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Koecher-Maaß Dirichlet series for Eisenstein series of Klingen type

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1 Introduction

Let $f(Z)$ be a Siegel modular form of weight k belonging to the symplectic group $\Gamma_n = Sp_n(\mathbf{Z})$. Then $f(Z)$ has a Fourier expansion of the form:

$$f(Z) = \sum_A a_f(A) exp(2\pi i\ tr(AZ)),$$

where A runs over all semi-positive definite half-integral matrices over $\mathbf{Z}$ of degree n and $tr(X)$ denotes the trace of a matrix X. We then define the Koecher-Maaß Dirichlet series $L(f,s)$ by

$$L(f,s) = \sum_A \frac{a_f(A)}{e(A)(\det A)^s},$$

where A runs over a complete set of representatives of $GL_n(\mathbf{Z})$-equivalence classes of positive definite half-integral matrices of degree n, and $e(A)$ denotes the order of the orthogonal group of A. The Koecher-Maaß Dirichlet series can also be obtained as the Mellin transform of F, and therefore its analytic properties are relatively known. As for this, we refer to Maaß [M], and Arakawa [Ar1],[Ar2]. However we had little knowledge about its arithmetic properties. Thus we present the following problem:

Problem 1: Investigate the arithmetic properties of $L(f,s)$.

To this problem, Böcherer and Shulze-Pillot have made a large contribution. As for this, we refer to [B-R1],[B-R2], and [B-R3]. In those papers, they mainly treat the case of Yoshida lifting. In this note, we take another approach to this problem. Namely we consider the Koecher-Maaß Dirichlet series for Eisensetin series of Klingen type; let f be a cusp form of weight k belonging to Γ_r $(0 \leq r \leq n,)$ and define $[f]_r^n(Z)$ as

$$[f]_r^n(Z) = \sum_{M \in \Delta_{n,r} \backslash \Gamma_n} f(M < Z >^*) j(M,Z)^{-k},$$

where $\Delta_{n,r} = \{\begin{pmatrix} * & * \\ O_{n-r,n+r} & * \end{pmatrix} \in \Gamma_n\}$, and for $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n$ let $M < Z >^*$ denote the upper left $r \times r-$block of the matrix $(AZ+B)(CZ+D)^{-1}$ and $j(M,Z) = \det(CZ+D)$. We note that $[1]_0^n(Z)$ is nothing but the Siegel Eisenstein series $E_{n,k}(Z)$ of weight k. We then propose the following problem:

Problem 2. Let $0 \leq r < n$. Then give an explicit form of $L([f]_r^n, s)$ in terms of f.

In [B2] among others Böcherer gave an explicit form of $L([f]_r^2, s)$ for $r = 0, 1$. In [I-K1] we gave an explicit form of $L(E_{n,k}(Z), s)$ for an arbitrary n. We note that $L(E_{n,k}(Z), s)$ is also regarded as the zeta function of prehomogeneous vetcor space. From this point of view, Saito gave a generalization of our result (cf. [Sa]). In relation to Problem 2 we should remark that a certain Dirichlet series attached to f appears in the explicit formula for $L([f]_1^2, s)$ by [B2]. This Dirichlet series is a modification of the Dirichlet series originally defined by Kohnen and Zagier [K-Z], and is of importance in its own right. Böcherer obtained a functionnal equation for the Dirichlet series from a general theory of the Koecher-Maaß Dirichlet series. Hence the following problem seems very interesting.

Problem 3. Investigate the analytic and arithmetic properties of the Dirichlet series related to f appearing in an explicit formula for $L([f]_r^n, s)$.

In this note, we give an answer to Problems 2 and 3 for the case $[f]_1^n$ with f a cusp form belonging to Γ_1 and n even. This also gives a certain generalization of Böcherer's result in [B2].

