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ON ABSOLUTE CM–PERIODS II

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Abstruct. For a CM-field K, Shimura defined the period symbol p_K by factorizing periods of abelian varieties with complex multiplication. We define the absolute period symbol g_K using division values of the multiple gamma function and conjecture that p_K coincides with g_K up to the multiplication by algebraic numbers. Taking the action of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ into account, we present a refined version of this conjecture. We show that these conjectures are consistently formulated and discuss various numerical examples which support our conjectures strongly.

In our previous paper [Y2], we formulated a conjecture which gives an expression of the derivatives of Artin L-functions at $s=0$ by CM-periods. However we could not express CM-periods themselves by such a conjecture.[1] In the present paper, we shall give a conjecture which expresses CM-periods by the values of the multiple gamma function at division points, and present various numerical examples which support it.

Let us explain our ideas and the contents of this paper more precisely. Let K be a CM-field, J_K be the set of all isomorphisms of K into $\mathbf{C}$ and I_K be the free abelian group generated by J_K. For every σ, $\tau \in I_K$, Shimura defined ([S2], [S3]) the CM-period $p_K(\sigma,\tau) \in \mathbf{C}^\times$, which is uniquely determined mod $\overline{\mathbf{Q}}^\times$. The fundamental properties of the period symbol p_K will be reviewed in §1. For a, $b \in \mathbf{C}$, let us write $a \sim b$ if $b \neq 0$ and $a/b \in \overline{\mathbf{Q}}$. Using p_K, we can write the Chowla-Selberg formula as

$$\pi p_K(\mathrm{id},\mathrm{id})^2 \sim \prod_{a=1}^{d-1} \Gamma(\frac{a}{d})^{w\chi(a)/2h}. \tag{1}$$

Here K is an imaginary quadratic field of discriminant $-d$, w is the number of roots of unity contained in K, h is the class number of K and χ is the Dirichlet character which corresponds to K.

Now assume that K is abelian over a totally real field F. Put $n=[F:\mathbf{Q}]$, $G = \mathrm{Gal}(K/F)$ and let $\rho \in G$ be (the induced map from) the complex conjugation. Let $\widehat{G}_-$ be the set of all characters χ of G such that $\chi(\rho)=-1$. Let $J_F=\{\sigma_1,\sigma_2,\cdots,\sigma_n\}$ and extend σ_i to an element of J_K, for which we use the same symbol σ_i. Our previous conjecture, that is essentially equivalent to a conjecture of Colmez [C], can be brought to the form

$$\prod_{i=1}^{n} p_K(\sigma_i,\tau\sigma_i) \sim \pi^{-n\mu(\tau)/2} \prod_{\chi\in\hat{G}_-} \exp(\frac{\chi(\tau)}{|G|}\frac{L'(0,\chi)}{L(0,\chi)}), \quad \tau \in G, \tag{2}$$

[1]This point will be shown explicitly by an example in §7. See also the discussion in the beginning of [Y2], §2.

where $\mu(\tau) = 1$ (resp. -1) if $\tau = 1$ (resp. $\tau = \rho$) and $\mu(\tau) = 0$ otherwise (cf. §3). A property of the period symbol implies that $\prod_{i=1}^n p_K(\sigma_i, \tau\sigma_i) \sim \prod_{i=1}^n p_{K^{\sigma_i}}(\mathrm{id}, \sigma_i^{-1}\tau\sigma_i)$. Shintani's results ([Sh1], [Sh2], [Sh3]) express $L'(0,\chi)$ in terms of the multiple gamma function introduced by Barnes ([Ba1], [Ba2]), which will be reviewed in §2. Inserting Shintani's formula for $L'(0,\chi)$, we shall realize that the right hand side of (2) can be factorized naturally in accord with the factorization of the left hand side. Thus, in §4, we shall define $g_K(\mathrm{id},\tau) \in \mathbf{C}^\times$ for $\tau \in \mathrm{Gal}(K/F)$ using the multiple gamma function so that $\prod_{i=1}^n g_{K^{\sigma_i}}(\mathrm{id}, \sigma_i^{-1}\tau\sigma_i)$ is equal to the right hand side of (2) and shall predict

Conjecture A. *$p_K(id,\tau) \sim g_K(id,\tau)$ for $\tau \in Gal(K/F)$.*

We note that if $F = \mathbf{Q}$, Conjecture A follows from a result of Anderson [A].

We can sharpen Conjecture A in a similar way as in [Y2], §5. By virtue of a theorem of Shimura (cf. §1), we can choose Grössencharacters $\lambda_1, \lambda_2, \cdots, \lambda_q$ of type A_0 of K and integers $\epsilon_1, \epsilon_2, \cdots, \epsilon_q, m$ so that

$$\prod_{i=1}^q L(m/2, \lambda_i)^{\epsilon_i} \sim \pi^A p_K(\mathrm{id},\tau)^e$$

holds with $A \in 2^{-1}\mathbf{Z}$, $e \in \mathbf{Z}$. Then we expect

Conjecture B. *For every $\sigma \in Aut(\mathbf{C})$, there exists a root of unity ζ such that*

$$\left(\frac{\prod_{i=1}^q L(m/2,\lambda_i)^{\epsilon_i}}{\pi^A g_K(id,\tau)^e}\right)^\sigma = \zeta \cdot \frac{\prod_{i=1}^q L(m/2,\lambda_i^\sigma)^{\epsilon_i}}{\pi^A g_{K^\sigma}(id,\sigma^{-1}\tau\sigma)^e}.$$

In §4, we shall also show that Conjecture A is sufficient to express $p_K(\sigma,\tau) \bmod \overline{\mathbf{Q}}^\times$ in terms of the multiple gamma function for all $\sigma, \tau \in I_K$.

Let us describe our factorization of (2) explicitly in the simplest case $n = 2$, $[K : F] = 2$, $\tau = 1$. Let $\epsilon > 1$ be the generator of the group of totally positive units of F. Let $\mathfrak{a}_1, \mathfrak{a}_2, \cdots, \mathfrak{a}_{h_0}$ be integral ideals which represent narrow ideal classes of F. Let $\mathfrak{f}\infty_1\infty_2$ be the conductor of K as a class field over F, where ∞_1 and ∞_2 are the archimedean primes of F. Let $C_\mathfrak{f}$ be the ideal class group of conductor $\mathfrak{f}\infty_1\infty_2$ of F. For $c \in C_\mathfrak{f}$, take $\mathfrak{a}_\mu$ so that c and $\mathfrak{a}_\mu\mathfrak{f}$ belong to the same narrow ideal class and put

$$R(\epsilon,c) = \{z = x + y\epsilon \in (\mathfrak{a}_\mu\mathfrak{f})^{-1} \mid x, y \in \mathbf{Q}, 0 < x \le 1, 0 \le y < 1, (z)\mathfrak{a}_\mu\mathfrak{f} = c \text{ in } C_\mathfrak{f}\}.$$

Regard $\chi \in \widehat{G}_-$ as a character of $C_\mathfrak{f}$ and let $\Gamma_2(z,(1,\epsilon))$ be the double gamma function. By Shintani's formula, (2) can be written as

$$\prod_{i=1}^2 p_K(\sigma_i,\sigma_i) \sim \pi^{-1}\exp\Big(\frac{1}{2L(0,\chi)}\sum_{c\in C_\mathfrak{f}}\chi(c)\sum_{z=x+y\epsilon\in R(\epsilon,c)}$$

$$\left[\log\frac{\Gamma_2(z,(1,\epsilon))\Gamma_2(z',(1,\epsilon'))}{\rho_2((1,\epsilon))\rho_2((1,\epsilon'))} + \frac{\epsilon'-\epsilon}{2}\log\epsilon\, B_2(x) - \log N(\mathfrak{a}_\mu\mathfrak{f})\left\{\frac{\epsilon+\epsilon'}{4}(B_2(x)+B_2(y)) + B_1(x)B_1(y)\right\}\right]\Big). \tag{3}$$

Here, for $\alpha \in F$, α' denotes its conjugate and $B_m(x)$ denotes the m-th Bernoulli polynomial. (For $\rho_2((1,\epsilon))$, see §2.) We define

$$g_K(\mathrm{id},\mathrm{id}) = \pi^{-1/2}\exp(\frac{1}{2L(0,\chi)}\sum_{c\in C_{\mathfrak{f}}}\chi(c)\sum_{z=x+y\epsilon\in R(\epsilon,c)}\Big[\log\frac{\Gamma_2(z,(1,\epsilon))}{\rho_2((1,\epsilon))}$$
$$+\frac{\epsilon'-\epsilon}{4}\log\epsilon\, B_2(x) - \frac{1}{4}\log N(\mathfrak{a}_\mu\mathfrak{f})\{\epsilon' B_2(x)+\epsilon B_2(y)+2B_1(x)B_1(y))\}\Big]),$$

$$g_{K^{\sigma_2}}(\mathrm{id},\mathrm{id}) = \pi^{-1/2}\exp(\frac{1}{2L(0,\chi)}\sum_{c\in C_{\mathfrak{f}}}\chi(c)\sum_{z=x+y\epsilon\in R(\epsilon,c)}\Big[\log\frac{\Gamma_2(z',(1,\epsilon'))}{\rho_2((1,\epsilon'))}$$
$$+\frac{\epsilon'-\epsilon}{4}\log\epsilon\, B_2(x) - \frac{1}{4}\log N(\mathfrak{a}_\mu\mathfrak{f})\{\epsilon B_2(x)+\epsilon' B_2(y)+2B_1(x)B_1(y))\}\Big]),$$

where $\sigma_2|F \neq \mathrm{id}$. Put

$$a = \frac{1}{L(0,\chi)}\sum_{c\in C_{\mathfrak{f}}}\chi(c)\sum_{z=x+y\epsilon\in R(\epsilon,c)}\frac{\epsilon'-\epsilon}{4}B_2(x).$$

Then Conjecture A predicts (cf. Lemma 4)

$$\pi p_K(\mathrm{id},\mathrm{id})^2 \sim \epsilon^a \prod_{c\in C_{\mathfrak{f}}}\left\{\prod_{z\in R(\epsilon,c)}\frac{\Gamma_2(z,(1,\epsilon))}{\rho_2((1,\epsilon))}\right\}^{\chi(c)/L(0,\chi)} \tag{4}$$

If $F=\mathbf{Q}(\sqrt{d})$, $0<d\in\mathbf{Q}$, a is of the form $b\sqrt{d}$ with $b\in\mathbf{Q}$. In our examples discussed in the text, this quantity a will play an important role. Note that $L(0,\chi)=2h/i$, where h is the relative class number of K/F and i is the index of the unit group of F in the unit group of K. The analogy of (4) to (1) is now manifest.

In the general case, Shintani's formula depends not only on the choice of representatives of narrow ideal classes but also on a cone decomposition of $\mathbf{R}^n_+$; so does our definition of "absolute period" $g_K(\mathrm{id},\tau)$. In §5, we shall study the dependence of $g_K(\mathrm{id},\tau)$ and show that the validity of Conjectures A and B does not depend on these auxiliary data. A careful reader will notice that our definition of $g_K(\mathrm{id},\tau)$ is almost uniquely forced to satisfy this demand of canonicality and the factorization of (2).

In §6 $\sim$ §8, we shall discuss numerical examples to convince ourselves of the truth of Conjectures A and B. In §6, we examine the case when F is real quadratic, $[K:F]=2$, K is not normal over $\mathbf{Q}$. In §7, we examine the normal closures over $\mathbf{Q}$ of CM-fields discussed in §6, which are cyclic extensions of degree 4 of real quadratic fields. These sections correpond to the classical case treated by Hecke and can be regarded as a sharpening of the experiments made in [Y2]. Basic procedures of present experiments are the same as before and though we shall describe our results fairly explicitly, the reader is advised to see [Y2], §3, §4 for more technical details. In §8, we shall present an example when F is a cubic field and $[K:F]=2$. This example is important for us to believe the validity of

Conjectures A and B in the general case, since the term $V(c)$ in the definition of $g_K(\mathrm{id}, \tau)$ (cf. (4.4)) is too simple when $n = 2$.

Our conjectures will produce a number of new problems. The author would like to discuss some of them on a future occasion.

Notation and Terminology. Throughout the paper, we fix an algebraic closure $\overline{\mathbf{Q}}$ of $\mathbf{Q}$ in $\mathbf{C}$. By an algebraic number field, we understand an algebraic extension of $\mathbf{Q}$ of finite degree contained in $\overline{\mathbf{Q}}$. We denote by ρ the complex conjugation. For an algebraic number field K, J_K denotes the set of all isomorphisms of K into $\mathbf{C}$ and I_K denotes the free abelian group generated by J_K. The ring of integers, the unit group and the class number of K are denoted by $\mathfrak{O}_K$, E_K and by h_K respectively. We denote by $I(K)$ the ideal group of K. For an integral ideal $\mathfrak{f}$ of K, $I_{\mathfrak{f}}(K)$ denotes the ideal group of K modulo $\mathfrak{f}$, i.e. the group of fractional ideals which are relatively prime to $\mathfrak{f}$. We abbreviate $\rho|K$ to ρ if no confusion is likely. For an extension L of K of finite degree, $\mathrm{Res}_{L/K}$ denotes the restriction homomorphism from I_L to I_K; $\mathrm{Inf}_{L/K}$ denotes the inflation homomorphism from I_K to I_L such that, for $\sigma \in J_K$, $\mathrm{Inf}_{L/K}(\sigma)$ is the sum of all elements of J_L whose restrictions to K coincide with σ. The norm (resp. trace) map from L to K is denoted by $N_{L/K}$ (resp. $\mathrm{Tr}_{L/K}$). We abbreviate $N_{K/\mathbf{Q}}$ (resp. $\mathrm{Tr}_{K/\mathbf{Q}}$) to N (resp. Tr). The relative discriminant of L over K is denoted by $D(L/K)$. For an abelian extension L of K of finite degree and for a fractional ideal $\mathfrak{X}$ of K, which is relatively prime to the conductor of L, $(\frac{L/K}{\mathfrak{X}})$ denotes the Artin symbol. We denote by $K_A^\times$ the idele group of K.

For a totally real algebraic number field F of degree n, we denote by $\infty_1, \infty_2, \cdots, \infty_n$ the archimedean primes of F. We identify every ∞_i with an element of J_F and choose ∞_1 so that it corresponds to the identity embedding of F into $\mathbf{C}$. For $a \in F$, $a \gg 0$ means that a is totally positive. We put $E_F^+ = \{\epsilon \in E_F \mid \epsilon \gg 0\}$. By a CM-field, we understand a totally imaginary quadratic extension of a totally real algebraic number field. For a CM-field K, $\Phi \in I_K$ is called a CM-type if $\Phi + \Phi\rho$ is the sum of all elements in J_K. A representation ψ of $\mathrm{Gal}(K/F)$, where K is a CM-field which is a finite Galois extension of a totally real algebraic number field F, is called odd if $\psi(\rho) = -\mathrm{id}$. For a finite group G, a subgroup H of G and a representation ψ of H, the induced representation from ψ is denoted by $\mathrm{Ind}_H^G\psi$. For elements $g_1, g_2, \cdots, g_n$ of a group, $\langle g_1, g_2, \cdots, g_n\rangle$ denotes the subgroup generated by $g_1, g_2, \cdots, g_n$. For a set S, $|S|$ denotes the cardinality of S. For a, $b \in \mathbf{C}$, we write $a \sim b$ if $b \neq 0$ and $a/b \in \overline{\mathbf{Q}}$. We put $\mathbf{R}_+ = \{x \in \mathbf{R} \mid x > 0\}$.

§1. Two theorems of Shimura

In this section, we quote two theorems of Shimura on CM-periods which are crucial for succeeding sections.

Theorem S1 ([S3], Theorem 1.1, [S5], Theorem 32.5). *For every CM-field K, there exists a map $p_K : I_K \times I_K \longrightarrow \mathbf{C}^\times$ with the following properties.*

(1) If Φ is a CM-type of K and $\sigma \in \Phi$, $\int_c \omega_\sigma \sim \pi\, p_K(\sigma, \Phi)$ for every $c \in H_1(A, \mathbf{Z})$, where A is any abelian variety defined over $\overline{\mathbf{Q}}$ of type (K, Φ) and ω_σ is any $\overline{\mathbf{Q}}$-rational holomorphic differential 1-form on A which is multiplied by a^σ under the action of $a \in K \cap End(A)$.

(2) $p_K(\xi_1+\xi_2,\eta) \sim p_K(\xi_1,\eta)p_K(\xi_2,\eta)$, $p_K(\xi,\eta_1+\eta_2) \sim p_K(\xi,\eta_1)p_K(\xi,\eta_2)$ *for every* ξ, ξ_1, ξ_2, η, η_1, $\eta_2 \in I_K$.
(3) $p_K(\xi\rho,\eta) \sim p_K(\xi,\eta\rho) \sim p_K(\xi,\eta)^{-1}$ *for every* ξ, $\eta \in I_K$.
(4) $p_K(\xi,\mathrm{Res}_{L/K}(\zeta)) \sim p_L(\mathrm{Inf}_{L/K}(\xi),\zeta)$ *if* $\xi \in I_K$, $\zeta \in I_L$ *and* $K \subset L$, L *is a CM-field.*
(5) $p_K(\mathrm{Res}_{L/K}(\zeta),\xi) \sim p_L(\zeta,\mathrm{Inf}_{L/K}(\xi))$ *if* $\xi \in I_K$, $\zeta \in I_L$ *and* $K \subset L$, L *is a CM-field.*
(6) $p_{K'}(\gamma\xi,\gamma\eta) \sim p_K(\xi,\eta)$ *if* γ *is an isomorphism of* K' *onto* K.

