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Theta liftings - a comparison between classical and representation theoretic results

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Introduction.

Theta liftings have been considered both from a classical and from a representation theoretic point of view. In the classical setting, one considers holomorphic theta series attached to integral quadratic forms as Siegel (or Hilbert-Siegel) modular forms. One is then interested in a description of the linear relations between the members of a given set of such theta series and in a characterization of the space of modular forms spanned by these theta series. The theta series in such a set are usually quite restricted in type, e.g., they belong to full lattices of some fixed level or they are theta series with characteristic (thetanullwerte) attached to a single lattice but with varying characteristic. The representation theoretic approach considers the more general theta correspondence between automorphic forms on adelic orthogonal and symplectic (or metaplectic) groups defined using the oscillator (or Weil-) representation of the metaplectic group. Here one discusses for an irreducible representation space of automorphic forms on one of the groups whether it is in the image under the theta correspondence of a representation space of automorphic forms on the other group respectively whether its image under the correspondence is zero or not.

Although both types of question appear to be extremely similar, they are not quite the same. It is the purpose of this note to discuss some cases in which a transfer of results between the two settings is made possible by recent results of Moeglin and to describe some of the difficulties that occur in other cases.

1. The problems.

Let (V, q) be a non-degenerate quadratic space of even dimension $m = 2k$ over $\mathbf{Q}$, denote by $B(x,y) = q(x+y) - q(x) - q(y)$ the associated symmetric bilinear form, let $L \subseteq V$ be an integral $\mathbf{Z}$-lattice of full rank on V (i.e., $q(L) \subseteq \mathbf{Z}$, $\mathbf{Q} \otimes L \cong V$) of level N (i.e., $q(L^{\#})\mathbf{Z} = N^{-1}\mathbf{Z}$, where $L^{\#} = \{y \in V \mid B(y, L) \subseteq \mathbf{Z}\}$ is the dual lattice of L). The genus of L consists of all lattices K on V with $L \otimes \mathbf{Z}_p$ isometric with respect to q to $K \otimes \mathbf{Z}_p$ for all primes p. It consists of finitely many isometry classes of lattices. We restrict attention to positive definite q, let $\mathfrak{H}_n$ denote the Siegel upper half space of degree (or genus) n, put

$$q(\mathbf{x}) = (\frac{1}{2}B(x_i, x_j)) \in M_n^{\mathrm{sym}}(\mathbf{Q}) \quad \text{for } \mathbf{x} = (x_1, \ldots, x_n) \in V^n$$

and consider the theta series

$$\vartheta_L^{(n)}(Z) = \sum_{\mathbf{x} \in L^n} \exp(2\pi i \operatorname{tr}(q(\mathbf{x})Z)).$$

Inhomogeneous theta series $\vartheta_{L,\mathbf{y}}^{(n)}(Z)$ are defined in the same way, but with summation over a coset $\mathbf{y} + L^n$ with $\mathbf{y} \in V^n$. More generally we can consider a harmonic form $P : V^n \otimes \mathbf{R} \longrightarrow W_\rho$ with values in some irreducible representation space W_ρ of $\mathrm{GL}_n(\mathbf{C})$ and put

$$\vartheta_{L,\mathbf{y}}^{(n)}(P, z) = \sum_{\mathbf{x} \in \mathbf{y} + L^n} P(\mathbf{x}) \exp(2\pi i \operatorname{tr}(q(\mathbf{x})Z))$$

[7].

For $\mathbf{Y} = \mathbf{0}$, $\vartheta_L^{(n)}(P, Z)$ is a W_ρ-valued Siegel modular form for $\Gamma_0^{(n)}(N)$ with character χ depending on the discriminant of L, for the inhomogenous theta series see [7]. We want to consider here the following questions:

A) Find the linear relations between the $\vartheta_{L_i}^{(n)}$ for L_i running through a set of representatives of the isometry classes of lattices in the genus of the lattice L.

B) Characterize the space of modular forms generated by the $\vartheta_{L_i}^{(n)}$ for L_i as above.