Now to state our main result, for a non-zero integer m such that $m \equiv 1 \bmod 4$ or $\equiv 0 \bmod 4$, let ψ_m denote the character of the quadratic field K whose discriminant is m. Here we understand that $\psi_1 = 1$. Put

$$\mathcal{F}_n = \{d_0 \in \mathbf{Z}_+; d_0 \text{ is the fundamental discriminant of a quadratic field or } 1\}.$$

For a positive integer $D = D_0 m^2$ with $D_0 \in \mathcal{F}_n$ and $m > 0$, put

$$L_D(s) = L(s, \psi_{(-1)^{n/2}D_0}) \sum_{d|m} \mu(d)\psi_{(-1)^{n/2}D_0}(d) d^{-s} \sum_{c|md^{-1}} c^{1-2s},$$

where $L(s, \psi_{(-1)^{n/2}D_0})$ is Dirichlet L-function attached to $\psi_{(-1)^{n/2}D_0}$, and μ is the Möbius function. Write $L_D(s)$ as

$$L_D(s) = \sum_{m=1}^{\infty} \epsilon_D(m) m^{-s},$$

and for a modular form $f(z) = \sum_{m=1}^{\infty} a(m) exp(2\pi i m z)$ of weight k with respect to Γ_1 put

$$L(f, s, D) = \sum_{m=1}^{\infty} a(m) \epsilon_D(m) m^{-s},$$

and

$$\mathcal{L}(f, \lambda, s) = \sum_{D} L(f, \lambda, s) D^{-s},$$

where D runs over all positive integers such that $(-1)^{n/2} D \equiv 1, 0 \bmod 4$. This type of Dirichlet series was originally introduced by Kohnen and Zagier [K-Z]. Further let $\zeta^+(f, s)$ denote the standard zeta function of f. Note that we have

$$\mathcal{L}(f, \lambda, s) = \zeta^+(f; 2s + 2\lambda - 1)\zeta(2s) \sum_{D_0 \in \mathcal{F}_n} D_0^{-s} \zeta(f; \psi_{(-1)^{n/2}D_0}; \lambda)$$

$$\times \prod_p \{(1 + p^{-2s+k-1-2\lambda} \psi_{(-1)^{n/2}D_0}(p)^2)(1 + p^{-2s+k-2\lambda})$$

$$-a(p)\psi_{(-1)^{n/2}D_0}(p) p^{-2s-\lambda}(1 + p^{k-2\lambda})\},$$

where $\zeta(f; \psi_{(-1)^{n/2}D_0}; s)$ denotes the twisted zeta function of f by $\psi_{(-1)^{n/2}D_0}$.

Theorem 1 *Let n be even. Then we have*

$$L([f]_1^n, s) = 2^{ns} \gamma_{n,k} [\frac{\zeta(f; k - n/2)}{\zeta^+(f; k-1)} \prod_{i=1}^{n/2} \zeta(2s - 2i + 1) \prod_{i=1}^{n/2-1} \zeta(2s - 2k + 2i + 2)$$

$$\times \mathcal{L}(f, k-1, s - k + 3/2)$$

$$+(-1)^{n(n-2)/8} \frac{\zeta(f; k-1)}{\zeta^+(f; k-1)} \zeta(2s - n + 1) \prod_{i=1}^{n/2-1} \zeta(2s - 2i) \prod_{i=1}^{n/2-1} \zeta(2s - 2k + 2i + 1)$$

$$\times \mathcal{L}(f, k - n/2, s - k + (n+1)/2)],$$

where $\gamma_{n,k}$ is a constant depending only on n and k.

By the above theorem combined with a general theory of $L([f]_1^n, s)$ by Maaß [M], we obtain

Corollary. *Put*

$$\mathbf{L}(f,n,s)$$
$$= \pi^{-2s}\zeta(2s+2k-2n)\Gamma(s+k-(n+1)/2)\Gamma(s+k-(n+2)/2)\mathcal{L}(f;k-n/2,s).$$

Then $\mathbf{L}(f,n,s)$ can be continued analytically to a meromorphic function of s in the whole complex plane, and has the following functional equation:

$$\mathbf{L}(f,n,n+1-s-k) = \mathbf{L}(f,n,s).$$

Remark 1. If $n=2$, the two terms inside the brackets in Theorem 1 coincide with each other, and unify in one term. This is nothing but Böcherer's result in [B2].