Let Φ be a CM-type and $\mathfrak{f}$ be an integral ideal of K. Let λ be a Grössencharacter of $I_{\mathfrak{f}}(K)$ such that

$$\lambda((\alpha)) = \prod_{\sigma\in\Phi}(\alpha^{\sigma\rho}/|\alpha^{\sigma}|)^{t_\sigma} \qquad \text{if} \quad \alpha \equiv 1 \mod^{\times}\mathfrak{f},$$

where t_σ, $\sigma \in \Phi$ are non-negative integers. Let $L(s,\lambda)$ denote the L-function attached to λ. We write $L(s,\lambda)$ as $L_K(s,\lambda)$ when we emphasize the dependence on K.

Theorem S2 ([S1], Theorem 2 combined with [S3], Theorem 1.1; or [S5], Theorem 32.12). *For every integer* m *such that* $m - t_\sigma \in 2\mathbf{Z}$ *and* $-t_\sigma < m \le t_\sigma$ *for every* $\sigma \in \Phi$, *we have*

$$L(m/2,\lambda) \sim \pi^{e/2}p_K(\sum_{\sigma\in\Phi} t_\sigma\cdot\sigma,\Phi),$$

where $e = m[F:\mathbf{Q}] + \sum_{\sigma\in\Phi} t_\sigma$.

§2. Review of Shintani's results

In this section, we recall Shintani's formula which expresses the derivative of a partial zeta function of a totally real algebraic number field at $s=0$ in terms of the multiple gamma function. We follow Shintani [Sh3], [Sh5] faithfully.

Let r be a natural number and let $\omega = (\omega_1,\omega_2,\cdots,\omega_r) \in \mathbf{R}_+^r$, $x > 0$. For $s \in \mathbf{C}$, we define the multiple Riemann zeta function by

$$\zeta_r(s,\omega,x) = \sum_{\Omega=m_1\omega_1+m_2\omega_2+\cdots+m_r\omega_r} (x+\Omega)^{-s}.$$

Here $(m_1,m_2,\cdots,m_r)$ extends over all r-tuples of non-negative integers. This series converges when $\Re(s) > r$ and can be continued meromorphically to the whole s-plane; $\zeta_r(s,\omega,x)$ is holomorphic except for simple poles at $s=1, 2, \cdots, r$. We put

$$-\log\rho_r(\omega) = \lim_{x\to+0}\{\frac{\partial}{\partial s}\zeta_r(s,\omega,x)|_{s=0} + \log x\},$$

$$\frac{\partial}{\partial s}\zeta_r(s,\omega,x)|_{s=0} = \log\{\frac{\Gamma_r(x,\omega)}{\rho_r(\omega)}\}.$$

$\Gamma_r(x,\omega)$ is the r-ple gamma function introduced by Barnes ([Ba2]). $\Gamma_r(x,\omega)^{-1}$ can be analytically continued as an entire function to the whole x-plane; $\Gamma_r(x,\omega)$ is holomorphic at x if x is not of the form $x = -(m_1\omega_1+m_2\omega_2+\cdots+m_r\omega_r)$ with non-negative integers

$m_1, m_2, \cdots, m_r$. For i, $1 \le i \le r$, put $\check{\omega}(i) = (\omega_1, \cdots, \omega_{i-1}, \omega_{i+1}, \cdots, \omega_r)$. Then we have the difference equation

$$\log \frac{\Gamma_r(x+\omega_i,\omega)}{\rho_r(\omega)} - \log \frac{\Gamma_r(x,\omega)}{\rho_r(\omega)} = -\log \frac{\Gamma_{r-1}(x,\check{\omega}(i))}{\rho_{r-1}(\check{\omega}(i))}. \tag{2.1}$$

Let $A = (a_{ij})$, $1 \le i \le n$, $1 \le j \le r$ be an $n \times r$-matrix. We assume $a_{ij} > 0$ for all i and j. Let x be a column vector such that ${}^t x = (x_1, x_2, \cdots, x_n)$, $x_i \ge 0$ for all i and that $x \ne 0$. We put

$$A = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{pmatrix}, \qquad Ax = \begin{pmatrix} (Ax)_1 \\ (Ax)_2 \\ \vdots \\ (Ax)_n \end{pmatrix}.$$

For $s \in \mathbf{C}$, we define

$$\zeta(s, A, x) = \sum_{z_1,\cdots,z_r=0}^{\infty} \prod_{i=1}^{n} \{\sum_{j=1}^{r} a_{ij}(z_j + x_j)\}^{-s}. \tag{2.2}$$

This series converges if $\Re(s) > r/n$ and can be continued meromorphically to the whole s-plane; it is holomorphic at $s = 0$. For $l = (l_1, l_2, \cdots, l_r)$, $0 \le l_i \in \mathbf{Z}$, $1 \le j, k \le n$, $j \ne k$, we put

$$C_{l,j,k}(A) = \int_0^1 \{\prod_{m=1}^{r} (a_{jm} + a_{km}u)^{l_m - 1} - \prod_{m=1}^{r} a_{jm}^{l_m - 1}\} \frac{du}{u}. \tag{2.3}$$

$$C_l(A) = \sum_{(j,k)} C_{l,j,k}(A).$$

Here (j, k) extends over all ordered pairs of integers such that $1 \le j, k \le n$, $j \ne k$. The Bernoulli polynomials are defined by $\dfrac{e^{xt}}{e^t - 1} = \sum_{m=0}^{\infty} \dfrac{B_m(x)}{m!} t^{m-1}$. Now Shintani's formulas are

$$\zeta(0, A, x) = \sum_{i=1}^{n} \zeta_r(0, A_i, (Ax)_i)/n = \frac{(-1)^r}{n} \sum_{l} \sum_{i=1}^{n} \prod_{j=1}^{r} \{\frac{a_{ij}^{l_j - 1} B_{l_j}(x_j)}{l_j!}\}. \tag{2.4}$$

$$\frac{\partial}{\partial s} \zeta(s, A, x)|_{s=0} = \log\{\prod_{i=1}^{n} \frac{\Gamma_r((Ax)_i, A_i)}{\rho_r(A_i)}\} + \frac{(-1)^r}{n} \sum_{l} C_l(A) \prod_{j=1}^{r} \{\frac{B_{l_j}(x_j)}{l_j!}\}. \tag{2.5}$$

Here $l = (l_1, l_2, \cdots, l_r)$ extends over all r-tuples of non-negative integers such that $l_1 + l_2 + \cdots + l_r = r$.

Now let F be a totally real algebraic number field of degree n. Let $J_F = \{\sigma_1, \sigma_2, \cdots, \sigma_n\}$. We put $x^{(i)} = x^{\sigma_i}$ for $x \in F$ and embed F into $\mathbf{R}^n$ by

$$F \ni x \longrightarrow (x^{(1)}, x^{(2)}, \cdots, x^{(n)}) \in \mathbf{R}^n.$$

For r linearly independent vectors $v_1, v_2, \cdots, v_r \in \mathbf{R}^n$, put

$$C(v_1, v_2, \cdots, v_r) = \{\sum_{i=1}^{r} t_i v_i \mid t_1, t_2, \cdots, t_r > 0\}$$

and call $C(v_1, v_2, \cdots, v_r)$ *an r-dimensional open simplicial cone with basis* $v_1, v_2, \cdots, v_r$. There exists a finite set J and an $r(j)$-dimensional open simplicial cone C_j with basis from $\mathfrak{O}_F$ for every j such that (cf. [Sh1], Proposition 4)

$$\mathbf{R}^n_+ = \cup_{\epsilon \in E_F^+} \epsilon(\cup_{j \in J} C_j) \qquad \text{(disjoint union)}. \tag{2.6}$$

We put

$$C_j = C(v_{j1}, v_{j2}, \cdots, v_{jr(j)}), \qquad v_{j1}, v_{j2}, \cdots, v_{jr(j)} \in \mathfrak{O}_F$$

and define an $n \times r(j)$-matrix A_j by $A_j = (v_{jm}^{(l)})$ (the (l, m) component of A_j is $v_{jm}^{(l)}$). For $z \in C_j$, we put

$$z = x_1(z)v_{j1} + x_2(z)v_{j2} + \cdots + x_{r(j)}(z)v_{jr(j)}, \quad {}^t x(z) = (x_1(z), x_2(z), \cdots, x_{r(j)}(z)) \in \mathbf{Q}^{r(j)}.$$

For a fractional ideal $\mathfrak{a}$ of F, we put

$$R(C_j, \mathfrak{a}) = \{z \in \mathfrak{a} \cap C_j \mid 0 < x_1(z), x_2(z), \cdots, x_{r(j)}(z) \leq 1\}. \tag{2.7}$$

Let h_0 be the class number of F in the narrow sense and let $\mathfrak{a}_1, \mathfrak{a}_2, \cdots, \mathfrak{a}_{h_0}$ be integral ideals which represent narrow ideal classes. Let $\mathfrak{f}$ be an integral ideal of F and let $C_\mathfrak{f}$ denote the ideal class group modulo $\mathfrak{f}\infty_1\infty_2\cdots\infty_n$. For $c \in C_\mathfrak{f}$, take $\mathfrak{a}_\mu$ so that c and $\mathfrak{a}_\mu\mathfrak{f}$ belong to the same narrow ideal class and put

$$R(C_j, c) = \{z \in R(C_j, (\mathfrak{a}_\mu\mathfrak{f})^{-1}) \mid (z)\mathfrak{a}_\mu\mathfrak{f} = c \ \text{ in } C_\mathfrak{f}\}. \tag{2.8}$$

Then we have

$$\begin{aligned} &\{z \in (\mathfrak{a}_\mu\mathfrak{f})^{-1} \cap C_j, \ (z)\mathfrak{a}_\mu\mathfrak{f} = c \ \text{ in } C_\mathfrak{f}\} \\ &= \cup_{z \in R(C_j, c)} \{z + x_1 v_{j1} + x_2 v_{j2} + \cdots + x_{r(j)} v_{jr(j)} \mid 0 \leq x_1, x_2, \cdots x_{r(j)} \in \mathbf{Z}\}. \end{aligned} \tag{2.9}$$

Let

$$\zeta_F(s, c) = \sum_{\mathfrak{a}, \mathfrak{a} = c \text{ in } C_\mathfrak{f}} N(\mathfrak{a})^{-s}$$

be the partial zeta function of the class c. By (2.9), we have

$$\zeta_F(s, c) = \sum_{j \in J} \sum_{z \in R(C_j, c)} N(\mathfrak{a}_\mu\mathfrak{f})^{-s} \zeta(s, A_j, x(z)). \tag{2.10}$$

From (2.4), (2.5) and (2.10), we obtain the following expressions of $\zeta_F(0,c)$ and $\zeta'_F(0,c)$. For $j \in J$ and $z \in R(C_j,c)$, we put

$$H_j(z) = \sum_l \left[\prod_{m=1}^{r(j)} \{\frac{B_{l_m}(x(z)_m)}{l_m!}\} \mathrm{Tr}_{F/\mathbf{Q}}\{\prod_{m=1}^{r(j)} (v_{jm}^{l_m-1})\} \right], \tag{2.11}$$

$$T_j(z) = \prod_{k=1}^{n} \left\{ \frac{\Gamma_{r(j)}(z^{(k)}, A_j^{(k)})}{\rho_{r(j)}(A_j^{(k)})} \right\}, \tag{2.12}$$

$$S_j(z) = \sum_l C_l(A_j) \prod_{m=1}^{r(j)} \left\{ \frac{B_{l_m}(x(z)_m)}{l_m!} \right\}. \tag{2.13}$$

Here in $\sum_l$, $l = (l_1, l_2, \cdots, l_{r(j)})$ extends over all $r(j)$-tuples of non-negative integers such that $l_1 + l_2 + \cdots + l_{r(j)} = r(j)$; $A_j^{(k)}$ denotes the k-th row of A_j. For $c \in C_{\mathfrak{f}}$, we put

$$H(c) = \sum_{j\in J} \frac{(-1)^{r(j)}}{n} \sum_{z\in R(C_j,c)} H_j(z), \tag{2.14}$$

$$T(c) = \prod_{j\in J} \prod_{z\in R(C_j,c)} T_j(z) \exp\{\sum_{j\in J} \frac{(-1)^{r(j)}}{n} \sum_{z\in R(C_j,c)} S_j(z)\}. \tag{2.15}$$

Then we have

$$\zeta_F(0,c) = H(c), \tag{2.16}$$

$$\zeta'_F(0,c) = \log T(c) - \log N(\mathfrak{a}_\mu \mathfrak{f}) \zeta_F(0,c). \tag{2.17}$$

§3. Reformulation of our previous conjecture

Let us begin by recalling a conjecture in [Y2]. Let L be a CM-field which is normal over $\mathbf{Q}$. Put $G = \mathrm{Gal}(L/\mathbf{Q})$ and let $\rho \in G$ be the complex conjugation. Let $\widehat{G}$ be the set of equivalence classes of all irreducible representations of G and let $\widehat{G}_-$ be the subset of $\widehat{G}$ which consists of the equivalence classes of all irreducible odd representations. For $\eta \in \widehat{G}$, let $L(s,\eta)$ be the Artin L-function attached to η. Conjecture 2.2 of [Y2] states

Conjecture 1. *Let c be a conjugacy class in G. Then*

$$\prod_{\sigma\in c} p_L(id,\sigma) \sim \pi^{-\mu(c)/2} \prod_{\eta\in\hat{G}_-} \exp(\frac{|c|\chi_\eta(c)}{[L:\mathbf{Q}]}\frac{L'(0,\eta)}{L(0,\eta)}),$$

where χ_η is the character of η and

$$\mu(c) = \begin{cases} 1 & \text{if } c=\{1\}, \\ -1 & \text{if } c=\{\rho\}, \\ 0 & \text{if } c\neq\{1\},\{\rho\}. \end{cases}$$

Now let F be a totally real algebraic number field such that $[F:\mathbf{Q}]=n$. Let K be a Galois extension of F of finite degree. We assume that K is a CM-field and let L be the normal closure of K over $\mathbf{Q}$. We put $G=\mathrm{Gal}(L/\mathbf{Q})$, $H=\mathrm{Gal}(L/F)$, $H_0=\mathrm{Gal}(L/K)$. For $\tau\in G$, we put

$$P_\tau = \prod_{x\in H_0\backslash G} p_K(x|K,(\tau x|K)).$$

We note that $P_\tau=\prod_{\sigma\in J_K} p_K(\sigma,\tau\sigma)$ if $\tau\in H$. We are going to compute P_τ assuming Conjecture 1. By Theorem S1, we have

$$\begin{aligned} P_\tau &\sim \prod_{x\in G} p_K(x|K,(\tau x|K))^{1/|H_0|} \sim \prod_{x\in G}\prod_{h\in H_0} p_L(x,h\tau x)^{1/|H_0|} \\ &\sim \prod_{x\in G}\prod_{h\in H_0} p_L(\mathrm{id},x^{-1}h\tau x)^{1/|H_0|} \sim \prod_{h\in H_0}\prod_{\sigma\in C(h\tau)} p_L(\mathrm{id},\sigma)^{|G|/|H_0||C(h\tau)|} \end{aligned}$$

Here for $g\in G$, $C(g)$ denotes the conjugacy class of g in G. Applying Conjecture 1 to this formula, we get

$$P_\tau \sim \prod_{h\in H_0}\left\{\pi^{-\mu(h\tau)|G|/2|C(h\tau)||H_0|}\prod_{\eta\in\hat{G}_-}\exp(\frac{|C(h\tau)|\chi_\eta(h\tau)}{[L:\mathbf{Q}]}\frac{L'(0,\eta)}{L(0,\eta)}\frac{|G|}{|H_0||C(h\tau)|})\right\}.$$

We put $\mu(\tau)=1$ (resp. $\mu(\tau)=-1$) if there exists an $h\in H_0$ such that $h\tau=1$ (resp. $h\tau=\rho$) and $\mu(\tau)=0$ otherwise, using the same letter μ. Then we have

$$P_\tau \sim \pi^{-\mu(\tau)|G|/2|H_0|}\prod_{\eta\in\hat{G}_-}\exp(\frac{1}{|H_0|}\sum_{h\in H_0}\chi_\eta(h\tau)\cdot\frac{L'(0,\eta)}{L(0,\eta)}). \tag{3.1}$$