The question B) (the basis problem) has been considered in many variants, for example

B') Let N, χ be given. Is the space $M_k^{(n)}(\Gamma_0(N), \chi)$ of modular forms of degree n, weight k and character χ (or its subspace of cusp forms) generated by

a) theta series (maybe inhomogeneous) attached to (arbitray) quadratic forms

b) by theta series attached to full lattices

c) by theta series attached to full lattices of level N

d) by theta series with spherical harmonics as above

In each of these subproblems, determine an explicit representation of a given $F \in M_k^{(n)}(\Gamma_0(N), \chi)$.

For a survey of some results concerning B), B') see [2].

For the representation theoretic view of the problem let $(U^{(n)}, A)$ denote a $2n$-dimensional space over $\mathbf{Q}$ with non-degenerate alternating form, $\psi : \mathbf{Q}_{\mathbf{A}}/\mathbf{Q} \longrightarrow S^1$ a nontrivial additive character. The symplectic group $G = G_n = \mathrm{Sp}(U^{(n)}, A) = \mathrm{Sp}(n)$ and the orthogonal group $H = O(V)$ form a dual reductive pair in the sense of Howe. Let $\omega = \omega_\psi$ denote the Weil representation of the adelic group $G(\mathbf{A}) \times H(\mathbf{A})$ on the space $S(V(\mathbf{A})^n)$ of Schwartz-Bruhat functions on $V(\mathbf{A})^n$. For $f \in S(V(\mathbf{A})^n)$, $g \in G(\mathbf{A})$, $h \in H(\mathbf{A})$ define the theta kernel

$$\theta(g, h; f) := \sum_{\mathbf{x} \in V(L)^n} \omega(g)\varphi(h^{-1}\mathbf{x}).$$

For a space Y of cuspidal automorphic forms on $G(\mathbf{A})$ we write $\Theta^V(Y)$ for the space of theta lifts

$$\Theta_f(\varphi')(h) := \int_{G(\mathbf{Q})\backslash G(\mathbf{A})} \varphi'(g)\Theta(g,h;f)dg$$

with $\varphi' \in Y$, $f \in S(V(\mathbf{A})^n)$, similarly for a space X of automorphic forms on $H(\mathbf{A})$ we write $\Theta^U(X)$ for the space of theta lifts

$$\Theta_f(\varphi) := \int_{H(\mathbf{Q})\backslash H(\mathbf{A})} \varphi(h)\Theta(g,h;f)dh$$

of functions $f \in X$ with respect to $f \in S(V(\mathbf{A})^n)$ to automorphic forms $\Theta_f(\varphi)$ on $G(\mathbf{A})$. We have then

$\tilde{\mathbf{A}}$) Given a (cuspidal) irreducible automorphic representation π of $H(\mathbf{A})$, decide whether $\Theta^{U^{(n)}}(\pi)$ is nonzero (find the first n for which it is nonzero).

$\tilde{\mathbf{B}}$) Given a cuspidal irreducible automorphic representation π' of $G(\mathbf{A})$, decide whether π' is a lift of some representation π of $H(\mathbf{A})$ as above.

With a lattice L on V with adelic orthogonal group $O_{\mathbf{A}}(L) \subseteq H(\mathbf{A})$ let $\mathcal{A}(H(\mathbf{A}), O_{\mathbf{A}}(L))$ be the space of functions $\varphi : H(\mathbf{A}) \longrightarrow \mathbf{C}$ with $\varphi(\gamma h u) = \varphi(h)$ for $\gamma \in H(\mathbf{Q})$, $u \in O_{\mathbf{A}}(L)$ and consider a double coset decomposition

$$H(\mathbf{A}) = \dot{\bigcup}_{i=1}^{t} H(\mathbf{Q})h_i O_{\mathbf{A}}(L).$$

Let $f^{(n)} \in S(V(\mathbf{A})^n)$ be given as $f^{(n)} = \prod_p f_p^{(n)}$ with $f_p^{(n)} = \mathbf{1}_{l_p^n}$ for finite p and

$$f_\infty^{(n)}(\mathbf{x}) = \exp(-2\pi(q(x_1) + \cdots + q(x_n))).$$

Then for $\varphi \in \mathcal{A}(H(\mathbf{A}), O_{\mathbf{A}}(L))$ we have

$$\Theta_{f^{(n)}}^{U^{(n)}}(\varphi)(g) = \sum_{j=1}^{t} \frac{\varphi(h_j)}{|O(h_j L)|}\omega(g) \sum_{\mathbf{x}\in(h_j L)^n} \exp(-2\pi \cdot \mathrm{tr}(q(\mathbf{x}))),$$