Remark 2. A similar formula holds for any $1 \leq r < n$. In particular we obtain an explicit formula for $r=1$ and n odd.

Theorem 1 cannot be derived directly from the commutativity of Siegel operator and Hecke operators. The main idea of the proof is to relate the Koecher-Maaß Dirichlet series for a modular form F to the standard zeta function for F. To be more precise, in Section 2 on the set of half-integral matrices we introduce a certain arithmetic function, which we call the squared Möbius function, and give a certain induction formula for the number of representaions of half-integral matrices (cf. Theorem 2). In Section 3, we express the Koecher-Maaß Dirichlet series $L(F,s)$ in terms of the squared Möbius function, the standard zeta function, and the "primitive coefficients" of F. (cf. Theorem 3.1). The primitive coeffcients of Eisenstein series of Klingen type is well-known (cf. Proposition 4.1). Thus, in Section 4, applying Theorem 3.1 to $F=[f]_1^n$ with f a cusp form belonging to Γ_1, we express $L([f]_1^n, s)$ as a sum of Euler products (cf. Theorem 4.2), and complete the proof in the final section. For the detail, see [I-K2] and[I-K3].

2 Squared Möbius function for half-integral matrices

For an integral domain R of characteristic 0 let $\mathcal{H}_n(R)$ denote the set of half-integral matrices over R. Further let $\mathcal{H}_n(\mathbf{Z})_{>0}$ (resp. $\mathcal{H}_n(\mathbf{Z})_{\geq 0}$) denote the set of positive definite (resp. semi-positive definite) half-integral matrices over $\mathbf{Z}$. Throughout this note, for two half-integral matrices A and B over $\mathbf{Z}_p$ of degree n we write $A \sim B$ if there is a unimodular matrix X of degree n with entries in $\mathbf{Z}_p$ such that ${}^tXAX = B$. Further for two square matrices U and V we write $U \perp V = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}$. A half-integral matrix A over $\mathbf{Z}_p$ is called non-degenerate modulo p if the quadratic form $\bar{A}[\mathbf{x}]$ over $\mathbf{Z}_p/p\mathbf{Z}_p$ associated with A is non-degenerate. We should remark that A is non-degenerate modulo p if and only if A is unimodular in the case of $p \neq 2$, where as it is non-degenerate modulo 2 if and only if $A = \frac{1}{2}U$ or $A \sim \frac{1}{2}U \perp c$ with U an even-integral unimodular matrix and $c \in \mathbf{Z}_2^*$ in the case of $p = 2$. To define the arithmetic function in the introduction, first we define

$$\mathcal{K}_n'(\mathbf{Z}_p) = \{A \in \mathcal{H}_n(\mathbf{Z}_p); A \sim V_0 \perp pV_1 \text{ with } V_0, V_1 \text{ non}-\text{degenerate modulo } p\}.$$

Next let $p = 2$. We then define a subset $\mathcal{K}_n''(\mathbf{Z}_2)$ of $\mathcal{H}_n(\mathbf{Z}_2)$ by

$$\mathcal{K}_n''(\mathbf{Z}_2) = \{A \in \mathcal{H}_n(\mathbf{Z}_2); A \sim \frac{1}{2}V_0 \perp V \perp V_1 \text{ with } V_0, V_1 \text{ even}-\text{integral matrices}$$

$$\text{and } V \text{ a diagonal unimodular matrix of degree 2 such that } \det V \equiv 1 \bmod 4\},$$

and

$$\mathcal{K}_n(\mathbf{Z}_p) = \mathcal{K}_n'(\mathbf{Z}_2) \cup \mathcal{K}_n''(\mathbf{Z}_2) \text{ or } \mathcal{K}_n'(\mathbf{Z}_p)$$