We have $\sum_{h\in H_0}\chi_\eta(h\tau)=0$ if $\eta|H_0$ does not contain the trivial representation. Let $\{\eta_1,\eta_2,\cdots,\eta_l\}$ be the set of all $\eta\in\widehat{G}_-$ such that $\eta|H_0$ contains the trivial representation. Then we get

$$P_\tau \sim \pi^{-[K:\mathbf{Q}]\mu(\tau)/2}\prod_{i=1}^{l}\exp(\frac{1}{|H_0|}\sum_{h\in H_0}\chi_{\eta_i}(h\tau)\cdot\frac{L'(0,\eta_i)}{L(0,\eta_i)}). \tag{3.2}$$

Take $\omega \in (\widehat{H/H_0})_-$ and regard ω as a representation of H. By the Frobenius reciprocity, we can write $\mathrm{Ind}_H^G\omega \cong \sum_{i=1}^l c_{i,\omega}\eta_i$, $c_{i,\omega} \in \mathbf{Z}$ and $\eta_i|H \cong \sum_{\omega\in(\widehat{H/H_0})_-} c_{i,\omega}\omega \oplus \xi_i$, where ξ_i is a representation of H such that $\xi_i|H_0$ does not contain the trivial representation. Now let us assume that $\tau \in H$. Then we have

$$\sum_{h\in H_0} \chi_{\eta_i}(h\tau) = |H_0| \sum_{\omega\in(\widehat{H/H_0})_-} c_{i,\omega}\chi_\omega(\tau).$$

Since $L(s,\omega) = \prod_{i=1}^l L(s,\eta_i)^{c_{i,\omega}}$, we have

$$\sum_{i=1}^l c_{i,\omega}\frac{L'(0,\eta_i)}{L(0,\eta_i)} = \frac{L'(0,\omega)}{L(0,\omega)}.$$

Hence we get

$$\sum_{i=1}^l \frac{1}{|H_0|}\sum_{h\in H_0}\chi_{\eta_i}(h\tau)\frac{L'(0,\eta_i)}{L(0,\eta_i)} = \sum_{\omega\in(\widehat{H/H_0})_-}\chi_\omega(\tau)\frac{L'(0,\omega)}{L(0,\omega)}.$$

Inserting this formula in (3.2), we get

$$P_\tau \sim \pi^{-[K:\mathbf{Q}]\mu(\tau)/2} \prod_{\omega\in(\widehat{H/H_0})_-} \exp\left(\chi_\omega(\tau)\frac{L'(0,\omega)}{L(0,\omega)}\right). \tag{3.3}$$

Summing up the calculations above, we obtain:

Proposition 1. *Let K be a CM-field which is a Galois extension of F of finite degree. Put $G = Gal(K/F)$ and let $\widehat{G}_-$ be the set of equivalence classes of all irreducible odd representations of G. For $\tau \in G$, let $\mu(\tau) = 1$ (resp. -1) if $\tau = 1$ (resp. $\tau = \rho$) and $\mu(\tau) = 0$ otherwise. If Conjecture 1 holds, then we have*

$$\prod_{\sigma\in J_K} p_K(\sigma,\tau\sigma) \sim \pi^{-[K:\mathbf{Q}]\mu(\tau)/2}\prod_{\omega\in\hat{G}_-}\exp\left(\chi_\omega(\tau)\frac{L'(0,\omega)}{L(0,\omega)}\right) \quad \text{for } \tau\in G. \tag{3.4}$$

Corollary. *Let $J_F = \{\sigma_1,\sigma_2,\cdots,\sigma_n\}$ and use the same letter σ_i for any extension of σ_i to J_K. We assume that K is abelian over F. Then we have*

$$\prod_{i=1}^n p_K(\sigma_i,\tau\sigma_i) \sim \pi^{-n\mu(\tau)/2}\prod_{\omega\in\hat{G}_-}\exp\left(\frac{\omega(\tau)}{|G|}\frac{L'(0,\omega)}{L(0,\omega)}\right). \tag{3.5}$$

We note that $p_K(\sigma_i,\tau\sigma_i) \bmod \overline{\mathbf{Q}}^\times$ in (3.5) does not depend on the choice of the extension σ_i in view of Theorem S1, (6). Clearly (3.4) includes Conjecture 1 as a special case where $F = \mathbf{Q}$. Actually we can prove that Corollary to Proposition 1 implies Conjecture 1 by a similar consideration to the above.

§4. Main conjectures

Throughout this section, let F be a totally real algebraic number field of degree n and K be an abelian extension of F; we assume that K is a CM-field. Let $\tilde{\mathfrak{f}}\infty_1\infty_2\cdots\infty_n$ be the conductor of K as a class field over F. Set $G = \mathrm{Gal}(K/F)$. We begin by writing the right hand side of (3.5) using Shintani's formula (2.17). For an integral ideal $\mathfrak{f}$ of F, $C_{\mathfrak{f}}$ denotes the ideal class group modulo $\mathfrak{f}\infty_1\infty_2\cdots\infty_n$. For $\omega \in \widehat{G}_-$, regard ω as a character of $C_{\tilde{\mathfrak{f}}}$ by the Artin map and let $\mathfrak{f}_\omega\infty_1\infty_2\cdots\infty_n$ be the conductor of ω. For an integral ideal $\mathfrak{f}\,|\,\tilde{\mathfrak{f}}$, we set $(\widehat{G}_-)_{\mathfrak{f}} = \{\omega \in \widehat{G}_- \mid \mathfrak{f}_\omega = \mathfrak{f}\}$ and regard $\omega \in (\widehat{G}_-)_{\mathfrak{f}}$ as a character of $C_{\mathfrak{f}}$. We have

$$\sum_{\omega\in\hat{G}_-}\frac{\omega(\tau)}{L(0,\omega)}L'(0,\omega) = \sum_{\mathfrak{f}|\tilde{\mathfrak{f}}}\sum_{\omega\in(\hat{G}_-)_{\mathfrak{f}}}\frac{\omega(\tau)}{L(0,\omega)}L'(0,\omega)$$
$$= \sum_{\mathfrak{f}|\tilde{\mathfrak{f}}}\sum_{\omega\in(\hat{G}_-)_{\mathfrak{f}}}\frac{\omega(\tau)}{L(0,\omega)}\sum_{c\in C_{\mathfrak{f}}}\omega(c)\zeta'_F(0,c).$$

(It can happen that $(\widehat{G}_-)_{\mathfrak{f}} = \emptyset$ for some $\mathfrak{f}\,|\,\tilde{\mathfrak{f}}$.) Thus (3.5) can be written as

$$\prod_{i=1}^{n} p_K(\sigma_i,\tau\sigma_i) \sim \pi^{-n\mu(\tau)/2}\exp\Big(\frac{1}{|G|}\sum_{\mathfrak{f}|\tilde{\mathfrak{f}}}\sum_{\omega\in(\hat{G}_-)_{\mathfrak{f}}}\frac{\omega(\tau)}{L(0,\omega)}\sum_{c\in C_{\mathfrak{f}}}\omega(c)\zeta'_F(0,c)\Big). \tag{4.1}$$

Writing $\zeta'_F(0,c)$ in full using (2.16) and (2.17), we have

$$\begin{aligned}\zeta'_F(0,c) = {}&\sum_{k=1}^{n}\sum_{j\in J}\sum_{z\in R(C_j,c)}\log\frac{\Gamma_{r(j)}(z^{(k)},A_j^{(k)})}{\rho_{r(j)}(A_j^{(k)})}\\ &+\sum_{j\in J}\frac{(-1)^{r(j)}}{n}\Big[\sum_{l}C_l(A_j)\sum_{z\in R(C_j,c)}\prod_{m=1}^{r(j)}\Big\{\frac{B_{l_m}(x(z)_m)}{l_m!}\Big\}\\ &-\log N(\mathfrak{a}_\mu\mathfrak{f})\sum_{l}\sum_{z\in R(C_j,c)}\prod_{m=1}^{r(j)}\Big\{\frac{B_{l_m}(x(z)_m)}{l_m!}\Big\}\mathrm{Tr}_{F/\mathbf{Q}}\Big\{\prod_{m=1}^{r(j)}(v_{jm}^{l_m-1})\Big\}\Big].\end{aligned} \tag{4.2}$$

We can factorize the right hand side of (4.1) naturally according as the factorization of the left hand side. Note that Shintani's formula (4.2) does not depend on the numbering of $J_F = \{\sigma_1,\sigma_2,\cdots,\sigma_n\}$.

Now we choose $\sigma_1 = \mathrm{id}$. For $c \in C_{\mathfrak{f}}$, put

$$G(c) = \sum_{j\in J}\sum_{z\in R(C_j,c)}\log\frac{\Gamma_{r(j)}(z^{(1)},A_j^{(1)})}{\rho_{r(j)}(A_j^{(1)})}, \tag{4.3}$$

$$V(c)=\sum_{j\in J}(-1)^{r(j)}\sum_{l}\Big[\frac{1}{n}\sum_{k=2}^{n}(C_{l,1,k}(A_j)+C_{l,k,1}(A_j))-\frac{1}{n^2}\sum_{1\le i,k\le n,i\ne k}C_{l,i,k}(A_j)\Big]$$
$$\times\sum_{z\in R(C_j,c)}\prod_{m=1}^{r(j)}\frac{B_{l_m}(x(z)_m)}{l_m!}, \tag{4.4}$$

$$W(c)=-\frac{1}{n}\log N(\mathfrak{a}_\mu\mathfrak{f})\sum_{j\in J}(-1)^{r(j)}\sum_{l}\sum_{z\in R(C_j,c)}\prod_{m=1}^{r(j)}\frac{B_{l_m}(x(z)_m)}{l_m!}(v_{jm}^{(1)})^{l_m-1}, \tag{4.5}$$

$$X(c)=G(c)+V(c)+W(c). \tag{4.6}$$

For $\tau\in G$, we put

$$g_K(\mathrm{id},\tau)=\pi^{-\mu(\tau)/2}\exp\Big(\frac{1}{|G|}\sum_{\mathfrak{f}|\tilde{\mathfrak{f}}}\sum_{\omega\in(\hat{G}_-)_{\mathfrak{f}}}\frac{\omega(\tau)}{L(0,\omega)}\sum_{c\in C_{\mathfrak{f}}}\omega(c)X(c)\Big). \tag{4.7}$$

In (4.2), (4.4) and (4.5), $\sum_l$ is taken over all $r(j)$-tuples of non-negative integers $l=(l_1,l_2,\cdots,l_{r(j)})$ such that $l_1+l_2+\cdots+l_{r(j)}=r(j)$. Hereafter $\sum_l$ will always have this meaning.

Conjecture A. *For $\tau\in G$, we have $p_K(id,\tau)\sim g_K(id,\tau)$.*

Roughly speaking, Conjecture A expresses $p_K(\mathrm{id},\tau)$ in terms of the multiple gamma function. Admitting Conjecture A, let us show that we can always express $p_K(\sigma,\tau)$ mod $\overline{\mathbf{Q}}^\times$, $\sigma,\tau\in I_K$ by the multiple gamma function. Let L be the normal closure of K over $\mathbf{Q}$. We may assume that $\sigma,\tau\in J_K$. Take $\tilde{\sigma}\in J_L$ so that $\tilde{\sigma}|K=\sigma$. By Theorem S1, we have $p_K(\sigma,\tau)\sim p_L(\tilde{\sigma},\mathrm{Inf}_{L/K}(\tau))\sim p_L(\mathrm{id},\tilde{\sigma}^{-1}\mathrm{Inf}_{L/K}(\tau))$. Hence it suffices to consider $p_L(\mathrm{id},\alpha)$ for $\alpha\in J_L$. We may regard α as an element of $\mathrm{Gal}(L/\mathbf{Q})$. Let F_α be the fixed field of $\langle\alpha,\rho\rangle$. Then F_α is totally real and L is abelian over F_α. Now Conjecture A expresses $p_L(\mathrm{id},\alpha)$ mod $\overline{\mathbf{Q}}^\times$ by the multiple gamma function.

We are going to formulate Conjecture A in covariant forms under the action of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ in a similar way as Conjecture 5.1 of [Y2]. Let K be as above and let $\mathfrak{q}$ be an integral ideal of K. For $1\le i\le q$, let Φ_i be a CM-type of K and λ_i be a character of $I_{\mathfrak{q}}(K)$ such that

$$\lambda_i((\alpha))=\prod_{\sigma\in\Phi_i}(\alpha^{\sigma\rho}/|\alpha^\sigma|)^{t_\sigma^{(i)}}\qquad\text{if}\quad\alpha\equiv1\mod^\times\mathfrak{q}.$$

Here $t_\sigma^{(i)}$ are positive integers. We assume that there exists an integer m which satisfies

$$m-t_\sigma^{(i)}\in2\mathbf{Z}\quad\text{and}\quad-t_\sigma^{(i)}<m\le t_\sigma^{(i)}$$

for every i and every $\sigma \in \Phi_i$. By Theorem S2, we have

$$(4.8) \qquad L(m/2, \lambda_i) \sim \pi^{e_i/2} p_K(\sum_{\sigma\in\Phi_i} t_\sigma^{(i)} \cdot \sigma,\, \Phi_i), \quad e_i = m[F:\mathbf{Q}] + \sum_{\sigma\in\Phi_i} t_\sigma^{(i)}.$$

Let L be the normal closure of K over $\mathbf{Q}$ and S be a CM-type of L. Take $\Phi_i^0 \in I_L$ so that $\mathrm{Res}_{L/K}(\Phi_i^0) = \Phi_i$ and put $\eta_i = \mathrm{Inf}_{L/K}(\sum_{\sigma\in\Phi_i} t_\sigma^{(i)} \cdot \sigma)$, $\eta_i^{-1}\Phi_i^0 = \sum_{\gamma\in\eta_i, \delta\in\Phi_i^0} \gamma^{-1}\delta = \sum_{\sigma\in S} l_\sigma^{(i)}\sigma + m_\sigma^{(i)}\sigma\rho$. Then for integers $\epsilon_1, \epsilon_2, \cdots, \epsilon_q$, we have

$$\prod_{i=1}^{q} p_K(\sum_{\sigma\in\Phi_i} t_\sigma^{(i)} \cdot \sigma,\, \Phi_i)^{\epsilon_i} \sim p_L(\mathrm{id}, \sum_{\sigma\in S} n_\sigma \cdot \sigma)$$

with $n_\sigma = \sum_{i=1}^{q} \epsilon_i (l_\sigma^{(i)} - m_\sigma^{(i)})$. For $\tau \in G$, put $\mathrm{Inf}_{L/K}(\tau) = \sum_{\sigma\in S} l'_\sigma\sigma + m'_\sigma\sigma\rho$, $n'_\sigma = l'_\sigma - m'_\sigma$. Then we have $p_K(\mathrm{id}, \tau) \sim p_L(\mathrm{id}, \sum_{\sigma\in S} n'_\sigma\sigma)$. We assume that there exists an integer e such that

$$(4.9) \qquad \sum_{\sigma\in S} n_\sigma\sigma = e \sum_{\sigma\in S} n'_\sigma\sigma.$$

Then we have

$$\prod_{i=1}^{q} L(m/2, \lambda_i)^{\epsilon_i} \sim \pi^A p_K(\mathrm{id}, \tau)^e, \qquad A = \sum_{i=1}^{q} \frac{\epsilon_i e_i}{2}$$

if the left hand side is meaningful.

For $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ and $C_j = C(v_{j1}, v_{j2}, \cdots, v_{jr(j)})$, put $C_j^\sigma = C(v_{j1}^\sigma, v_{j2}^\sigma, \cdots, v_{jr(j)}^\sigma)$. (To fix our idea, we define C_j^σ including the ordering of the basis though the ordering has no effects on our definition as will be shown in §5.) Then we have the decomposition

$$\mathbf{R}_+^n = \cup_{\epsilon\in E_{F^\sigma}^+} \epsilon(\cup_{j\in J} C_j^\sigma) \qquad \text{(disjoint union)}.$$

A natural way to define $g_{K^\sigma}(\mathrm{id}, \sigma^{-1}\tau\sigma)$ for $\tau \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ is to use the data $\{C_j^\sigma\}_{j\in J}$ and $\mathfrak{a}_\mu^\sigma$, $1 \le \mu \le h_0$. We then say that $\{g_{K^\sigma}(\mathrm{id}, \sigma^{-1}\tau\sigma)\}_{\sigma\in\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})}$ *is covariantly defined.* When $\{C_j\}_{j\in J}$ and $\mathfrak{a}_\mu$ are given, $g_{K^\sigma}(\mathrm{id}, \sigma^{-1}\tau\sigma)$ depends only on $\sigma|F$.

Conjecture B. *Assume that $L(m/2, \lambda_i)^{\epsilon_i} \neq 0$ if $\epsilon_i < 0$ and that $\{g_{K^\sigma}(id, \sigma^{-1}\tau\sigma)\}_{\sigma\in\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})}$ is covariantly defined. Then for every $\sigma \in Gal(\overline{\mathbf{Q}}/\mathbf{Q})$, there exists a root of unity ζ such that*

$$\left(\frac{\prod_{i=1}^{q} L(m/2, \lambda_i)^{\epsilon_i}}{\pi^A g_K(id, \tau)^e}\right)^\sigma = \zeta \cdot \frac{\prod_{i=1}^{q} L(m/2, \lambda_i^\sigma)^{\epsilon_i}}{\pi^A g_{K^\sigma}(id, \sigma^{-1}\tau\sigma)^e}.$$

Here $\lambda_i^\sigma(\mathfrak{a}) = (\lambda_i(\mathfrak{a}))^\sigma$ if all $t_\sigma^{(i)}$ are even. For the general case, see [Y2], §3. In Conjecture B, ζ depends on σ and the determination of ζ can well be called a reciprocity law. A special case of this problem is discussed in [Y2], §6, §7.

