarity variables v and v' properly behave. It is to be noted that both variables are not quite independent but have a cross kurtosis since $<\Delta u_r^2 \Delta v_r^2>$ is not trivial.

Chen et al. [7] who had not consider the scaling of similarity variable, however, proposed the RSH tranverse (RSHT) that implies

$$\Delta v_r = v'[r(\nu\omega^2)_r]^{1/3} \tag{27}$$

where ω is the magnitude of vorticity and the suffix r means the average of the braced quantity over a domain of scale r. They presumed that the possible scale-similarity [27] of $(\nu\omega^2)_r$ is different from that of ε_r in order to explain the gap of $\zeta_L(n)$ and $\zeta_T(n)$. Fig. 4 in [7] seems to verify this presumption pretty well, only if similarity variable has no such scaling as discussed above. However, it should be pointed out that $(\nu\omega^2)_r$ and ε_r are not so independent of each other in reality; according to our DNS [10], the correlation coefficient of the two quantities ranges from 0.9987 to 0.9579 in the inertial range and drops to 0.8466, 0.6468 and 0.4835 for about 4, 2 and 1 times Kolmogorov scale. At the present stage, it might be hard to say which is better, RSHT or GRSH. But judging from the success of GRSH for longitudinal velocity increment [12], neglecting the scale-ratio dependence of the statistics of similarity variables seems to be too restrictive.

In summary, the scale-ratio dependent statistics of similarity variable v in isotropic turbulence has been investigated based on the GRSH ansatz and the 3D binomial Cantor set model for dissipation measure. In this investigation Pullin and Saffman's theory [15] and the 3/8 rule [19] play the largest role. As a result, plausible values of the Kolmogorov prefactor and scaling indices of longitudinal velocity structure functions have been obtained. In a similar way, scaling indices of transverse velocity structure functions have been evaluated. These are well comparable with those from the recent DNS by Chen et al. and experiment by Dhruva et al. Although it has semi-empirical factors, the present consideration reveals a new conceptual possibility in the expected universal structure of isotropic turbulence. This is the first trial on the ansatz and so there would be room for improvement of the whole process as precise knowledges increase.

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Figure Caption

Fig. 1 Scaling indices of velocity structure functions. Those based on the 3D binomial Cantor set model and She-Leveque model are indicated by the solid line and dotted line, respectively. Upper and lower dots show those for longitudinal velocity structure functions (17,18) and transverse velocity structure functions (20), respectively, obtained by the present method.

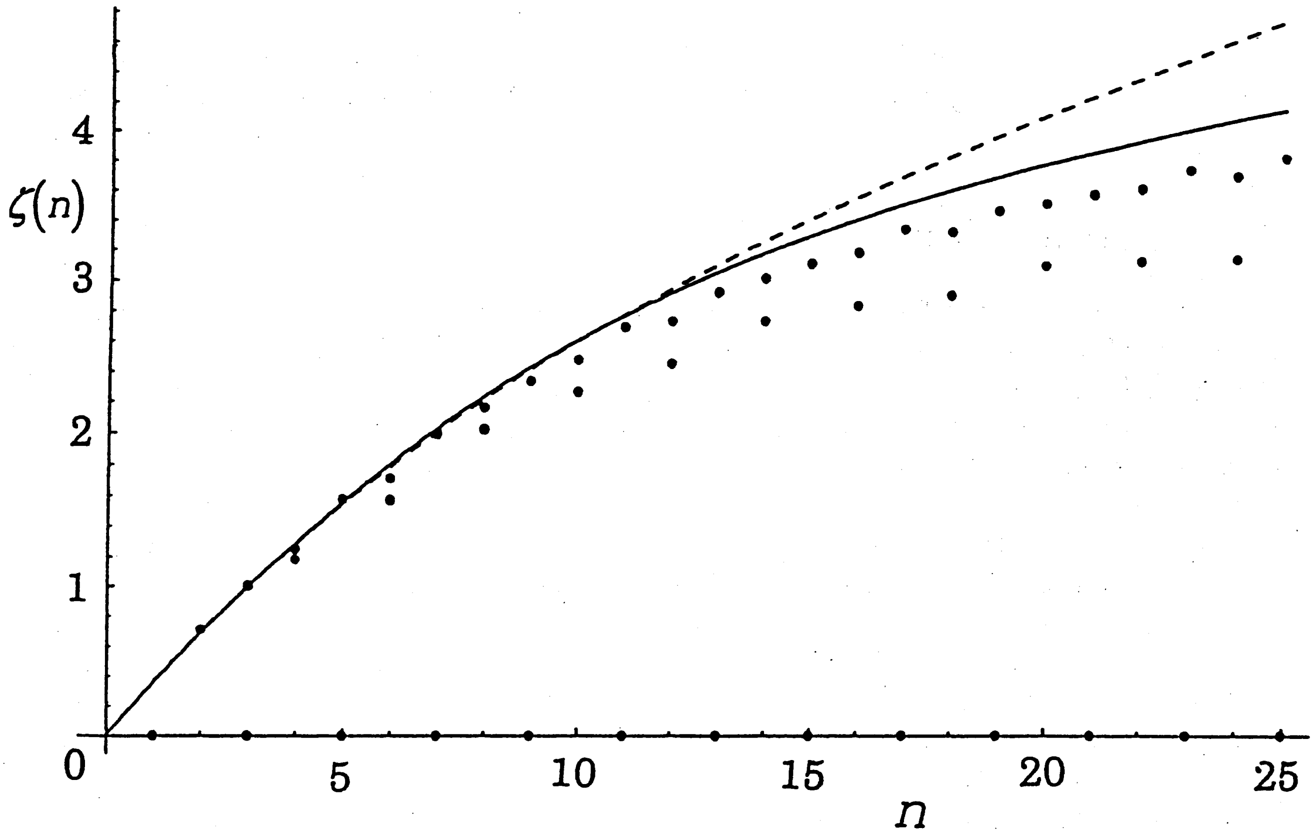

Fig.1