according as $p = 2$ or not. For a p-adic number c put

$$\chi_p(c) = 1, -1 \text{ or } 0$$

according as $\mathbf{Q}_p(\sqrt{c}) = \mathbf{Q}_p$, $\mathbf{Q}_p(\sqrt{c})/\mathbf{Q}_p$ is quadratic unramified, or $\mathbf{Q}_p(\sqrt{c})/\mathbf{Q}_p$ is quadratic ramified. Further for a symmetric matrix A of even degree n with entries in $\mathbf{Q}_p$ we put

$$\xi_p(A) = \chi_p((-1)^{n/2} \det A).$$

For a non-degenerate half-integral matrix A we define $\sigma_p(A)$ as follows; first assume that A belongs to $\mathcal{K}_n'(\mathbf{Z}_p)$. Then we have $A \sim \frac{1}{2}V_0 \perp \frac{p}{2}V_1$ with V_0, V_1 non-degnerate matrices modulo p of dgree n_0 and n_1, respectively. Then we put

$$\sigma_p(A) = \begin{cases} (-1)^{n_1/2}\xi_p(V_1)p^{(n_1^2-2n_1)/4} & \text{if } n_1 \text{ is even} \\ (-1)^{(n_1-1)/2}p^{(n_1-1)^2/4} & \text{if } n_1 \text{ is odd.} \end{cases}$$

Next let $p = 2$ and assume that A belongs to $\mathcal{K}''_n(\mathbf{Z}_2)$. Then we have $A \sim \frac{1}{2}V_0 \perp V \perp V_1$ with V_0, V_1 even-integral unimodular matrices of dgree n_0 and n_1, respectively, and V a unimodular diagonal matrix of degree 2 such that $\det V \equiv 1 \bmod 4$. Then we put

$$\sigma_p(A) = (-1)^{n_1/2}p^{n_1^2/4}.$$

Finally if A does not belong to $\mathcal{K}_n(\mathbf{Z}_p)$ we put $\sigma_p(A) = 0$. For a non-degenerate half-integral matrix A over $\mathbf{Z}$ put

$$\sigma(A) = \prod_p \sigma_p(A).$$

By definition $\sigma(A)$ depends only on the genus of A. Put $\mathcal{K}_n(\mathbf{Z}) = \mathcal{H}_n(\mathbf{Z}) \cap \prod_p \mathcal{K}_n(\mathbf{Z}_p)$. Then by definition we have $\sigma(A) = 0$ for $A \notin \mathcal{K}_n(\mathbf{Z})$. We remark that $\mathcal{H}_1(\mathbf{Z}) \cap GL_1(\mathbf{Q})$ can be identified with the set of all non-zero integers. Further by definition we have $\sigma(a) = 1$ or 0 according as a is square free or not, and therefore, σ is nothing but the square of the usual Möbius function in case $n = 1$. Thus we call σ the squared Möbius function over $\mathcal{H}_n(\mathbf{Z})$. Now for a non-degenerate positive definite half-integral matrices A and B of degree n over $\mathbf{Z}$ put

$$G(A, B) = \sum_{A' \in \mathcal{G}(A)} \frac{a(A', B)}{a(A', A')},$$

where $\mathcal{G}(A)$ denotes the set of equivalence classes belonging to the genus of A, and $a(A, B)$ the representation number of B by A. As is well-known $G(A, B)$ is determined by $\mathcal{G}(A)$ and $\mathcal{G}(B)$. Then we have

Theorem 1. *Let A be a positive definite half-integral matrix of degree n over $\mathbf{Z}$. Then we have*

$$\sum_{A_0} \sigma(A_0)G(A_0, A) = 1,$$

where A_0 runs over all genera of positive definite half-integral matrices of degree n.

For a proof, see [I-K2].