By Theorems S1 and S2, we can easily derive the relation[2]

$$\prod_{i=1}^{q} L(m/2, \lambda_i^{\sigma})^{\epsilon_i} \sim \pi^A p_{K^\sigma}(\mathrm{id}, \sigma^{-1}\tau\sigma)^e$$

from the assumption (4.9). Hence Conjecture B is not only consistent with Conjecture A but also refines it.

§5. On consistency of the main conjectures

In the formulation of Conjectures A and B, $g_K(\mathrm{id}, \tau)$ depends on the choice of a cone decomposition $\{C_j\}_{j\in J}$ and the representatives $\{\mathfrak{a}_\mu\}$ of the narrow ideal classes. In this section, we shall show that if Conjecture A (resp. B) holds for one choice of these data, then it holds for any other choices.

Let us first show that if $\{g_{K^\sigma}(\mathrm{id}, \sigma^{-1}\tau\sigma)\}_{\sigma\in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})}$ is covariantly defined, then

$$\prod_{i=1}^{n} g_{K^{\sigma_i}}(\mathrm{id}, \sigma_i^{-1}\tau\sigma_i) = \pi^{-n\mu(\tau)/2} \exp\Big(\frac{1}{|G|}\sum_{\mathfrak{f}|\tilde{\mathfrak{f}}}\sum_{\omega\in(\hat{G}_-)_{\mathfrak{f}}} \frac{\omega(\tau)}{L(0,\omega)}\sum_{c\in C_{\mathfrak{f}}}\omega(c)\zeta_F'(0,c)\Big). \tag{5.1}$$

We put $G_i = \mathrm{Gal}(K^{\sigma_i}/F^{\sigma_i})$. Then we have $G_i = \sigma_i^{-1}G\sigma_i$. For $\omega\in\widehat{G}_-$, define $\omega^{(i)}\in(\widehat{G}_i)_-$ by $\omega^{(i)}(x) = \omega(\sigma_i x \sigma_i^{-1})$, $x\in G_i$. We have

$$\omega^{(i)}(c^{\sigma_i}) = \omega(c), \quad c\in C_{\mathfrak{f}}, \qquad L(s,\omega^{(i)}) = L(s,\omega).$$

By definition, we have

$$\begin{aligned} g_{K^{\sigma_i}}(\mathrm{id}, \sigma_i^{-1}\tau\sigma_i) &= \pi^{-\mu(\sigma_i^{-1}\tau\sigma_i)/2}\exp\Big(\frac{1}{|G_i|}\sum_{\mathfrak{f}^{\sigma_i}|\tilde{\mathfrak{f}}^{\sigma_i}}\sum_{\omega\in(\hat{G}_{i,-})_{\mathfrak{f}^{\sigma_i}}}\frac{\omega(\sigma_i^{-1}\tau\sigma_i)}{L(0,\omega)}\sum_{c\in C_{\mathfrak{f}^{\sigma_i}}}\omega(c)X(c)\Big) \\ &= \pi^{-\mu(\tau)/2}\exp\Big(\frac{1}{|G|}\sum_{\mathfrak{f}|\tilde{\mathfrak{f}}}\sum_{\omega\in(\hat{G}_-)_{\mathfrak{f}}}\frac{\omega(\tau)}{L(0,\omega)}\sum_{c\in C_{\mathfrak{f}^{\sigma_i}}}\omega(c^{\sigma_i^{-1}})X(c)\Big). \end{aligned}$$

Hence

$$g_{K^{\sigma_i}}(\mathrm{id}, \sigma_i^{-1}\tau\sigma_i) = \pi^{-\mu(\tau)/2}\exp\Big(\frac{1}{|G|}\sum_{\mathfrak{f}|\tilde{\mathfrak{f}}}\sum_{\omega\in(\hat{G}_-)_{\mathfrak{f}}}\frac{\omega(\tau)}{L(0,\omega)}\sum_{c\in C_{\mathfrak{f}}}\omega(c)X(c^{\sigma_i})\Big) \tag{5.2}$$

and it suffices to show

$$\sum_{i=1}^{n} X(c^{\sigma_i}) = \zeta_F'(0,c). \tag{5.3}$$

[2] $L(m/2,\lambda_i)\neq 0$ implies $L(m/2,\lambda_i^{\sigma})\neq 0$. (cf. [S4], Corollary 6.2.)

Using (2.16), we have

$$\sum_{i=1}^{n} W(c^{\sigma_i}) = -\log N(\mathfrak{a}_\mu \mathfrak{f})\zeta_F(0,c).$$

Hence (5.3) is reduced to

$$\sum_{i=1}^{n} V(c^{\sigma_i}) = \sum_{j\in J} \frac{(-1)^{r(j)}}{n} \sum_{l} C_l(A_j) \sum_{z\in R(C_j,c)} \prod_{m=1}^{r(j)} \left\{ \frac{B_{l_m}(x(z)_m)}{l_m!} \right\}. \tag{5.4}$$

Let $A_{j,i}$ be the $n \times r(j)$-matrix determined by the cone $C_j^{\sigma_i}$. Fix $j \in J$ and $l = (l_1, l_2, \cdots, l_{r(j)})$. Then (5.4) follows from

$$\sum_{i=1}^{n}(\sum_{k=2}^{n}(C_{l,1,k}(A_{j,i}) + C_{l,k,1}(A_{j,i})) - \frac{1}{n}\sum_{1\le p,k\le n, p\ne k} C_{l,p,k}(A_{j,i})) = C_l(A_j). \tag{5.5}$$

We easily have

$$\frac{1}{n}\sum_{i=1}^{n}\sum_{1\le p,k\le n,p\ne k} C_{l,p,k}(A_{j,i}) = C_l(A_j),$$

$$\sum_{i=1}^{n}\sum_{k=2}^{n} C_{l,1,k}(A_{j,i}) + C_{l,k,1}(A_{j,i}) = \sum_{i=1}^{n}(\sum_{1\le k\le n,k\ne i}(C_{l,k,i}(A_j) + C_{l,i,k}(A_j))) = 2C_l(A_j).$$

Hence (5.1) is established.

Now let us consider the dependence of $g_K(\mathrm{id}, \tau)$ on $\{C_j\}_{j\in J}$ and $\{\mathfrak{a}_\mu\}$. Let $\{C'_l\}_{l\in L}$ be a finite family of open simplicial cones with basis in $\mathfrak{O}_F$. We call $\{C'_l\}_{l\in L}$ *a refinement* of $\{C_j\}_{j\in J}$ if every C'_l is contained in some C_j and if

$$\mathbf{R}_+^n = \cup_{\epsilon\in E_F^+}\epsilon(\cup_{l\in L}C'_l) \qquad \text{(disjoint union)}.$$

In this case, we have $\cup_{j\in J}C_j = \cup_{l\in L}C'_l$.

Lemma 2. *Let $\{C_j\}_{j\in J}$ and $\{D_i\}_{i\in I}$ be two finite families of open simplicial cones with basis in $\mathfrak{O}_F$ such that*

$$\mathbf{R}_+^n = \cup_{\epsilon\in E_F^+}\epsilon(\cup_{j\in J}C_j) \quad \textit{(disjoint union)}, \qquad \mathbf{R}_+^n = \cup_{\epsilon\in E_F^+}\epsilon(\cup_{i\in I}D_i) \quad \textit{(disjoint union)}.$$

Then there exists a finite family $\{C'_l\}_{l\in L}$ of open simplicial cones with basis in $\mathfrak{O}_F$ and a family of units $\{\epsilon_l\}_{l\in L}$, $\epsilon_l \in E_F^+$ such that $\{C'_l\}_{l\in L}$ is a refinement of $\{C_j\}_{j\in J}$ and that $\{\epsilon_l C'_l\}_{l\in L}$ is a refinement of $\{D_i\}_{i\in I}$.

Proof. We have

$$C_j = \cup_{i\in I}\cup_{\epsilon\in E_F^+, C_j\cap\epsilon D_i\ne\emptyset}(C_j\cap\epsilon D_i) \qquad \text{(disjoint union)}, \tag{5.6}$$

(5.7) $$D_i = \cup_{j\in J} \cup_{\epsilon\in E_F^+, D_i\cap\epsilon^{-1}C_j\neq\emptyset} (D_i \cap \epsilon^{-1}C_j) \quad \text{(disjoint union)}.$$

Fix $j \in J$ and $i \in I$. Let us show that there are only finitely many $\epsilon \in E_F^+$ which satisfy $C_j \cap \epsilon D_i \neq \emptyset$. Set $H = \{(x_1, x_2, \cdots, x_n) \in \mathbf{R}_+^n \mid x_1 x_2 \cdots x_n = 1\}$. Then E_F^+ is a discrete subgroup of H. For an open simplicial cone C, put $C^1 = C \cap H$. Clearly $C_j \cap \epsilon D_i \neq \emptyset$ if and only if $C_j^1 \cap \epsilon D_i^1 \neq \emptyset$. It suffices to show that the closures $\overline{C_j^1}$ and $\overline{D_i^1}$ are compact. Let $C = C(v_1, v_2, \cdots, v_r)$ be an r-dimensional open simplicial cone with $v_k = (v_{k1}, v_{k2}, \cdots, v_{kn}) \in \mathbf{R}_+^n$, $1 \leq k \leq r$. If $x \in \overline{C^1}$, then we see immediately that x is of the form $x = \sum_{k=1}^r t_k v_k$, $0 \leq t_k \leq 1/\sqrt[n]{v_{k1}v_{k2}\cdots v_{kn}}$. Hence $\overline{C^1}$ is compact.

By Shintani [Sh1], Corollary to Lemma 2, $C_j \cap \epsilon D_i$ can be written as a disjoint union of finitely many open simplicial cones with basis in $\mathfrak{O}_F$. Now the assertion follows from (5.6) and (5.7).

By Lemma 2, to see the dependence of $g_K(\mathrm{id}, \tau)$ on $\{C_j\}_{j\in J}$, it suffices to consider the following two operations.
(1) Replace $\{C_j\}_{j\in J}$ by a refinement $\{C_l'\}_{l\in L}$.
(2) Replace one member C_j of $\{C_j\}_{j\in J}$ by $\epsilon_j C_j$ with $\epsilon_j \in E_F^+$.

Now fix $c \in C_{\mathfrak{f}}$ and take $\mathfrak{a}_\mu$ so that the narrow ideal class of c is the same as that of $\mathfrak{a}_\mu \mathfrak{f}$. For $j \in J$ and for $1 \leq i \leq n$, put

$$\zeta^{(i)}(s, C_j, \mathfrak{a}_\mu) = \sum_{z\in(\mathfrak{a}_\mu\mathfrak{f})^{-1}\cap C_j, (z)\mathfrak{a}_\mu\mathfrak{f}\equiv c} (z^{(i)})^{-s}.$$

Here, since $z^{(i)} > 0$, $(z^{(i)})^{-s}$ is naturally defined by $\exp(-s\log z^{(i)})$ taking the principal branch of $\log z^{(i)}$ and $\equiv$ means the equality in $C_{\mathfrak{f}}$. By (2.9), we have

$$\zeta^{(i)}(s, C_j, \mathfrak{a}_\mu) = \sum_{z\in R(C_j, c)} \zeta(s, A_j^{(i)}, x(z^{(i)})).$$

By (2.4) and (2.5), we obtain

(5.8) $$\zeta^{(i)}(0, C_j, \mathfrak{a}_\mu) = (-1)^{r(j)} \sum_{z\in R(C_j,c)} \sum_l \prod_{m=1}^{r(j)} \frac{B_{l_m}(x(z)_m)}{l_m!} (v_{jm}^{(i)})^{l_m-1},$$

(5.9) $$\frac{d}{ds}\zeta^{(i)}(s, C_j, \mathfrak{a}_\mu)|_{s=0} = \sum_{z\in R(C_j,c)} \log \frac{\Gamma_{r(j)}(z^{(i)}, A_j^{(i)})}{\rho_{r(j)}(A_j^{(i)})}.$$

For $1 \leq i, k \leq n$, $i \neq k$, we put

$$\zeta^{(i,k)}(s, C_j, \mathfrak{a}_\mu) = \sum_{z\in(\mathfrak{a}_\mu\mathfrak{f})^{-1}\cap C_j, (z)\mathfrak{a}_\mu\mathfrak{f}\equiv c} (z^{(i)}z^{(k)})^{-s}.$$

Using (2.9), (2.4) and (2.5) similarly to the above, we obtain

$$(5.10)\quad \zeta^{(i,k)}(0,C_j,\mathfrak{a}_\mu)=\frac{(-1)^{r(j)}}{2}\sum_{z\in R(C_j,c)}\sum_{l}\prod_{m=1}^{r(j)}\frac{B_{l_m}(x(z)_m)}{l_m!}\{(v_{jm}^{(i)})^{l_m-1}+(v_{jm}^{(k)})^{l_m-1}\},$$

$$(5.11)\quad \begin{aligned}\frac{d}{ds}\zeta^{(i,k)}(s,C_j,\mathfrak{a}_\mu)|_{s=0}=&\sum_{z\in R(C_j,c)}\Bigg[\log\frac{\Gamma_{r(j)}(z^{(i)},A_j^{(i)})}{\rho_{r(j)}(A_j^{(i)})}+\log\frac{\Gamma_{r(j)}(z^{(k)},A_j^{(k)})}{\rho_{r(j)}(A_j^{(k)})}\\&+\frac{(-1)^{r(j)}}{2}\sum_{l}(C_{l,i,k}(A_j)+C_{l,k,i}(A_j))\prod_{m=1}^{r(j)}\frac{B_{l_m}(x(z)_m)}{l_m!}\Bigg].\end{aligned}$$

Therefore we have

$$(5.12)\qquad G(c)=\sum_{j\in J}\frac{d}{ds}\zeta^{(1)}(s,C_j,\mathfrak{a}_\mu)|_{s=0},$$

(5.13)

$$\begin{aligned}V(c)=&\sum_{j\in J}\Big(\frac{2}{n}\sum_{k=2}^{n}\Big(\frac{d}{ds}\zeta^{(1,k)}(s,C_j,\mathfrak{a}_\mu)|_{s=0}-\frac{d}{ds}\zeta^{(1)}(s,C_j,\mathfrak{a}_\mu)|_{s=0}-\frac{d}{ds}\zeta^{(k)}(s,C_j,\mathfrak{a}_\mu)|_{s=0}\Big)\\&-\frac{1}{n^2}\sum_{1\le i,k\le n,i\ne k}\Big(\frac{d}{ds}\zeta^{(i,k)}(s,C_j,\mathfrak{a}_\mu)|_{s=0}-\frac{d}{ds}\zeta^{(i)}(s,C_j,\mathfrak{a}_\mu)|_{s=0}-\frac{d}{ds}\zeta^{(k)}(s,C_j,\mathfrak{a}_\mu)|_{s=0}\Big)\Big),\end{aligned}$$

$$(5.14)\qquad W(c)=-\frac{1}{n}\log N(\mathfrak{a}_\mu\mathfrak{f})\sum_{j\in J}\zeta^{(1)}(0,C_j,\mathfrak{a}_\mu).$$

By (5.12) $\sim$ (5.14), it is evident that $g_K(\mathrm{id},\tau)$ does not change when we replace $\{C_j\}_{j\in J}$ by its refinement. Also when we change the ordered choice of the basis of C_j, $g_K(\mathrm{id},\tau)$ does not change.

Now assume that C_j is replaced by ϵC_j with $\epsilon\in E_F^+$. We have

$$\zeta^{(i)}(s,\epsilon C_j,\mathfrak{a}_\mu)=(\epsilon^{(i)})^{-s}\zeta^{(i)}(s,C_j,\mathfrak{a}_\mu),$$

$$\zeta^{(i,k)}(s,\epsilon C_j,\mathfrak{a}_\mu)=(\epsilon^{(i)}\epsilon^{(k)})^{-s}\zeta^{(i,k)}(s,C_j,\mathfrak{a}_\mu).$$

Note that we have $\zeta^{(i,k)}(0,C_j,\mathfrak{a}_\mu)=\frac{1}{2}(\zeta^{(i)}(0,C_j,\mathfrak{a}_\mu)+\zeta^{(k)}(0,C_j,\mathfrak{a}_\mu))$ by (5.8) and (5.10). By a direct computation using (5.12),(5.13) and (5.14), we see that $G(c)+V(c)$ is changed to

$$G(c)+V(c)-\frac{1}{n}\log\epsilon^{(1)}\sum_{k=1}^{n}\zeta^{(k)}(0,C_j,\mathfrak{a}_\mu)$$

and that $W(c)$ does not change. Therefore $g_K(\mathrm{id},\tau)$ is multiplied by $(\epsilon^{(1)})^{-a}$ where

$$(5.15)\qquad \begin{aligned} a &= \frac{1}{|G|}\sum_{\mathfrak{f}|\tilde{\mathfrak{f}}}\sum_{\omega\in(\hat{G}_-)_{\mathfrak{f}}}\frac{\omega(\tau)}{L(0,\omega)}\sum_{c\in C_{\mathfrak{f}}}\omega(c)\zeta(0,C_j,\mathfrak{a}_\mu),\\ \zeta(s,C_j,\mathfrak{a}_\mu) &= \sum_{z\in(\mathfrak{a}_\mu\mathfrak{f})^{-1}\cap C_j,(z)\mathfrak{a}_\mu\mathfrak{f}\equiv c} N(z)^{-s} = \sum_{z\in R(C_j,c)}\zeta(s,A_j,x(z)).\end{aligned}$$

Since $\zeta(0,C_j,\mathfrak{a}_\mu)\in\mathbf{Q}$, we see easily that $a\in\mathbf{Q}$; we find similarly that $g_{K^{\sigma_i}}(\mathrm{id},\sigma_i^{-1}\tau\sigma_i)$ is multiplied by $(\epsilon^{(i)})^{-a}$. Hence if Conjecture B holds for $\{C_j\}_{j\in J}$, it also holds when we replace a member C_j by ϵC_j.