and under the usual correspondence of automorphic forms on $G(\mathbf{A})$ and on the Siegel half space $\mathfrak{H}_n$ of degree n this function corresponds to

$$\Theta_L^{(n)}(\varphi)(Z) := \Theta_{f^{(n)}}^{U^{(n)}}(\varphi)(Z) = \sum_{j=1}^{h} \frac{\varphi(h_j)}{|O(h_j L)|} \vartheta^{(n)}(h_j L, Z),$$

where $L_j := h_j L$ runs through a set of representatives of the isometry classes in the genus of L. Similar expressions involving theta series with harmonic forms arise for φ from a space of functions on $O_{\mathbf{A}}(V)$ that are right invariant under $\prod_{p\neq\infty} O(L_p)$ and transform according to some fixed irreducible representation of $H(\mathbf{R})$ under right translation by elements of $H(\mathbf{R})$, see [7, 4, 5].

On the other hand, given φ' on $G(\mathbf{A})$ that corresponds to a Siegel modular cusp form F of weight $k = \frac{m}{2}$ we have

$$\Theta_{f^{(n)}}^{V}(\varphi')(h_j) = C \cdot \langle F, \vartheta^{(n)}(L_j) \rangle$$

with some constant $C \neq 0$ depending on the normalization chosen and with $\langle \ , \ \rangle$ denoting the Petersson inner product.

Our questions concerning the relation between results in the classical and in the representation theoretic setting are then

?A) Let π be an irreducible cuspidal representation of $H(\mathbf{A})$ with $\pi \cap \mathcal{A}(H(\mathbf{A}), O_{\mathbf{A}}(L)) \neq \{0\}$ and with $\Theta^{U^{(n)}}(\pi) \neq \{0\}$. Then there are some $\varphi \in \pi$, $f \in S(V_{\mathbf{A}}^n)$ such that

$$\Theta_f^{U^{(n)}}(\varphi) \neq 0,$$

but the lift may of course vanish for individual φ, f. Under which conditions on $\varphi \in \mathcal{A}(H(\mathbf{A}), O_{\mathbf{A}}(L)) \cap \pi$ and on π can we conclude that

$$\Theta_L^{(n)}(\varphi) \neq 0 \ ?$$

?B) Let π' be an irreducible cuspidal automorphic representation of $G_n(\mathbf{A})$ such that $\pi' = \Theta^{U^{(n)}}(\pi)$ for some π of $H(\mathbf{A})$ and let F be a Siegel modular form of level N and weight $k = \frac{m}{2}$ corresponding to some $\varphi' \in \pi'$. Then there exist $\varphi \in \pi$ and $f \in S(V_{\mathbf{A}}^n)$ with

$$\varphi' = \Theta_f^{U^{(n)}}(\varphi).$$

Under which conditions on F and on π' can we conclude that F can more specifically be written as $F = \Theta_L^{(n)}(\varphi)(Z)$ for some lattice L on V of level N and $\varphi \in \mathcal{A}(H(\mathbf{A}), O_{\mathbf{A}}(L))$?

We remark that question B) is important if we want to write down F explicitly (e.g. by giving part of its Fourier expansion): It is for moderate dimensions and levels computationally feasible to tabulate the lattices in a given genus and to compute their theta series. On the other hand, the formulation $\pi' = \Theta^{U^{(n)}}(\pi)$ gives only the existence of an expression of F by possibly inhomogenous theta series attached to lattices of some levels (which are in no way restricted).

2. Results.

Our main tool will be the following theorem of Moeglin [15]:

Theorem (Moeglin). *Let π be an irreducible subrepresentation of the space of (cuspidal) automorphic forms on the adelic orthogonal group $H(\mathbf{A})$ and suppose that $\Theta^{U^{(n)}}(\pi)$ contains a nonzero cusp form. Then $\Theta^{U^{(n)}}(\pi)$ is an irreducible cuspidal representation, and one has $\Theta^V(\Theta^{U^{(n)}}(\pi)) = \pi$. Conversely for π' a cuspidal irreducible automorphic representation of $G(\mathbf{A})$, $\Theta^{(V)}(\pi')$ is irreducible (cuspidal) (if it contains a nonzero cusp form) and*

$$\Theta^{U^{(n)}}\Theta^V(\pi') = \pi'.$$

Remark. For our present situation of positive definite V the cuspidality assumption is vacuous for forms on $H(\mathbf{A})$ and can hence be omitted. Moeglin's result is valid for general (V, q).