3 Koecher-Maaß Dirichlet series and the standard zeta function

From now on for a $p-$adic number c let $\nu(c) = \nu_p(c)$ denote the normalized additive valuation on $\mathbf{Q}_p$. Now for a Siegel modular form f of weight k belonging to Γ_n we define the Koecher-Maaß Dirichlet series $L(f,s)$ for f as in Introduction. For a non-degenerate half-integral matrix A over $\mathbf{Z}_p$ let $r = r_p(A)$ denote the rank of a maximal totally singular subspace of the quadratic space over $\mathbf{Z}_p/p\mathbf{Z}_p$ associated with A. If $n-r$ is even, we have $A \sim \frac{1}{2}U_0 \perp \frac{p}{2}U_1$ with U_0 an even integral unimodular matrix of degree $n-r$ and U_1 an even integral matrix. We then put $\eta_p(A) = \xi_p(\frac{1}{2}U_0)$. Now we define a polynomial $B_p(v;A)$ by

$$B_p(v,A)$$
$$= \begin{cases} (1+v)(1-\eta_p(A)p^{-(n-r)/2}v)\prod_{i=1}^{(n-r)/2-1}(1-p^{-2i}v^2) & \text{if } n-r \text{ is even} \\ (1+v)\prod_{i=1}^{(n-r-1)/2}(1-p^{-2i}v^2) & \text{if } n-r \text{ is odd.} \end{cases}$$

Here we make the convention that $B_p(v,A) = 1$ if $r = n$. For a non-degenerate half-integral matrix A over $\mathbf{Z}$ put

$$B(s;A) = \prod_p B_p(p^{-s};A).$$

For a positive definite half-integral matrix A of degree n over $\mathbf{Z}$, put

$$M(A) = \sum_{A' \in \mathcal{G}(A)} \frac{1}{a(A',A')}.$$

Now let A and B be non-degenerate half-integral matrices of degree n over $\mathbf{Z}$. We say A dominates B over $\mathbf{Z}$ if there is a square matrix D with entries in R such that $B = {}^tDAD$, and define a finite Euler product $T(s;A,B)$ by

$$T(s;A,B) = \prod_p \prod_{i=1}^{m_p}(1-p^{-s-n+i}).$$

where $m_p = 1/2(\nu_p(\det B) - \nu_p(\det A))$. We also put $T(s;A,B) = 1$ if A does not dominate B over $\mathbf{Z}$.

Now following [B-R], we define a "primitive" Fourier coefficient $a_f^*(A)$ by means of the relation:

$$a_f(A) = \sum_D a_f^*({}^tD^{-1}AD^{-1}),$$

where D runs over a complete set of representatives of left $GL_n(\mathbf{Z})$-equivalence classes of non-degenerate square matrices of degree n, and put

$$G_f^*(C_0) = \sum_{C \in \mathcal{G}(C_0)} \frac{a_f^*(C)}{a(C,C)}.$$

Put

$$K(f,s) = \sum_{A_0} \frac{\sigma(A_0)B(2s+1-k, A_0)}{(\det A_0)^s}$$
$$\times M(A_0) \sum_{C_0} \frac{G(C_0,A_0)G_f^*(C_0)}{M(C_0)} T(2s+2-2k; C_0, A),$$

where A_0 and C_0 run over all genera of positive definite matrices of degree n.

Now let $\mathbf{L}_{np} = \mathbf{L}(GSp_n(\mathbf{Q}_p), Sp_n(\mathbf{Z}_p))$ be the Hecke algebra associated with the pair $(GSp_n(\mathbf{Q}_p), Sp_n(\mathbf{Z}_p))$ for each prime p. Assume that f is an eigen function for all the Hecke operators, and for each prime p let $\alpha_{0,p}, \alpha_{1,p}, \ldots, \alpha_{n,p}$ denote the Satake parameters of $\mathbf{L}_{np}$ determined by f. We then define the standard zeta function $\zeta^+(f,s)$ of f by

$$\zeta^+(f,s) = \prod_p \{\prod_{i=1}^n (1-\alpha_{i,p}p^{-s})(1-\alpha_{i,p}^{-1}p^{-s})\}^{-1}.$$