Next suppose that $\mathfrak{a}_\mu$ is replaced by $(\alpha_\mu)\mathfrak{a}_\mu$ where α_μ is a totally positive element of F. Since the consistency of the main conjectures with the change of a cone decomposition is already established, it suffices to consider the simultaneous change[3]

$$\mathfrak{a}_\mu \longrightarrow (\alpha_\mu)\mathfrak{a}_\mu,\qquad \{C_j\}_{j\in J}\longrightarrow\{(\alpha_\mu)^{-1}C_j\}_{j\in J}.$$

We have

$$\begin{aligned}\zeta^{(i)}(s,\alpha_\mu^{-1}C_j,(\alpha_\mu)\mathfrak{a}_\mu) &= \sum_{z\in((\alpha_\mu)\mathfrak{a}_\mu\mathfrak{f})^{-1}\cap\alpha_\mu^{-1}C_j,(z)(\alpha_\mu)\mathfrak{a}_\mu\mathfrak{f}\equiv c}(z^{(i)})^{-s}\\ &= \sum_{z\in(\mathfrak{a}_\mu\mathfrak{f})^{-1}\cap C_j,(z)\mathfrak{a}_\mu\mathfrak{f}\equiv c}((\alpha_\mu^{(i)})^{-1}z^{(i)})^{-s} = (\alpha_\mu^{(i)})^s\zeta^{(i)}(s,C_j,\mathfrak{a}_\mu).\end{aligned}$$

Similarly we obtain

$$\zeta^{(i,k)}(s,\alpha_\mu^{-1}C_j,(\alpha_\mu)\mathfrak{a}_\mu) = (\alpha_\mu^{(i)}\alpha_\mu^{(k)})^s\zeta^{(i,k)}(s,C_j,\mathfrak{a}_\mu).$$

By (5.12), we see that $G(c)$ changes to

$$G(c)+\sum_{j\in J}\log\alpha_\mu^{(1)}\zeta^{(1)}(0,C_j,\mathfrak{a}_\mu).$$

By (5.13), we see that $V(c)$ changes to

$$\begin{aligned} V(c)-\sum_{j\in J}\Bigg[&\log\alpha_\mu^{(1)}\zeta^{(1)}(0,C_j,\mathfrak{a}_\mu)-\frac{1}{n}\log\alpha_\mu^{(1)}\sum_{k=1}^{n}\zeta^{(k)}(0,C_j,\mathfrak{a}_\mu)\\ &-\frac{1}{n}\log N(\alpha_\mu)\zeta^{(1)}(0,C_j,\mathfrak{a}_\mu)+\frac{1}{n^2}\log N(\alpha_\mu)\sum_{k=1}^{n}\zeta^{(k)}(0,C_j,\mathfrak{a}_\mu)\Bigg].\end{aligned}$$

By (5.14), we see that $W(c)$ changes to

$$W(c)-\frac{1}{n}\sum_{j\in J}\log N(\alpha_\mu)\zeta^{(1)}(0,C_j,\mathfrak{a}_\mu).$$

[3] We may assume that $(\alpha_\mu)^{-1}C_j$ has a basis in $\mathfrak{O}_F$.

Hence $X(c)$ changes to

$$X(c)+\log\alpha_\mu^{(1)}\zeta_F(0,c)-\frac{1}{n}\log N(\alpha_\mu)\zeta_F(0,c),$$

since $\zeta_F(0,c)=\frac{1}{n}\sum_{j\in J}\sum_{k=1}^{n}\zeta^{(k)}(0,C_j,\mathfrak{a}_\mu)$. Therefore $g_K(\mathrm{id},\tau)$ is multiplied by $(\alpha_\mu^{(1)}N(\alpha_\mu)^{-1/n})^b$ where

(5.16)
$$\begin{aligned}b&=\frac{1}{|G|}\sum_{\mathfrak{f}|\tilde{\mathfrak{f}}}\sum_{\omega\in(\hat{G}_-)_\mathfrak{f}}\frac{\omega(\tau)}{L(0,\omega)}\sum_{c\in C_\mathfrak{f}}\omega(c)\zeta_F(0,c)\\&=\frac{1}{|G|}\sum_{\omega\in\hat{G}_-}\omega(\tau)=\begin{cases}1/2 & \text{if}\quad \tau=\mathrm{id},\\ -1/2 & \text{if}\quad \tau=\rho,\\ 0 & \text{if}\quad \tau\neq\mathrm{id},\rho,\end{cases}\end{aligned}$$

which is the sum of a given by (5.15) over all $j\in J$. In other words, $b=\mu(\tau)/2$. We find similarly that $g_{K^{\sigma_i}}(\mathrm{id},\sigma_i^{-1}\tau\sigma_i)$ is multiplied by $(\alpha_\mu^{(i)}N(\alpha_\mu)^{-1/n})^b$. Therefore Conjecture B holds also for the new data $\{\alpha_\mu^{-1}C_j\}_{j\in J}$ and $(\alpha_\mu)\mathfrak{a}_\mu$.

Let us call the map $\sigma\longrightarrow\zeta=\zeta(\sigma)$ of Conjecture B *a reciprocity map.* We deduce from the above considerations the following:

Proposition 3.

(1) *The reciprocity map does not depend on the ordered choice of the basis of C_j and does not change when we replace $\{C_j\}_{j\in J}$ by its refinement.*

(2) *When we change a member C_j of $\{C_j\}_{j\in J}$ by ϵC_j with $\epsilon\in E_F^+$, $\zeta(\sigma)$ changes to $\{\frac{(\epsilon^a)^\sigma}{(\epsilon^\sigma)^a}\}^e\zeta(\sigma)$, where $a\in\mathbf{Q}$ is given by (5.15).*

(3) *When we change an ideal $\mathfrak{a}_\mu$ to $(\alpha_\mu)\mathfrak{a}_\mu$ and $\{C_j\}_{j\in J}$ to $\{(\alpha_\mu)^{-1}C_j\}_{j\in J}$ with a totally positive element α_μ of F, $\zeta(\sigma)$ changes to $\{\frac{(\alpha_\mu^\sigma)^b(N(\alpha_\mu)^{b/n})^\sigma}{(\alpha_\mu^b)^\sigma N(\alpha_\mu)^{b/n}}\}^e\zeta(\sigma)$, where $b\in\mathbf{Q}$ is given by (5.16).*

We also have to consider the dependence on the choice of m and λ_i in Conjecture B. The consistency with changes of m and λ_i follows easily from considerations in [Y2], §7, using Deligne's conjecture proved by Blasius [Bl].

§6. Numerical examples I: Quadratic extensions of real quadratic fields

Let F be a real quadratic field. We are going to write Conjecture B more explicitly. For an element or an ideal z of F, z' denotes its conjugate. Let $\epsilon>1$ be the generator of E_F^+. We may take $C_1=C(1)$, $C_2=C(1,\epsilon)$. Let $\mathfrak{f}$ be an integral ideal of F and $c\in C_\mathfrak{f}$. Following Shintani [Sh2], we put

$$R(\epsilon,c)=\{z=x+y\epsilon\in(\mathfrak{a}_\mu\mathfrak{f})^{-1}\mid x,y\in\mathbf{Q},0<x\leq 1,0\leq y<1,(z)\mathfrak{a}_\mu\mathfrak{f}=c\ \text{ in } C_\mathfrak{f}\},$$

where $\mathfrak{a}_\mu$ is taken as before. We have

$$\sum_{z\in R(C_1,c)} \log\frac{\Gamma_1(z,1)}{\rho_1(1)} = \sum_{z=x+y\epsilon\in R(\epsilon,c),y=0} \log\frac{\Gamma_1(z,1)}{\rho_1(1)},$$

$$\sum_{z\in R(C_2,c)} \log\frac{\Gamma_2(z,(1,\epsilon))}{\rho_2((1,\epsilon))} = \sum_{z=x+y\epsilon\in R(\epsilon,c),0<x\le 1,0<y<1} \log\frac{\Gamma_2(z,(1,\epsilon))}{\rho_2((1,\epsilon))} + \sum_{z=x+y\epsilon\in R(\epsilon,c),y=0} \log\frac{\Gamma_2(z+\epsilon,(1,\epsilon))}{\rho_2((1,\epsilon))}.$$

By the difference equation (2.1), we obtain

$$G(c) = \sum_{z\in R(\epsilon,c)} \log\frac{\Gamma_2(z,(1,\epsilon))}{\rho_2((1,\epsilon))}. \tag{6.1}$$

Here, for simplicity, we write z for $z^{(1)}$. We have $C_1(A_1)=0$. Hence

$$V(c) = \frac{1}{4}\sum_l C_l(A_2) \sum_{z\in R(C_2,c)} \prod_{m=1}^{2} \frac{B_{l_m}(x(z)_m)}{l_m!}.$$

If $A=\begin{pmatrix}\alpha & \beta\\ \alpha' & \beta'\end{pmatrix}$, $\alpha,\ \beta\in F$, $\alpha,\ \beta\gg 0$, then

$$C_{(2,0)}(A) = \frac{\alpha'\beta-\alpha\beta'}{\beta\beta'}\log\frac{\beta'}{\beta},\quad C_{(1,1)}(A)=0,\quad C_{(0,2)}(A) = \frac{\beta'\alpha-\beta\alpha'}{\alpha\alpha'}\log\frac{\alpha'}{\alpha}.$$

Hence we obtain

$$V(c) = \frac{1}{2}(\epsilon'-\epsilon)\log\epsilon \sum_{z\in R(C_2,c)} \frac{B_2(x(z)_1)}{2}.$$

Then we easily get

$$V(c) = \frac{1}{4}(\epsilon'-\epsilon)\log\epsilon \sum_{z=x+y\epsilon\in R(\epsilon,c)} B_2(x). \tag{6.2}$$

We also see easily that

$$W(c) = -\frac{1}{4}\log N(\mathfrak{a}_\mu\mathfrak{f}) \sum_{z=x+y\epsilon\in R(\epsilon,c)} (\epsilon' B_2(x)+\epsilon B_2(y)+2B_1(x)B_1(y)). \tag{6.3}$$

Summing up, we have

$$X(c)=\sum_{z=x+y\epsilon\in R(\epsilon,c)}\left\{\log\frac{\Gamma_2(z,(1,\epsilon))}{\rho_2((1,\epsilon))}+\frac{1}{4}(\epsilon'-\epsilon)\log\epsilon\, B_2(x)\right.$$
$$\left.-\frac{1}{4}\log N(\mathfrak{a}_\mu\mathfrak{f})(\epsilon' B_2(x)+\epsilon B_2(y)+2B_1(x)B_1(y))\right\}. \tag{6.4}$$

Shintani's formulas (2.16) and (2.17) simplify in this case to ([Sh4], (16), (17))

$$\zeta_F(0,c)=\sum_{z=x+y\epsilon\in R(\epsilon,c)}\left\{\frac{\epsilon+\epsilon'}{4}(B_2(x)+B_2(y))+B_1(x)B_1(y))\right\}, \tag{6.5}$$

$$\zeta_F'(0,c)=\sum_{z=x+y\epsilon\in R(\epsilon,c)}\left\{\log\frac{\Gamma_2(z,(1,\epsilon))\Gamma_2(z',(1,\epsilon'))}{\rho_2((1,\epsilon))\rho_2((1,\epsilon'))}+\frac{\epsilon'-\epsilon}{2}\log\epsilon\, B_2(x)\right\}$$
$$-\log N(\mathfrak{a}_\mu\mathfrak{f})\zeta_F(0,c). \tag{6.6}$$

By (5.3), we get

$$X(c')=\sum_{z=x+y\epsilon\in R(\epsilon,c)}\left\{\log\frac{\Gamma_2(z',(1,\epsilon))}{\rho_2((1,\epsilon'))}+\frac{1}{4}(\epsilon'-\epsilon)\log\epsilon\, B_2(x)\right.$$
$$\left.-\frac{1}{4}\log N(\mathfrak{a}_\mu\mathfrak{f})(\epsilon B_2(x)+\epsilon' B_2(y)+2B_1(x)B_1(y))\right\}. \tag{6.7}$$

The following Lemma would be of some interest.

Lemma 4. *We have* $\sum_{z=x+y\epsilon\in R(\epsilon,c)}B_2(x)=\sum_{z=x+y\epsilon\in R(\epsilon,c)}B_2(y)$ *and* $W(c)=-\frac{1}{2}\log N(\mathfrak{a}_\mu\mathfrak{f})\zeta_F(0,c)$.

Proof. It suffices to show the first assertion. Let $z=x+y\epsilon\in R(\epsilon,c)$. Then $\epsilon^{-1}z+\epsilon=y+x(\epsilon+\epsilon')+(1-x)\epsilon$ satisfies $(\epsilon^{-1}z+\epsilon)\mathfrak{a}_\mu\mathfrak{f}=c$ in $C_\mathfrak{f}$. We take $x_1\in\mathbf{Q}$ so that $0<x_1\le 1$, $x_1\equiv y+x(\epsilon+\epsilon') \mod \mathbf{Z}$ and put $y_1=1-x$, $z_1=x_1+y_1\epsilon$. Then we have $z_1\in R(\epsilon,c)$. Put $\varphi(z)=z_1$. Clearly the mapping φ is injective, hence bijective. Now the assertion follows from $B_2(1-x)=B_2(x)$.

Though it can well be the case that $W(c)=-\frac{1}{n}\log N(\mathfrak{a}_\mu\mathfrak{f})\zeta_F(0,c)$ in general, we have not examined this problem yet.