Proposition 1. *Let π be an irreducible subrepresentation of the space of automorphic forms on $H(\mathbf{A})$ such that $\Theta^{(n)}(\pi)$ contains a nonzero cusp form. Let L be a lattice on V satisfying*

> $(*)$ *For $r \in \mathbf{N}$ and for all $\varphi \in \mathcal{A}(H(\mathbf{A}), O_{\mathbf{A}}(L))$ with $\Theta_L^r(\varphi) = 0$ the lift $\Theta_L^{(r+1)}(\varphi)$ is cuspidal.*

Then $\Theta_L^{(n)}(\varphi) \neq 0$ for all $\varphi \neq 0$ in $\mathcal{A}(H(\mathbf{A}), O_{\mathbf{A}}(L)) \cap \pi$.

Proof. Ist is known [16] that there is r with $\Theta^{U^{(r)}}(\pi) \neq \{\mathbf{0}\}$ and that for the smallest such r the lifting $\Theta^{(r)}(\pi)$ is cuspidal, $\Theta^{(s)}(\pi)$ non-cuspidal for $s > r$. Also, the linear independence of theta series of degree bigger than the dimension implies that $\Theta_L^{(s)}(\varphi) \neq 0$ for some s. If $\Theta_L^{(n)}(\varphi)$ were 0, our assumption $(*)$ would imply that $\Theta_L^{(s)}(\varphi)$ is nonzero cuspidal for some $s > n$, hence $\Theta^{U^{(s)}}(\pi)$ is cuspidal by Moeglin's theorem. But then Rallis' result quoted above contradicts $\Theta^{(n)}(\pi) \neq 0$.

Remark.

a) Condition $(*)$ is obviously true for L of level 1 (even unimodular L) and by [3] for square free N. For general N, little seems to be known. Classically this condition says that $\Theta_L^{(s)}(\varphi)$ is cuspidal if and only if its image under Siegel's Φ-operator is zero.

b) If there is $r \in \mathbf{N}$ with $\Theta_L^{(r)}(\varphi) = 0$, $\theta_L^{(r+1)}(\varphi)$ noncuspidal, then $\theta^{(r+1)}(\pi)$ is not cuspidal and hence by Rallis' result $\Theta^{(r)}(\pi) \neq 0$. Condition $(*)$ is therefore necessary for the validity of the conclusion of Proposition 1.

c) Statement and proof of the proposition can be transferred to theta series with spherical harmonics (liftings of $\varphi \in \mathcal{A}(H(\mathbf{A}), O_{\mathbf{A}}(L), \tau)$ for an irreducible representation (Z_τ, τ) of $H(\mathbf{R})$).

d) Let V be a definite quaternion algebra over $\mathbf{Q}$, equipped with the (reduced) norm form as quadratic form, L an Eichler order in V of square free level N. Then in [3, 5] examples have been given of φ on $H(\mathbf{A})$ with $\theta_L^{(2)}(\varphi) = 0$. The proposition shows that then indeed $\Theta^{U^{(2)}}(\pi) = 0$ for the irreducible representation π generated by φ, hence $\Theta^{U^{(3)}}(\pi)$ is cuspidal. It has been shown by Roberts [18] that the representation (extended to the group of similitudes) has a nonzero theta lifting to to GSp_2 locally everywhere. This is in contrast to the situation in [11], where the nonvanishing of the theta lift to GSp_2 is decided by purely local conditions, using the result of [8].

Proposition 2. *Let π' be an irreducible subrepresentation of the space of cuspidal automorphic forms on $G_n(\mathbf{A})$ containing a function φ' corresponding to the Siegel modular form F of weight k for $\Gamma_0^{(n)}(N)$. Assume that $\pi := \Theta^V(\pi') \neq \{0\}$ and that*

> (**) *there is a lattice L on V of level $N'|N$ such that $\mathcal{A}(H(\mathbf{A}), O_{\mathbf{A}}(L)) \cap \pi \neq \{0\}$ and such that (*) from Proposition 1 holds.*

Then π' contains a Siegel cusp form $\tilde{F}$ of weight k for $\Gamma_0^{(n)}(N')$ such that $\tilde{F}$ is a linear combination of theta series attached to lattices in the genus of L.