We note that the analytic and arithmetic properties of $\zeta^+(f,s)$ are fairly well known (cf. [An2],[B1],[Sh]). Then by [An1, Theorem 1],[An2, Theorem 4.3.19] and Theorem 1 we obtain

Theoerem. 3.1 *Let $A \in \mathcal{K}_n(\mathbf{Z})$. We have*

$$L(f,s) = \zeta^+(f, 2s+1-k)K(f,s)$$

An explicit form of $M(A)$ is well known (cf. [Ki2, Theorem 5.6.3]). To give an explicit formula of $G(A,B)$ for $A, B \in \mathcal{K}_n(\mathbf{Z}) \cap \mathcal{H}_n(\mathbf{Z})_{>0}$, let $\alpha_p(A,B)$ be the local density representing B by A over $\mathbf{Z}_p$, and put

$$G_p(A,B) = \frac{\alpha_p(A,B)}{\alpha(A,A)} p^{(-\nu_p(\det B)+\nu_p(\det A))/2}.$$

Then by Siegel's main theorem on quadratic forms we have

$$G(A,B) = \prod_p G_p(A,B)$$

(cf. [Ki2, Theorem 6.8.1]).

Now for a non-degenerate matrix U modulo p of degree n put

$$J(i,U,p)$$
$$=\begin{cases} (p^{n/2}-\xi_p(U))(p^{n/2-i}+\xi_p(U))\prod_{j=1}^{i-1}(p^{n-2j}-1) & \text{if } n \text{ is even} \\ \prod_{j=1}^{i}(p^{n-2j+1}-1) & \text{if } n \text{ is odd.} \end{cases}$$

Further put $\phi_i(x)=\prod_{j=1}^{i}(x^j-1)$. Then the following proposition gives us an explicit formula of $G_p(A,B)$, and therefore that of $G(A,B)$:

Proposition 3.2 *Let $A,B\in\mathcal{K}_n(\mathbf{Z}_p)$, and $i=(\nu_p(\det B)-\nu_p(\det A))/2$. Assume that A dominates B over $\mathbf{Z}_p$.*

(1) *Let $B\sim\frac{1}{2}U_0\perp\frac{p}{2}U_1$ with U_0,U_1 non-degenerate modulo p. Then we have*

$$G_p(A,B)=\frac{J(i;\frac{1}{2}U_1;p)}{\phi_i(p)}.$$

(2) *Let $p=2$ and $B\sim\frac{1}{2}U_0\perp V\perp U_1$ with U_0,U_1 even unimodular and V a diagonal unimodular matrix of degree 2 such that* $\det V\equiv 1 \bmod 4$. *Then we have*

$$G_p(A,B)=\frac{J(i;\frac{1}{2}U_1\perp 1;p)}{\phi_i(p)}.$$

Thus, if we get an explicit form of $G_f^*(C_0)$, we will know a lot of information on $L(f,s)$. In fact, in the case where f is Klingen-Eisenstein series, by [B-R] or [Ki1], we know an explicit form of $G_f^*(C_0)$, and therefore give an explicit form of $L(f,s)$ by the above theorem. We also remark that we have given an explicit form of $L(f,s)$ for Siegel-Eisenstein series f by a different method from this note (cf. [I-K1]).

4 Koecher-Maaß Dirichlet series for Eisenstein series of Klingen type

Let f be a Siegel cusp form of weight k belonging to Γ_r and $[f]_r^n$ the Klingen's Eisenstein series of degree n attached to f. Then f and $[f]_r^n$ have the following Fourier expansions:

$$f(z)=\sum_{C\in\mathcal{H}_r(\mathbf{Z})_{\geq 0}} b(C)exp(2\pi i tr(Cz)),$$

$$[f]_r^n(Z) = \sum_{T \in \mathcal{H}_n(\mathbf{Z})_{\geq 0}} a_{n,f}(T) exp(2\pi i \ tr(TZ)).$$

For two positive definite half-integral matrices B and C of degree m and n, respectively, over $\mathbf{Z}$ put

$$G(B,C)^* = \sum_{B' \in \mathcal{G}(B)} \frac{a(B',C)^*}{a(B',B')},$$

where $a(B',C)^*$ denotes the number of primitive represention of C by B. Then rewriting [B-R, Theorem 1] we have