Let K be a totally imaginary quadratic extension of F and let $\mathfrak{f}\infty_1\infty_2$ be the conductor of K as a class field over F. Let χ be the Hecke character of $F_A^\times$ which corresponds to K. Then (4.7) can be written as

$$g_K(\mathrm{id},\mathrm{id})=\pi^{-1/2}\exp\left(\frac{1}{2L(0,\chi)}\sum_{c\in C_\mathfrak{f}}\chi(c)X(c)\right), \tag{6.8}$$

$$g_{K^\sigma}(\mathrm{id},\mathrm{id}) = \pi^{-1/2}\exp(\frac{1}{2L(0,\chi)}\sum_{c\in C_{\mathfrak{f}}}\chi(c)X(c')) \tag{6.9}$$

if $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ induces the nontrivial automorphism on F. In this case, we have

$$g_K(\mathrm{id},\mathrm{id})g_{K^\sigma}(\mathrm{id},\mathrm{id}) = \pi^{-1}\exp(\frac{L'(0,\chi)}{2L(0,\chi)}).$$

Let $F = \mathbf{Q}(\sqrt{d})$, $0 < d \in \mathbf{Q}$ and $x + y\sqrt{d}$, x, $y \in \mathbf{Q}$ be a totally positive element. Set $\xi = \sqrt{x + y\sqrt{d}}\, i$, $\xi' = \sqrt{x - y\sqrt{d}}\, i$, $K = \mathbf{Q}(\xi)$. We assume that K is not normal over $\mathbf{Q}$. Define $\sigma \in J_K$ by $\xi^\sigma = \xi'$. Let $\mathfrak{q}$ be an integral ideal of K. For positive integers a and b, let $\lambda_{a,b}^{(1)}$ and $\lambda_{a,b}^{(2)}$ be characters of $I_{\mathfrak{q}}(K)$ such that

$$\lambda_{a,b}^{(1)}((\alpha)) = (\frac{\alpha^\rho}{|\alpha|})^a(\frac{\alpha^{\sigma\rho}}{|\alpha^\sigma|})^b, \qquad \alpha \equiv 1 \mod^\times \mathfrak{q},$$

$$\lambda_{a,b}^{(2)}((\alpha)) = (\frac{\alpha^\rho}{|\alpha|})^a(\frac{\alpha^{\sigma}}{|\alpha^{\sigma\rho}|})^b, \qquad \alpha \equiv 1 \mod^\times \mathfrak{q}.$$

By Theorem S2, we have

$$L_K(1,\lambda_{a,b}^{(1)}) \sim \pi^{(4+a+b)/2}p_K(a\cdot\mathrm{id} + b\cdot\sigma, \mathrm{id}+\sigma),$$

$$L_K(1,\lambda_{a,b}^{(2)}) \sim \pi^{(4+a+b)/2}p_K(a\cdot\mathrm{id} + b\cdot\sigma\rho, \mathrm{id}+\sigma\rho).$$

We note that $L_K(1,\lambda_{a,b}^{(i)}) \neq 0$ for every i, a, b. By Theorem S1, we have

$$\frac{L_K(1,\lambda_{4,2}^{(1)})L_K(1,\lambda_{4,2}^{(2)})}{L_K(1,\lambda_{2,2}^{(1)})L_K(1,\lambda_{2,2}^{(2)})} \sim \pi^2 p_K(\mathrm{id},\mathrm{id})^4, \tag{6.10}$$

$$\frac{L_K(1,\lambda_{2,4}^{(1)})L_K(1,\lambda_{2,4}^{(2)})}{L_K(1,\lambda_{2,2}^{(1)})L_K(1,\lambda_{2,2}^{(2)})} \sim \pi^2 p_K(\sigma,\sigma)^4. \tag{6.11}$$

In this case, Conjecture B states that

$$\begin{aligned}&\left(\frac{L_K(1,\lambda_{4,2}^{(1)})L_K(1,\lambda_{4,2}^{(2)})L_K(1,\lambda_{2,2}^{(1)})^{-1}L_K(1,\lambda_{2,2}^{(2)})^{-1}}{\pi^2 g_K(\mathrm{id},\mathrm{id})^4}\right)^\alpha \\ =&\zeta\cdot\frac{L_K(1,(\lambda_{4,2}^{(1)})^\alpha)L_K(1,(\lambda_{4,2}^{(2)})^\alpha)L_K(1,(\lambda_{2,2}^{(1)})^\alpha)^{-1}\,L_K(1,(\lambda_{2,2}^{(2)})^\alpha)^{-1}}{\pi^2 g_{K^\alpha}(\mathrm{id},\mathrm{id})^4}\end{aligned} \tag{6.12}$$

for every $\alpha \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ with a root of unity ζ. Put

$$\begin{aligned}Q_1 &= \frac{L_K(1,\lambda_{4,2}^{(1)})L_K(1,\lambda_{4,2}^{(2)})L_K(1,\lambda_{2,2}^{(1)})^{-1}L_K(1,\lambda_{2,2}^{(2)})^{-1}}{\pi^2 g_K(\mathrm{id},\mathrm{id})^4},\\ Q_2 &= \frac{L_K(1,\lambda_{2,4}^{(1)})L_K(1,\lambda_{2,4}^{(2)})L_K(1,\lambda_{2,2}^{(1)})^{-1}L_K(1,\lambda_{2,2}^{(2)})^{-1}}{\pi^2 g_{K^\sigma}(\mathrm{id},\mathrm{id})^4}.\end{aligned} \tag{6.13}$$

If $h_K = 1$ and the conductors of $\lambda_{a,b}^{(i)}$ are (1) in (6.13), then we see easily that (6.12) can be written as

$$(6.14) \qquad \begin{cases} (Q_1)^\alpha = \zeta_1 \cdot Q_1, & (Q_2)^\alpha = \zeta_2 \cdot Q_2, \quad \text{if} \quad \alpha|F = \text{id}, \\ (Q_1)^\alpha = \zeta_1 \cdot Q_2, & (Q_2)^\alpha = \zeta_2 \cdot Q_1, \quad \text{if} \quad \alpha|F \neq \text{id}, \end{cases}$$

with roots of unity ζ_1 and ζ_2 which depend on α.

Example 1. Let $F = \mathbf{Q}(\sqrt{5})$, $K = \mathbf{Q}(\sqrt{\frac{13+\sqrt{5}}{2}}\, i)$. Then we have $\epsilon = \frac{3+\sqrt{5}}{2}$, $\mathfrak{f} = (\frac{13+\sqrt{5}}{2})$, $N(\mathfrak{f}) = 41$, $L(0,\chi) = 2$, $h_F = h_K = 1$, $|C_\mathfrak{f}| = 2$. Since $h_0 = 1$, we may take $\mathfrak{a}_1 = (1)$. Take $\mathfrak{q} = (1)$. Then for even positive integers a and b, $\lambda_{a,b}^{(1)}$ and $\lambda_{a,b}^{(2)}$ exist and are uniquely determined. Define Q_1 and Q_2 by (6.13). By a numerical experiment, we find $Q_1 Q_2 = \frac{25}{9}$. (The numerical value coincide to the 45-th decimal place. This quantity is equal to ABQ in the notation of [Y2], §4, Example 1.) We also find

$$\epsilon^{-22/41} Q_1 + \epsilon^{22/41} Q_2 = \frac{490}{3 \cdot 41}.$$

Solving these equations and comparing with numerical values to determine the sign of square roots, we get

$$Q_1 = \epsilon^{22/41} \cdot \frac{245 + 60\sqrt{5}}{3 \cdot 41}, \qquad Q_2 = \epsilon^{-22/41} \cdot \frac{245 - 60\sqrt{5}}{3 \cdot 41}.$$

The numerical coincidence is to the 45-th decimal place.[4] In this example, ζ_1 is a 41-st root of unity given by $(\epsilon^{22/41})^\alpha = \zeta_1((\epsilon^\alpha)^{22/41})$. Here the involvement of the fractional power of ϵ is curious and this was the main difficulty of the numerical experiment. Let $C_\mathfrak{f} = \{c_1, c_2\}$ where c_1 is the identity. We may take $c_2 =$ the class of (6). We have $|R(\epsilon, c_i)| = 20$, $i = 1$, 2 and

$$V(c_1) = (\epsilon' - \epsilon) \log \epsilon \cdot \frac{19}{2 \cdot 3 \cdot 41}, \qquad V(c_2) = (\epsilon' - \epsilon) \log \epsilon \cdot \frac{-29}{2 \cdot 3 \cdot 41},$$
$$V(c_1) - V(c_2) = -\frac{8}{41}\sqrt{5} \log \epsilon, \qquad W(c_1) = -\frac{1}{2}\log(41), \qquad W(c_2) = \frac{1}{2}\log(41).$$

Thus we have

$$\pi^2 g_K(\text{id}, \text{id})^4 = \frac{1}{41}\epsilon^{-8\sqrt{5}/41} \Big(\prod_{z \in R(\epsilon, c_1)} \frac{\Gamma_2(z, (1,\epsilon))}{\rho_2((1,\epsilon))} \Big) \Big(\prod_{z \in R(\epsilon, c_2)} \frac{\Gamma_2(z, (1,\epsilon))}{\rho_2((1,\epsilon))} \Big)^{-1}.$$

We note that $\epsilon^{\sqrt{5}}$ is a transcendental number by the Gelfond-Schneider theorem. It is remarkable that the power $\epsilon^{22/41}$ can be "explained" by $22 - 8\sqrt{5} \equiv 0 \mod \mathfrak{f}'$. Similar phenomena will be noticed in the following examples.

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[4]For the sake of completeness, we carried out the second computation with higher precision and improved the numerical coincidence to the 112-th decimal place.

Example 2. Let $F = \mathbf{Q}(\sqrt{41})$, $K = \mathbf{Q}(\sqrt{13 + 2\sqrt{41}}\, i)$. Then we have $\epsilon = 2049 + 320\sqrt{41}$, $\mathfrak{f} = (13 + 2\sqrt{41})$, $N(\mathfrak{f}) = 5$, $L(0, \chi) = 2$, $h_F = h_K = 1$, $|C_{\mathfrak{f}}| = 2$. We take $\mathfrak{a}_1 = (1)$, $\mathfrak{q} = (1)$. Then we find

$$Q_1 = \epsilon^{4/25} \cdot \frac{546718 + 85383\sqrt{41}}{2^2 \cdot 3 \cdot 5}, \qquad Q_2 = \epsilon^{-4/25} \cdot \frac{546718 - 85383\sqrt{41}}{2^2 \cdot 3 \cdot 5}.$$

The numerical coincidence is to the 42-nd decimal place. Let $C_{\mathfrak{f}} = \{c_1, c_2\}$, $c_1 = 1$. Then $|R(\epsilon, c_i)| = 1280$, $i = 1, 2$. We have $V(c_1) - V(c_2) = -\frac{1}{25}\sqrt{41} \log \epsilon$ and $4 - \sqrt{41} \equiv 0 \mod \mathfrak{f}'^2$ holds. (The "reason" of square is that 5^2 appears in the denominator of $V(c_1) - V(c_2)$. Compare with Example 11.)

Example 3. Let $F = \mathbf{Q}(\sqrt{29})$, $K = \mathbf{Q}(\sqrt{\frac{9 + \sqrt{29}}{2}}\, i)$. Then we have $\epsilon = \frac{27 + 5\sqrt{29}}{2}$, $\mathfrak{f} = (\frac{9 + \sqrt{29}}{2})$, $N(\mathfrak{f}) = 13$, $L(0, \chi) = 2$, $h_F = h_K = 1$, $|C_{\mathfrak{f}}| = 2$. We take $\mathfrak{a}_1 = (1)$, $\mathfrak{q} = (1)$. Then we find

$$Q_1 = \epsilon^{9/13} \cdot \frac{2935 + 503\sqrt{29}}{2^4 \cdot 3 \cdot 13}, \qquad Q_2 = \epsilon^{-9/13} \cdot \frac{2935 - 503\sqrt{29}}{2^4 \cdot 3 \cdot 13}.$$

The numerical coincidence is to the 29-th decimal place. Let $C_{\mathfrak{f}} = \{c_1, c_2\}$, $c_1 = 1$. We have $V(c_1) - V(c_2) = -\frac{1}{13}\sqrt{29} \log \epsilon$ and $9 - \sqrt{29} \equiv 0 \mod \mathfrak{f}'$ holds.

Example 4. Let $F = \mathbf{Q}(\sqrt{13})$, $K = \mathbf{Q}(\sqrt{9 + 2\sqrt{13}}\, i)$. Then we have $\epsilon = \frac{11 + 3\sqrt{13}}{2}$, $\mathfrak{f} = (9 + 2\sqrt{13})$, $N(\mathfrak{f}) = 29$, $L(0, \chi) = 2$, $h_F = h_K = 1$, $|C_{\mathfrak{f}}| = 2$. We take $\mathfrak{a}_1 = (1)$, $\mathfrak{q} = (1)$. Then we find

$$Q_1 = \epsilon^{28/29} \cdot \frac{5669 + 274\sqrt{13}}{2^3 \cdot 3^2 \cdot 29}, \qquad Q_2 = \epsilon^{-28/29} \cdot \frac{5669 - 274\sqrt{13}}{2^3 \cdot 3^2 \cdot 29},$$

The numerical coincidence is to the 45-th decimal place. Let $C_{\mathfrak{f}} = \{c_1, c_2\}$, $c_1 = 1$. We have $V(c_1) - V(c_2) = -\frac{3}{29}\sqrt{13} \log \epsilon$ and $28 - 3\sqrt{13} \equiv 0 \mod \mathfrak{f}'$ holds.

Example 5. Let $F = \mathbf{Q}(\sqrt{17})$, $K = \mathbf{Q}(\sqrt{9 + 2\sqrt{17}}\, i)$. Then we have $\epsilon = 33 + 8\sqrt{17}$, $\mathfrak{f} = (9 + 2\sqrt{17})$, $N(\mathfrak{f}) = 13$, $L(0, \chi) = 2$, $h_F = h_K = 1$, $|C_{\mathfrak{f}}| = 2$. We take $\mathfrak{a}_1 = (1)$, $\mathfrak{q} = (1)$. Then we find

$$Q_1 = \epsilon^{11/13} \cdot \frac{285 + 6\sqrt{17}}{2^2 \cdot 13}, \qquad Q_2 = \epsilon^{-11/13} \cdot \frac{285 - 6\sqrt{17}}{2^2 \cdot 13}.$$

The numerical coincidence is to the 29-th decimal place. Let $C_{\mathfrak{f}} = \{c_1, c_2\}$, $c_1 = 1$. We have $V(c_1) - V(c_2) = -\frac{1}{13}\sqrt{17} \log \epsilon$ and $11 - \sqrt{17} \equiv 0 \mod \mathfrak{f}'$ holds.

Example 6. Let $F = \mathbf{Q}(\sqrt{13})$, $K = \mathbf{Q}(\sqrt{\frac{9+\sqrt{13}}{2}}\, i)$. Then we have $\epsilon = \frac{11+3\sqrt{13}}{2}$, $\mathfrak{f} = (\frac{9+\sqrt{13}}{2})$, $N(\mathfrak{f}) = 17$, $L(0,\chi) = 2$, $h_F = h_K = 1$, $|C_\mathfrak{f}| = 2$. We take $\mathfrak{a}_1 = (1)$, $\mathfrak{q} = (1)$. Then we find

$$Q_1 = \epsilon^{1/17} \cdot \frac{5522 + 1519\sqrt{13}}{3^2 \cdot 17}, \qquad Q_2 = \epsilon^{-1/17} \cdot \frac{5522 - 1519\sqrt{13}}{3^2 \cdot 17}.$$

The numerical coincidence is to the 44-th decimal place. Let $C_\mathfrak{f} = \{c_1, c_2\}$, $c_1 = 1$. We have $V(c_1) - V(c_2) = -\frac{2}{17}\sqrt{13}\log\epsilon$ and $1 - 2\sqrt{13} \equiv 0 \mod \mathfrak{f}'$ holds.

Example 7. Let $F = \mathbf{Q}(\sqrt{5})$, $K = \mathbf{Q}(\sqrt{7+2\sqrt{5}}\, i)$. Then we have $\epsilon = \frac{3+\sqrt{5}}{2}$, $\mathfrak{f} = (4)(7+2\sqrt{5})$, $N(\mathfrak{f}) = 2^4 \cdot 29$, $L(0,\chi) = 4$, $h_F = 1, h_K = 2$, $|C_\mathfrak{f}| = 16$. We take $\mathfrak{a}_1 = (1)$, $\mathfrak{q} = (1)$. The prime ideal (2) of F ramifies in K. Put $(2) = \mathfrak{P}_2^2$. Let $i = 1$ or 2. For every positive even integers a and b, there exists two Grössencharacters $\lambda_{a,b}^{(i)}$ of conductor (1). They are determined by $\lambda_{a,b}^{(i)}(\mathfrak{P}_2) = 1$ or -1. First determine $\lambda_{a,b}^{(i)}$ taking $\lambda_{a,b}^{(i)}(\mathfrak{P}_2) = 1$ for every a, b and i. Define Q_1 and Q_2 by (6.13). Though $h_K = 2$, (6.14) still follows from (6.12). We find

$$Q_1 = \epsilon^{19/29} \cdot \frac{49675 - 4569\sqrt{5}}{2 \cdot 3 \cdot 13 \cdot 29}, \qquad Q_2 = \epsilon^{-19/29} \cdot \frac{49675 + 4569\sqrt{5}}{2 \cdot 3 \cdot 13 \cdot 29}.$$

Next determine $\lambda_{a,b}^{(i)}$ taking $\lambda_{a,b}^{(i)}(\mathfrak{P}_2) = -1$ for every a, b and i. Then we find

$$Q_1 = \epsilon^{19/29} \cdot \frac{24645 - 823\sqrt{5}}{2 \cdot 5^2 \cdot 29}, \qquad Q_2 = \epsilon^{-19/29} \cdot \frac{24645 + 823\sqrt{5}}{2 \cdot 5^2 \cdot 29}.$$

The numerical coincidence is to the 44-th decimal place. Let $C_\mathfrak{f} = \{c_i \mid 1 \le i \le 16\}$. We have $\frac{1}{2}\sum_{i=1}^{16} \chi(c_i)V(c_i) = -\frac{15}{58}\sqrt{5}\log\epsilon$ and $38 - 15\sqrt{5} \equiv 0 \mod (7 - 2\sqrt{5})$ holds.