Proof. By Moeglin's theorem π is irreducible. For $0 \neq \varphi \in \mathcal{A}(H(\mathbf{A}), O_{\mathbf{A}}(L)) \cap \pi$ by Proposition 1 we have $\tilde{F} := \theta_L^{(n)}(\varphi) \neq 0$. Again by Moeglin's theorem $\tilde{F} \in \pi'$ holds.

Remark.

a) If the $\mathbf{Z}$-maximal lattices on V are even unimodular and N is square free and odd, the condition $(**)$ is satisfied: By the results of Aubert [1] for each $p|N$ the local representation π_p contains a vector invariant under the orthogonal group of some lattice of level dividing p von V_p. The resulting function $\varphi \in \mathcal{A}(H(\mathbf{A}), O_{\mathbf{A}}(L))$ satisfies $(*)$ as in the remark to Proposition 1. A modified version of Aubert's result should be true for the local theta correspondence with respect to arbitrary V_p (Aubert requires V_p to carry a self-dual lattice and $p \neq 2$). This would give the validity of $(**)$ for arbitrary V and square free N.

b) The condition $(**)$ is necessary for the validity of the conclusion of the proposition: If $\tilde{F}$ is as described, then $\tilde{F} = \Theta_L^{(n)}(\varphi)$ for some $\varphi \in \mathcal{A}(H(\mathbf{A}), O_{\mathbf{A}}(L))$, and $\varphi \in \Theta^V(\pi') = \pi$ follows from Moeglin's theorem, hence $\mathcal{A}(H(\mathbf{A}), O_{\mathbf{A}}(L)) \cap \pi \neq \{\mathbf{0}\}$. Then $\Theta_L^{(n)}(\varphi) = \tilde{F}$ is cuspidal by assumption, hence $\Theta_L^{(r)}(\varphi) = 0$ for all $r < n$, and $(*)$ is satisfied for φ as well.

c) If $N = 1$, the cusp form F of level 1 in π' is unique (for each p, the $\mathrm{Sp}_n(\mathbf{Z}_p)$-fixed vectors in an irreducible local representation is unique). Hence $\tilde{F}$ is proportional to F, that is, F itself is a linear combination of theta series of even unimodular lattices. In this case we have therefore proved that a representation theoretic solution of the basis problem implies a classical solution.

d) If $n = 1$ and N is the level of π' then by [6] $\tilde{F}$ is again proportional to F. The condition $(**)$ can then be obtained from the local Jacquet-Langlands correspondence [13], except for the additional requirement of $(*)$. In particular, our result seems not be strong enough to show that the correspondence of Jacquet-Langlands implies the solution of the basis problem due to Hijikata, Pizer and Shemanske [12], except for the case of square free level N. In this case, a different way to derive the implications has been sketched in Section 9 of [12].

Proposition 3. *Let π be an irreducible subrepresentation of the space of automorphic forms on $H(\mathbf{A})$ and let $\varphi \in \pi$, $f \in S(V(\mathbf{A})^n)$ be such that $\Theta_f^{U(n)}(\varphi)$ corresponds to a Siegel modular form F of weight k with respect to $\mathrm{Sp}_n(\mathbf{Z})$. Assume that the $\mathbf{Z}$-maximal lattices on V are even unimodular. Then F is a linear combination of theta series attached to even unimodular lattices on V.*

Proof. By Moeglin's theorem $\Theta^{U(n)}(\pi) =: \pi'$ is cuspidal irreducible with $\Theta_V(\pi') = \pi$. By the local theta correspondence [14, 17] π has to contain a vector $\tilde{\varphi}$ invariant under the group $O_{\mathbf{A}}(L)$ for a lattice L of level 1. The assertion follows from Proposition 2.

Remark. If $\varphi \in \mathcal{A}(H(\mathbf{A}), O_{\mathbf{A}}(K)) \cap \pi$ for some lattice K on V is such that $\Theta_{f_n(K)}^{U(n)}(\varphi)$ corresponds to F with the special test function $f_n(K)$ associated to K the assertion can also be deduced using the computations of traces of theta series in [10].

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