Prposition 4.1. *We have*

$$G_{n,f}(B)^* = a_{n,k}(B)^* \sum_C \frac{G(B,C)^* b(C)^*}{(\det C)^{k-(r+1)/2} a(C,C) a_{r,k}(C)^*},$$

where $a_{n,k}(B)^*$ and $a_{r,k}(C)^*$ denote the primitive Fourier coefficients of Siegel-Eisenstein series of degree n and r, respectively.

Now let $r=1$. For an element A of $\mathcal{H}_n(\mathbf{Z}_p)$ and a non-zero p-adic integer, put

$$\begin{aligned} &H_p(s;A;e) \\ &= \frac{p^{((n+1)/2-s)\nu(\det A)} \sigma_p(A) B_p(p^{-(2s-k+1)};A)}{\alpha_p(A,A)} \sum_{C_0} p^{(2k-1-n)\nu(\det C_0)/2} G_p(C_0,A) \\ &\quad \times T_p(p^{-(2s-k+1)};C_0,A)\alpha_p(H_k,C_0)^* p^{-\nu(\det C_0)/2} \alpha_p(C_0,e)^*, \end{aligned}$$

and for a non-zero p-adic number d_0, and a function ω on $\mathcal{H}_n(\mathbf{Z}_p)$ put

$$H_p(s;d_0;\omega,e) = \sum_{r=0}^{\infty} \sum_{\det A = p^{2r-2[n/2]\delta_{2p}} d_0} \omega(A) H_p(s;A;e),$$

where for a half-integral matrix U and $V, \alpha_p(U,V)^*$ denotes the primitive local density representing V by U, and $G_p(C_0,A)$ is the one defined in Section 1. Let ι_p be a constant function on $\mathcal{H}_n(\mathbf{Z}_p)$ taking the value 1, and h_p the Hasse invariant on $\mathcal{H}_n(\mathbf{Z}_p)$. We note that $h_p(C_0)$ for $C_0 \in \mathcal{H}_n(\mathbf{Z}_p)$ is the same as that of A if C_0 dominates A over $\mathbf{Z}_p$. Let A be a positive definite half-integral matrix of degree n over $\mathbf{Z}$. If n is even, then $\det A$ can be expressed as $d_0 f^2$ with positive integers d_0 and f such that $\nu_p(d_0) \leq 1$ for $p \neq 2$, and $(-1)^{n/2} d_0 \equiv 1$ or $\equiv 0 \bmod 4$. If n is odd, $\det A$ can be expressed as $d_0 f^2$ with a positive integer f and a square free positive integer d_0. Thus by Theorem 3.1 and Proposition 4.1, we have

Theorem 4.2. (1) *Let n be even. Then we have*

$$K([f]_1^n, s) = \alpha_{nk} \sum_{e=1}^{\infty} b(e)^* B(k-1, e) e^{n/2-k} \sum_{d_0 \in \mathcal{F}_n} (\prod_p H_p(s; d_0; \iota_p; e) + \prod_p H_p(s; d_0; h_p; e)),$$

where $\mathcal{F}_n$ is the set defined in Section 1, and α_{nk} is a constant depending only on n and k.
(2) *Let n be odd. Then we have*

$$K([f]_1^n, s) = \beta_{nk} \sum_{e=1}^{\infty} b(e)^* B(k-1, e) e^{n/2-k} \sum_{d_0} (\prod_p H_p(s; d_0; \iota_p; e) + \prod_p H_p(s; d_0; h_p; e)),$$

where d_0 runs over all square free positive integers, and β_{nk} is a constant depending only on n and k.