§7. Numerical examples II: Quartic extensions of real quadratic fields

We keep the notation in the previous section and proceed to consider the normal closure $L = \mathbf{Q}(\xi, \xi')$ of K over $\mathbf{Q}$. We have $[L : \mathbf{Q}] = 8$. Define σ, $\tau \in \mathrm{Gal}(L/\mathbf{Q})$ by

$$\sigma : (\xi, \xi') \longrightarrow (\xi', -\xi), \qquad \tau : (\xi, \xi') \longrightarrow (\xi', \xi).$$

(Previous σ is extended.) Then $\mathrm{Gal}(L/\mathbf{Q})$ is the dihedral group generated by σ and τ which are subject to the relations

$$\sigma^4 = \tau^2 = 1, \qquad \tau\sigma = \sigma^3\tau, \qquad \sigma^2 = \rho.$$

The reflex field K' of $(K, \{\mathrm{id}, \sigma\})$ is contained in L and given by $K' = \mathbf{Q}(\sqrt{2(x+\sqrt{d'}\, i)}$, $d' = x^2 - y^2 d$. The maximal real subfield of K' is $F' = \mathbf{Q}(\sqrt{d'})$. We have

$$\mathrm{Gal}(L/K) = \{1, \sigma\tau\}, \qquad \mathrm{Gal}(L/K') = \{1, \tau\}.$$

Let us consider the problem to give $p_L(\mathrm{id}, \alpha)$ for all $\alpha \in \mathrm{Gal}(L/\mathbf{Q})$ in terms of the multiple gamma function. $\mathrm{Gal}(L/\mathbf{Q})$ has five conjugacy classes $\{\mathrm{id}\}$, $\{\rho\}$, $\{\sigma, \sigma\rho\}$, $\{\tau, \tau\rho\}$, $\{\sigma\tau, \sigma\tau\rho\}$. Therefore, in view of Theorem S1, (3), our previous Conjecture 1 can give essentially only $p_L(\mathrm{id}, \mathrm{id})$. We have

$$p_K(\mathrm{id}, \mathrm{id}) \sim p_L(\mathrm{id}, \mathrm{id}) p_L(\mathrm{id}, \sigma\tau), \qquad p_K(\sigma, \sigma) \sim p_L(\mathrm{id}, \mathrm{id}) p_L(\mathrm{id}, \sigma\tau)^{-1},$$
$$p_{K'}(\mathrm{id}, \mathrm{id}) \sim p_L(\mathrm{id}, \mathrm{id}) p_L(\mathrm{id}, \tau), \qquad p_{K'}(\sigma\tau, \sigma\tau) \sim p_L(\mathrm{id}, \mathrm{id}) p_L(\mathrm{id}, \tau)^{-1}$$

and these quantities were examined in examples discussed in §6. In other words, we examined the validity of our conjectures for $p_L(\mathrm{id}, \mathrm{id})$, $p_L(\mathrm{id}, \tau)$ and for $p_L(\mathrm{id}, \sigma\tau)$. What remains to be considered is $p_L(\mathrm{id}, \sigma)$. Put $F_* = \mathbf{Q}(\sqrt{dd'}) \subset L$. We have $\mathrm{Gal}(L/F_*) = \langle \sigma \rangle \cong \mathbf{Z}/4\mathbf{Z}$. By Theorems S1 and S2, we can easily derive the relation

$$\frac{L_K(1, \lambda_{4,2}^{(1)}) L_K(1, \lambda_{2,4}^{(2)})}{L_K(1, \lambda_{2,4}^{(1)}) L_K(1, \lambda_{4,2}^{(2)})} \sim p_L(\mathrm{id}, \sigma)^8.$$

Hence we put

$$Q = \frac{L_K(1, \lambda_{4,2}^{(1)}) L_K(1, \lambda_{2,4}^{(2)}) L_K(1, \lambda_{2,4}^{(1)})^{-1} L_K(1, \lambda_{4,2}^{(2)})^{-1}}{g_L(\mathrm{id}, \sigma)^8} \tag{7.1}$$

and are going to examine the algebraicity of Q. Here $g_L(\mathrm{id}, \sigma)$ is defined by (4.7) (σ in place of τ) and (6.4), L being an abelian extension of F_*.

Let $\mathfrak{q}'$ be an integral ideal of K'. For positive integers a and b, let $\mu_{a,b}^{(1)}$ and $\mu_{a,b}^{(2)}$ be characters of $I_{\mathfrak{q}'}(K')$ such that

$$\mu_{a,b}^{(1)}((\alpha)) = (\frac{\alpha^\rho}{|\alpha|})^a (\frac{\alpha^{\sigma\tau\rho}}{|\alpha^{\sigma\tau}|})^b, \qquad \alpha \equiv 1 \mod^\times \mathfrak{q}',$$
$$\mu_{a,b}^{(2)}((\alpha)) = (\frac{\alpha^\rho}{|\alpha|})^a (\frac{\alpha^{\sigma\tau}}{|\alpha^{\sigma\tau\rho}|})^b, \qquad \alpha \equiv 1 \mod^\times \mathfrak{q}'.$$

Then we have

$$\frac{L_{K'}(1, \mu_{2,4}^{(1)}) L_{K'}(1, \mu_{4,2}^{(2)})}{L_{K'}(1, \mu_{4,2}^{(1)}) L_{K'}(1, \mu_{2,4}^{(2)})} \sim p_L(\mathrm{id}, \sigma)^8.$$

We put

$$R = \frac{L_{K'}(1, \mu_{2,4}^{(1)}) L_{K'}(1, \mu_{4,2}^{(2)}) L_{K'}(1, \mu_{4,2}^{(1)})^{-1} L_{K'}(1, \mu_{2,4}^{(2)})^{-1}}{g_L(\mathrm{id}, \sigma)^8}. \tag{7.2}$$

Example 8. Let $F = \mathbf{Q}(\sqrt{5})$, $K = \mathbf{Q}(\sqrt{\frac{13+\sqrt{5}}{2}}\, i)$. Then $F' = \mathbf{Q}(\sqrt{41})$, $K' = \mathbf{Q}(\sqrt{13+2\sqrt{41}}\, i)$. These fields are considered in Examples 1 and 2. We have $F_* = \mathbf{Q}(\sqrt{225})$. The fundamental unit ϵ_0 of F_* is $\frac{43+3\sqrt{205}}{2}$, which is totally positive. Set $\epsilon = \epsilon_0$. We also have $h_{F_*} = 2$, $h_0 = 4$. We have $D(K/F) = (\frac{13+\sqrt{5}}{2})$, $D(K'/F') = (13+2\sqrt{41})$. Hence $D(L/\mathbf{Q}) = N_{K/\mathbf{Q}}(D(L/K)) \cdot (5^4 \cdot 41^2) = N_{K'/\mathbf{Q}}(D(L/K')) \cdot (5^2 \cdot 41^4)$. Since $L = K \vee \mathbf{Q}(\sqrt{41}) = K' \vee \mathbf{Q}(\sqrt{5})$, only prime factors of (41) (resp. (5)) can ramify in L/K (resp. L/K'). Here $\vee$ denotes the operation to make the composite field. Then we easily get $D(L/\mathbf{Q}) = 5^4 \cdot 41^4$. Since $D(L/\mathbf{Q}) = N_{F_*/\mathbf{Q}}(D(L/F_*))D(F_*/\mathbf{Q})^4$, we must have $D(L/F_*) = (1)$. Therefore L is unramified over F_*. Hence L is the maximal ray class field of conductor $(1)\infty_1\infty_2$ of F_*. Set

$$\mathfrak{p}_3 = 3\mathbf{Z} + \frac{1+\sqrt{205}}{2}\mathbf{Z}, \qquad \mathfrak{p}_3' = 3\mathbf{Z} + \frac{1-\sqrt{205}}{2}\mathbf{Z}.$$

Then $\mathfrak{p}_3$ and $\mathfrak{p}_3'$ are prime ideals of F_* and we have $(3) = \mathfrak{p}_3\mathfrak{p}_3'$. As representatives of the narrow ideal class group of F_*, we can take

$$\mathfrak{a}_1 = (1), \qquad \mathfrak{a}_2 = \mathfrak{p}_3, \qquad \mathfrak{a}_3 = \mathfrak{p}_3^2 = (14-\sqrt{205}), \qquad \mathfrak{a}_4 = \mathfrak{p}_3'.$$

We have $(\frac{L/F_*}{\mathfrak{p}_3}) = \sigma$. For $1 \le i \le 4$, let c_i be the class of $\mathfrak{a}_i$. We have

$$|R(\epsilon, c_1)| = 3, \qquad |R(\epsilon, c_2)| = |R(\epsilon, c_4)| = 9, \qquad |R(\epsilon, c_3)| = 27.$$

By (6.5), we get

$$\zeta_{F_*}(0, c_1) = 1, \qquad \zeta_{F_*}(0, c_2) = \zeta_{F_*}(0, c_4) = 0, \qquad \zeta_{F_*}(0, c_3) = -1.$$

Hence

$$g_L(\mathrm{id}, \sigma) = \exp(\frac{1}{4}(X(c_4) - X(c_2))).$$

Define $\lambda_{a,b}^{(i)}$ as in Example 1 and let us consider the quantity Q defined by (7.1). Conjecture B does not apply directly to Q since $\lambda_{a,b}^{(i)}$ is not a Grössencharacter of L. To reduce to the situation of Conjecture B, regard $\lambda_{a,b}^{(i)}$ as a Hecke character of $K_A^\times$ and let ω be the Hecke character of $K_A^\times$ which corresponds to the quadratic extension L/K. We have $L_L(s, \lambda_{a,b}^{(i)} \circ N_{L/K}) = L_K(s, \lambda_{a,b}^{(i)})L_K(s, \lambda_{a,b}^{(i)}\omega)$. Set

$$\Lambda = \frac{L_K(1, \lambda_{4,2}^{(1)})L_K(1, \lambda_{2,4}^{(2)})}{L_K(1, \lambda_{2,4}^{(1)})L_K(1, \lambda_{4,2}^{(2)})}, \qquad \Lambda' = \frac{L_K(1, \lambda_{4,2}^{(1)}\omega)L_K(1, \lambda_{2,4}^{(2)}\omega)}{L_K(1, \lambda_{2,4}^{(1)}\omega)L_K(1, \lambda_{4,2}^{(2)}\omega)}$$

and for $\alpha \in \mathrm{Gal}(L/\mathbf{Q})$, set

$$\Lambda_\alpha = \frac{L_K(1,(\lambda_{4,2}^{(1)})^\alpha)L_K(1,(\lambda_{2,4}^{(2)})^\alpha)}{L_K(1,(\lambda_{2,4}^{(1)})^\alpha)L_K(1,(\lambda_{4,2}^{(2)})^\alpha)}, \qquad \Lambda'_\alpha = \frac{L_K(1,(\lambda_{4,2}^{(1)}\omega)^\alpha)L_K(1,(\lambda_{2,4}^{(2)}\omega)^\alpha)}{L_K(1,(\lambda_{2,4}^{(1)}\omega)^\alpha)L_K(1,(\lambda_{4,2}^{(2)}\omega)^\alpha)}.$$

We can easily verify that

$$\Lambda_\sigma = \Lambda, \quad \Lambda_\tau = \Lambda^{-1}, \qquad \Lambda'_\sigma = \Lambda', \quad \Lambda'_\tau = \Lambda'^{-1}.$$

Put $Q_* = \dfrac{\Lambda\Lambda'}{g_L(\mathrm{id},\sigma)^{16}}$. Conjecture B can be applied to Q_*. Note that $g_L(\mathrm{id},\tau^{-1}\sigma\tau) = g_L(\mathrm{id},\sigma\rho) = g_L(\mathrm{id},\sigma)^{-1}$. We use the same letter σ (resp. τ) for any extension of σ (resp. τ) to $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. The conjecture implies that $Q_*^\sigma = \zeta_\sigma Q_*$, $Q_*^\tau = \zeta_\tau Q_*^{-1}$ with roots of unity ζ_σ and ζ_τ. On the other hand, by Proposition 7.3 of [Y2], we immediately get $(\dfrac{\Lambda'}{\Lambda})^\sigma = \pm\dfrac{\Lambda'}{\Lambda}$, $(\dfrac{\Lambda'}{\Lambda})^\tau = \pm\dfrac{\Lambda}{\Lambda'}$. Since $Q^2 = Q_* \cdot \dfrac{\Lambda'}{\Lambda}$, we conclude that

$$Q^\sigma = \zeta'_\sigma Q, \qquad Q^\tau = \zeta'_\tau Q^{-1}$$

with roots of unity ζ'_σ and ζ'_τ. We search $a \in \mathbf{Z}$ so that $(\epsilon^{a/27}Q)^2 + (\epsilon^{a/27}Q)^{-2}$ is close to a rational number with a simple denominator.[5] We find $a = 6$ fits this scheme and

$$(\epsilon^{2/9}Q)^2 + \frac{1}{(\epsilon^{2/9}Q)^2} = \frac{367}{3^4}.$$

Solving this equation and comparing with the numerical value, we find

$$Q = \epsilon^{-2/9} \cdot \frac{1}{3^2}\sqrt{\frac{367 + 23\sqrt{205}}{2}}.$$

The numerical coincidence is to the 47-th decimal place. Similarly, taking $\mathfrak{q}' = (1)$ for $\mu_{a,b}^{(i)}$, we find

$$R = \epsilon^{-2/9} \cdot \frac{\sqrt{637530967 - 18592639\sqrt{205}}}{3^2 \cdot 31 \cdot 61}.$$

The numerical coincidence is to the 48-th decimal place.

Example 9. Let $F = \mathbf{Q}(\sqrt{17})$, $K = \mathbf{Q}(\sqrt{9 + 2\sqrt{17}}\ i)$. Then $F' = \mathbf{Q}(\sqrt{13})$, $K' = \mathbf{Q}(\sqrt{\dfrac{9+\sqrt{13}}{2}}\ i)$. These fields are considered in Examples 5 and 6. We have $F_* = \mathbf{Q}(\sqrt{221})$.

[5]Practically we can do such test using the expansion of a real number into the continued fraction.

The fundamental unit ϵ_0 of F_* is $\dfrac{15+\sqrt{221}}{2}$. Set $\epsilon = \epsilon_0$. We have $h_{F_*} = 2$, $h_0 = 4$ and L is the maximal ray class field of conductor $(1)\infty_1\infty_2$ of F_*. Set

$$\mathfrak{p}_5 = 5\mathbf{Z} + \frac{1+\sqrt{221}}{2}\mathbf{Z}, \qquad \mathfrak{p}_5' = 5\mathbf{Z} + \frac{1-\sqrt{221}}{2}\mathbf{Z}.$$

Then we have $(5) = \mathfrak{p}_5\mathfrak{p}_5'$, $(\dfrac{L/F_*}{\mathfrak{p}_5}) = \sigma$. As representatives of the narrow ideal class group of F_*, we take

$$\mathfrak{a}_1 = (1), \qquad \mathfrak{a}_2 = \mathfrak{p}_5, \qquad \mathfrak{a}_3 = \mathfrak{p}_5^2 = (14-\sqrt{221}), \qquad \mathfrak{a}_4 = \mathfrak{p}_5'.$$

Define $\lambda_{a,b}^{(i)}$ (resp. $\mu_{a,b}^{(i)}$) taking $\mathfrak{q} = (1)$ (resp. $\mathfrak{q}' = (1)$). Then we find

$$Q = \epsilon^{-1/5} \cdot \frac{1}{53}\sqrt{\frac{276343 + 18585\sqrt{221}}{2}}.$$

The numerical coincidence is to the 30-th decimal place.

$$R = \epsilon^{-1/5} \cdot \frac{\sqrt{64769 + 3600\sqrt{221}}}{191}.$$

The numerical coincidence is to the 47-th decimal place.

Example 10. Let $F = \mathbf{Q}(\sqrt{29})$, $K = \mathbf{Q}(\sqrt{\dfrac{9+\sqrt{29}}{2}}\, i)$. Then $F' = \mathbf{Q}(\sqrt{13})$, $K' = \mathbf{Q}(\sqrt{9+2\sqrt{13}}\, i)$. These fields are considered in Examples 3 and 4. We have $F_* = \mathbf{Q}(\sqrt{377})$. The fundamental unit ϵ_0 of F_* is $233 + 12\sqrt{377}$. Set $\epsilon = \epsilon_0$. We have $h_{F_*} = 2$, $h_0 = 4$ and L is the maximal ray class field of conductor $(1)\infty_1\infty_2$ of F_*. Set

$$\mathfrak{p}_2 = 2\mathbf{Z} + \frac{1+\sqrt{377}}{2}\mathbf{Z}, \qquad \mathfrak{p}_2' = 2\mathbf{Z} + \frac{1-\sqrt{377}}{2}\mathbf{Z}.$$

Then we have $(2) = \mathfrak{p}_2\mathfrak{p}_2'$, $(\dfrac{L/F_*}{\mathfrak{p}_2}) = \sigma$. As representatives of the narrow ideal class group of F_*, we take

$$\mathfrak{a}_1 = (1), \qquad \mathfrak{a}_2 = \mathfrak{p}_2, \qquad \mathfrak{a}_3 = \mathfrak{p}_2^2 = (\frac{19-\sqrt{377}}{2}), \qquad \mathfrak{a}_4 = \mathfrak{p}_2'.$$

Define $\lambda_{a,b}^{(i)}$ (resp. $\mu_{a,b}^{(i)}$) taking $\mathfrak{q} = (1)$ (resp. $\mathfrak{q}' = (1)$). Then we find

$$Q = \epsilon^{-1/4} \cdot \frac{1}{2^2 \cdot 1889}\sqrt{\frac{490432649 - 24564411\sqrt{377}}{2}}.$$

The numerical coincidence is to the 32-nd decimal place.

$$R = \epsilon^{-1/4} \cdot \frac{1}{2^2 \cdot 23 \cdot 179} \sqrt{\frac{2459444473 - 123549195\sqrt{377}}{2}}.$$

The numerical coincidence is to the 49-th decimal place.

Remark. There exist intricate relations between quantities considered in §6 and §7. Let us consider the quantity Q_1 in Example 5. The ideals (2) and (53) decompose completely in $\mathbf{Q}(\sqrt{17})$; so let $(2) = \mathfrak{p}_2\mathfrak{p}_2'$, $(53) = \mathfrak{p}_{53}\mathfrak{p}_{53}'$ be the decompositions into prime ideals. Then we have the factorization into prime ideals:

$$(*) \qquad (\epsilon^{-11/13}Q_1) = \mathfrak{p}_2^{-2}\mathfrak{p}_2'^{-2}(3)\mathfrak{p}_{53}\mathfrak{f}'\mathfrak{f}^{-1}$$

where $\mathfrak{f} = (9+2\sqrt{17})$. We can check that $\mathfrak{f}'\mathfrak{f}^{-1}$ always appears in the factorizations of similar quantities in Examples 1 $\sim$ 7. (In Example 7, $\mathfrak{f}$ is not prime, but a similar fact holds.) Now consider the quantity Q in Example 9. We have the factorization into prime ideals:

$$(\epsilon^{1/5}Q)^2 = \mathfrak{q}_{53}^2(\mathfrak{q}_{53}')^{-2},$$

where $(53) = \mathfrak{q}_{53}\mathfrak{q}_{53}'$ in $\mathbf{Q}(\sqrt{221})$. The appearance of 53 in the denominator of Q seems to be explained by (*). Similar relations exist for all the other examples.

§8. Numerical examples III: Quadratic extension of a totally real cubic field

Let F be a totally real cubic field and let K be a totally imaginary quadratic extension of F. Let $J_F = \{\sigma_1, \sigma_2, \sigma_3\}$ with $\sigma_1 = \mathrm{id}$ and extend each σ_i to an element of J_K. Let $\mathfrak{q}$ be an integral ideal of K. For positive integers a, b and c, let $\lambda_{a,b}^{(i)}$, $1 \le i \le 4$ be characters of $I_{\mathfrak{q}}(K)$ such that

$$\lambda_{a,b,c}^{(1)}((\alpha)) = (\frac{\alpha^\rho}{|\alpha|})^a(\frac{\alpha^{\sigma_2\rho}}{|\alpha^{\sigma_2}|})^b(\frac{\alpha^{\sigma_3\rho}}{|\alpha^{\sigma_3}|})^c,$$
$$\lambda_{a,b,c}^{(2)}((\alpha)) = (\frac{\alpha^\rho}{|\alpha|})^a(\frac{\alpha^{\sigma_2\rho}}{|\alpha^{\sigma_2}|})^b(\frac{\alpha^{\sigma_3}}{|\alpha^{\sigma_3\rho}|})^c,$$
$$\lambda_{a,b,c}^{(3)}((\alpha)) = (\frac{\alpha^\rho}{|\alpha|})^a(\frac{\alpha^{\sigma_2}}{|\alpha^{\sigma_2\rho}|})^b(\frac{\alpha^{\sigma_3\rho}}{|\alpha^{\sigma_3}|})^c,$$
$$\lambda_{a,b,c}^{(4)}((\alpha)) = (\frac{\alpha^\rho}{|\alpha|})^a(\frac{\alpha^{\sigma_2}}{|\alpha^{\sigma_2\rho}|})^b(\frac{\alpha^{\sigma_3}}{|\alpha^{\sigma_3\rho}|})^c.$$

Since

$$\prod_{i=1}^4 \frac{L_K(1,\lambda_{4,2,2}^{(i)})}{L_K(1,\lambda_{2,2,2}^{(i)})} \sim \pi^4 p_K(\mathrm{id},\mathrm{id})^8$$

holds (independently of the extensions σ_i), Conjecture B states that

$$(8.1) \qquad \begin{aligned}&(\frac{\prod_{i=1}^4 L_K(1,\lambda_{4,2,2}^{(i)})L_K(1,\lambda_{2,2,2}^{(i)})^{-1}}{\pi^4 g_K(\mathrm{id},\mathrm{id})^8})^\alpha \\ &= \zeta \cdot \frac{\prod_{i=1}^4 L_K(1,(\lambda_{4,2,2}^{(i)})^\alpha)L_K(1,(\lambda_{2,2,2}^{(i)})^\alpha)^{-1}}{\pi^4 g_{K^\alpha}(\mathrm{id},\mathrm{id})^8}\end{aligned}$$

for every $\alpha \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ with a root of unity ζ. Put

$$(8.2)\qquad \begin{aligned} Q_1 &= \frac{\prod_{i=1}^4 L_K(1,\lambda_{4,2,2}^{(i)})L_K(1,\lambda_{2,2,2}^{(i)})^{-1}}{\pi^4 g_K(\mathrm{id},\mathrm{id})^8}, \\ Q_2 &= \frac{\prod_{i=1}^4 L_K(1,\lambda_{2,4,2}^{(i)})L_K(1,\lambda_{2,2,2}^{(i)})^{-1}}{\pi^4 g_{K^{\sigma_2}}(\mathrm{id},\mathrm{id})^8}, \\ Q_3 &= \frac{\prod_{i=1}^4 L_K(1,\lambda_{2,2,4}^{(i)})L_K(1,\lambda_{2,2,2}^{(i)})^{-1}}{\pi^4 g_{K^{\sigma_3}}(\mathrm{id},\mathrm{id})^8}. \end{aligned}$$

Example 11. Let $F = \mathbf{Q}(\zeta_7 + \zeta_7^{-1})$, where $\zeta_7 = e^{2\pi i/7}$. Set

$$\omega_1 = 2\cos\frac{2}{7}\pi = \zeta_7 + \zeta_7^{-1}, \quad \omega_2 = 2\cos\frac{4}{7}\pi = \zeta_7^2 + \zeta_7^{-2}, \quad \omega_3 = 2\cos\frac{8}{7}\pi = \zeta_7^4 + \zeta_7^{-4}.$$

We have $\mathfrak{O}_F = \mathbf{Z} + \mathbf{Z}\omega_1 + \mathbf{Z}\omega_2$. Define $\sigma_1, \sigma_2, \sigma_3 \in \mathrm{Gal}(F/\mathbf{Q})$ by $\sigma_1 = \mathrm{id}$, $\omega_1^{\sigma_2} = \omega_2$, $\omega_1^{\sigma_3} = \omega_3$. Set $\epsilon = (1+\omega_1)^2\omega_1^{-2} = 2+\omega_1$, $\eta = \omega_1^2 = 2+\omega_2$. Then it is known that $h_F = 1$ and that

$$E_F = \langle\, -1,\ \omega_1,\ 1+\omega_1 \,\rangle, \qquad E_F^+ = \langle\, \epsilon,\ \eta \,\rangle.$$

Since E_F has an arbitrary signature distribution, the class number of F in the narrow sense is also 1. By Shintani [Sh1], p. 415, $\mathbf{R}_+^3 = \cup_{u\in E_F^+} u(\cup_{j=1}^6 C_j)$ holds by taking (cf. also Thomas and Vasquez [TV], Theorem 1)

$$\begin{aligned} &C_1 = C(1,\epsilon,\epsilon\eta), \qquad &&C_2 = C(1,\eta,\eta\epsilon), \qquad &&C_3 = C(1,\epsilon), \\ &C_4 = C(1,\eta), &&C_5 = C(1,\epsilon\eta), &&C_6 = C(1). \end{aligned}$$

Now put $\delta = 6+\omega_1-\omega_2$, $K = F(\sqrt{\delta}\, i)$. We see that δ is totally positive and $N(\delta) = 167$. We have $(167) = \mathfrak{p}\mathfrak{p}'\mathfrak{p}''$ in F with conjugate prime ideals $\mathfrak{p}$, $\mathfrak{p}'$, $\mathfrak{p}''$. We specify $\mathfrak{p}$ by $\omega_1 \equiv 19$ mod $\mathfrak{p}$. Then we can verify that $\mathfrak{p} = (\delta)$. We check that $\dfrac{\sqrt{\delta}\, i + 1 + \omega_2}{2}$ is integral over $\mathfrak{O}_F$. Therefore we see that K is a CM-field, $\mathfrak{p}$ is the only prime ideal which ramifies in K/F and that $\mathfrak{O}_K = \mathfrak{O}_F \dfrac{\sqrt{\delta}\, i + 1 + \omega_2}{2} \oplus \mathfrak{O}_F$. We can also verify that K is the maximal ray class field of conductor $\mathfrak{p}\infty_1\infty_2\infty_3$ of F. We have $C_{\mathfrak{p}} = \{c_1, c_2\}$ where $c_1 =$ the class of (1), $c_2 =$ the class of (5). We see that

$$|R(C_1,c_1)| = |R(C_1,c_2)| = 83, \qquad |R(C_2,c_1)| = |R(C_2,c_2)| = 166$$

and $R(C_j,c_1) = R(C_j,c_2) = \emptyset$ if $j \geq 3$. We compute $\zeta_F(0,c_i)$ by (2.16) and obtain

$$\zeta_F(0,c_1) = 2, \qquad \zeta_F(0,c_2) = -2.$$

Let χ be the Hecke character of $F_A^\times$ which corresponds to the quadratic extension K/F. We have $L(0,\chi) = 4$. Since $E_K = E_F$, we get $h_K = 1$ (cf. [Y2], §3). We have

$$g_{K^\sigma}(\mathrm{id},\mathrm{id}) = \pi^{-1/2}\exp\Big(\frac{1}{8}(X(c_1^\sigma) - X(c_2^\sigma))\Big), \qquad \sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}).$$

Here to define $g_{K^\sigma}(\mathrm{id},\mathrm{id})$, we use $\{C_j^\sigma\}_{1\le j\le 6}$ and $\mathfrak{a}_1^\sigma = (1)$. We extend σ_1, σ_2, $\sigma_3 \in \mathrm{Gal}(F/\mathbf{Q})$ to σ_1, σ_2, $\sigma_3 \in J_K$ by $\sigma_1 = \mathrm{id}$,

$$(\sqrt{\delta}\, i)^{\sigma_2} = \sqrt{\delta^{(2)}}\, i, \quad \delta^{(2)} = 6 + \omega_2 - \omega_3, \qquad (\sqrt{\delta}\, i)^{\sigma_3} = \sqrt{\delta^{(3)}}\, i, \quad \delta^{(3)} = 6 + \omega_3 - \omega_1.$$

We compute $g_{K^\sigma}(\mathrm{id},\mathrm{id})$ by the asymptotic expansion of the multiple gamma function given in Barnes [Ba2], p. 424[6] and by the difference equation (2.1). We take $\mathfrak{q} = (1)$ and consider the quantities Q_i defined by (8.2). (Note that for every positive even integers a, b, c and $1 \le i \le 4$, there exists unique $\lambda_{a,b,c}^{(i)}$ of conductor (1)). Then (8.1) implies

$$(8.3) \quad \begin{cases} (Q_1)^\alpha = \zeta_1 \cdot Q_1, & (Q_2)^\alpha = \zeta_2 \cdot Q_2, \quad (Q_3)^\alpha = \zeta_3 \cdot Q_3, \quad \text{if} \quad \alpha|F = \mathrm{id}, \\ (Q_1)^\alpha = \zeta_1 \cdot Q_2, & (Q_2)^\alpha = \zeta_2 \cdot Q_3, \quad (Q_3)^\alpha = \zeta_3 \cdot Q_1, \quad \text{if} \quad \alpha|F = \sigma_2, \\ (Q_1)^\alpha = \zeta_1 \cdot Q_3, & (Q_2)^\alpha = \zeta_2 \cdot Q_1, \quad (Q_3)^\alpha = \zeta_3 \cdot Q_2, \quad \text{if} \quad \alpha|F = \sigma_3 \end{cases}$$

with roots of unity ζ_1, ζ_2 and ζ_3 which depend on α. We compute $L_K(1, \lambda_{a,b,c}^{(i)})$ by Shimura's method explained in [Y2], §3. We obtain (listing to the 20-th decimal place)

$$Q_1 = 19.59569210489183464935\cdots, \qquad Q_2 = 0.04269484846107960286\cdots,$$
$$Q_3 = 4.61030480431463779839\cdots.$$

We find easily that $Q_1Q_2Q_3 = \dfrac{27}{7}$. The coincidence is to the 33-rd decimal place. We have $V(c_2) = -V(c_1)$, $W(c_2) = -W(c_1)$,

$$(8.4) \quad \begin{aligned} V(c_1) - V(c_2) &= \frac{1}{2^2 \cdot 3^2 \cdot 167^2}\{(-85174 - 389193\,\omega_1 + 133671\omega_2)\log\epsilon \\ &\qquad + (-313883 - 283680\,\omega_1 - 657969\,\omega_2)\log\eta\}, \\ W(c_1) - W(c_2) &= -\frac{4}{3}\log 167. \end{aligned}$$

To identify Q_i's with algebraic numbers is more difficult than the case of quadratic fields. We proceed in the following way. Note that $(7) = (2-\omega_1)^3$ in F. From the experience in the quadratic field case, we suppose that

$$(*) \qquad 167^{-4/3} Q_1 = \epsilon^a \eta^b \cdot \frac{3}{2-\omega_1} (\delta^{(1)})^\alpha (\delta^{(2)})^\beta (\delta^{(3)})^\gamma$$

[6]Barnes did not estimate the remainder term explicitly. Probably this is the reason for that Shintani [Sh2] gave another asymptotic expansion of $\log \frac{\Gamma_2(z,\omega)}{\rho_2(\omega)}$ with an explicit estimate of the remainder term. The author finds an elementary method to estimate the remainder term of Barnes' asymptotic expansion of $\log \frac{\Gamma_r(z,\omega)}{\rho_r(\omega)}$ if $\omega_i/\omega_j \in \overline{\mathbf{Q}} - \mathbf{Q}$ for all $1 \le i < j \le r$. This result includes classical estimate of the remainder term of Stirling's series ([WW], p. 252) as a special case and is quite practical for our present purpose. The details will be exposed elsewhere.

where α, β, $\gamma \in \mathbf{Z}$ satisfy $\alpha+\beta+\gamma=-4$ and a, $b \in (2^2\cdot 3^2\cdot 167^2)^{-1}\mathbf{Z}$. Then, by the action of σ_2, we would have

$$(**)\qquad 167^{-4/3}Q_2=\eta^a(\epsilon\eta)^{-b}\cdot\frac{3}{2-\omega_2}(\delta^{(2)})^\alpha(\delta^{(3)})^\beta(\delta^{(1)})^\gamma.$$

When α, β and γ are given, we can solve $(*)$ and $(**)$ and obtain a and b. We search α, β, $\gamma\in\mathbf{Z}$ so that $2^2\cdot 3^2\cdot 167^2 a$ is close to an integer for the solution (a,b). This procedure works well for $\alpha=-4$, $\beta=\gamma=0$ and we find

$$a=\frac{901097}{2\cdot 3^2\cdot 167^2},\qquad b=\frac{927649}{2\cdot 3^2\cdot 167^2}.$$

Thus we obtain identifications

$$\begin{aligned}
Q_1&=167^{4/3}\cdot\epsilon^{901097/2\cdot 3^2\cdot 167^2}\eta^{927649/2\cdot 3^2\cdot 167^2}\cdot\frac{3}{2-\omega_1}\frac{1}{\delta^4}\\
&=\sqrt[3]{167}\cdot\epsilon^{901097/2\cdot 3^2\cdot 167^2}\eta^{927649/2\cdot 3^2\cdot 167^2}\frac{23395929-8832486\,\omega_1+13899651\omega_2}{7\cdot 167^3},\\
Q_2&=167^{4/3}\cdot\epsilon^{-927649/2\cdot 3^2\cdot 167^2}\eta^{-26552/2\cdot 3^2\cdot 167^2}\cdot\frac{3}{2-\omega_2}\frac{1}{(\delta^{(2)})^4}\\
&=\sqrt[3]{167}\cdot\epsilon^{-927649/2\cdot 3^2\cdot 167^2}\eta^{-26552/2\cdot 3^2\cdot 167^2}\frac{9496278-13899651\omega_1-22732137\omega_2}{7\cdot 167^3},\\
Q_3&=167^{4/3}\cdot\epsilon^{26552/2\cdot 3^2\cdot 167^2}\eta^{-901097/2\cdot 3^2\cdot 167^2}\cdot\frac{3}{2-\omega_3}\frac{1}{(\delta^{(3)})^4}\\
&=\sqrt[3]{167}\cdot\epsilon^{26552/2\cdot 3^2\cdot 167^2}\eta^{-901097/2\cdot 3^2\cdot 167^2}\frac{32228415+22732137\omega_1+8832486\,\omega_2}{7\cdot 167^3}.
\end{aligned}$$

The numerical values coincide with these algebraic numbers to the 33-rd decimal place. Concerning the exponents of ϵ and η, we find (cf. (8.4))

$$\begin{aligned}
&2\cdot 901097+(-85174-389193\,\omega_1+133671\omega_2)\equiv 0\mod \mathfrak{p}'^2\mathfrak{p}''^2,\\
&2\cdot 927649+(-313883-283680\,\omega_1-657969\omega_2)\equiv 0\mod \mathfrak{p}'^2\mathfrak{p}''^2,
\end{aligned}$$

where $\mathfrak{p}'=(\delta^{(2)})$, $\mathfrak{p}''=(\delta^{(3)})$. This phenomenon is quite similar to Examples 1 ~ 7, especially to Example 2.

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