5 Proof of Theorem 1

In this section let n be even. Then by Proposition 3.2, Theorem 4.2, and [Ki2, Theorem 5.6.3] combined with some combinatorial technique, we obtain

Theorem 5.1 *Let n be even, and $D_0 \in \mathbf{Z}_p^*$ with p odd, or $D_0 \in \mathbf{Z}_2^*$ such that $(-1)^{n/2} D_0 \equiv 1 \bmod 4$. Put $Q(e, D_0) = 1 - p^{-n+2}$ or 1 according as $e \equiv 0 \bmod p^2$ or not, and $R(e, D_0) = 1 + \delta p^{-n/2+1}$ or 1 according as $e \equiv 0 \bmod p$ or not, where $\delta = \chi_p((-1)^{n/2} D_0)$. Further put $\Phi_{nk} = \frac{(1-p^{-k}) \prod_{i=1}^{n/2-1} (1-p^{-2k+2i})}{\phi_{n/2-1}(p^{-2})}$. Then we have*
(1)

$$H_p(s; D_0; \iota_p, e) = 2^{\delta_{2,p} ns} p^{(n-2)\nu(e)/2} \Phi_{nk}$$
$$\times [Q(e, D_0) p^{-2s+2k-3} (1 - p^{n-2k})(1 + p^{-k+2}) \prod_{i=0}^{n/2-2} (1 - p^{2i-n-1+2k-2s})(1 - p^{2i+2-2s})$$
$$+ R(e, D_0)(1 + \delta p^{n/2-k} \delta) \prod_{i=0}^{n/2-1} (1 - p^{2i-n-1+2k-2s})(1 - p^{2i-2s})].$$

(2)

$$H_p(s; D_0; h_p, e) = (-1, -1)_p^{n(n+2)/8} 2^{\delta_{2,p} ns} p^{(n-2)\nu(e)/2} (1 + \delta p^{n/2-k}) \Phi_{nk}$$
$$\times [Q(e, D_0) \delta p^{-2s+2k-n/2-2} (1 - p^{n/2-k} \delta)(1 + p^{n-k}) \prod_{i=0}^{n/2-2} (1 - p^{2i-n+2k-2s})(1 - p^{2i+1-2s})$$
$$+ R(e, D_0) \{ \prod_{i=0}^{n/2-1} (1 - p^{2i-n+2k-2s})(1 - p^{2i+1-2s})$$

$$+(1+p^{-k+1})p^{-2s+2k-2}(1-p^{n/2-k}\delta)(1-\delta p^{-n/2})\prod_{i=0}^{n/2-2}(1-p^{2i-n+2k-2s})(1-p^{2i+1-2s})\}].$$

Theorem 5.2 *Let n be even, and $D_0 \in p\mathbf{Z}_p^*$ with p odd, or $D_0 \in 4\mathbf{Z}_2^*$ such that $(-1)^{n/2}4^{-1}D_0 \equiv 3 \bmod 4$ or $D_0 \in 8\mathbf{Z}_2^*$. Put $l_0 = \nu(D_0)$ and $d_0 = 2^{-\delta_{2,p}n}p^{-l_0}D_0$. Further put $\Psi_{nk} = \frac{(1-p^{-k})\prod_{i=1}^{n/2}(1-p^{-2k+2i})}{\phi_{n/2-1}(p^{-2})}$.*
(1) Put $Q(e,D_0) = 1-p^{-n+2}$ or 1 according as $e \equiv 0 \bmod p^2$ or not. Then we have

$$H_p(s;D_0;\iota_p,e) = 2^{\delta_{2,p}ns}p^{(-s+k-3/2)l_0}(1+p^{-2s+k-1})\Psi_{nk}$$
$$\times Q(e,D_0)\prod_{i=0}^{n/2-1}(1-p^{2i-n-1+2k-2s})(1-p^{2i-2s}).$$

(2.1) Let $p \neq 2$, and $R(e,D_0) = (\frac{(-1)^{n/2}e}{p}), (\frac{-p^{-2}eD_0}{p})$ or 0 according as $e \in \mathbf{Z}_p^*, \in p\mathbf{Z}_p^*$ or not, where $(\frac{*}{p})$ denotes Legendre symbol. Then we have

$$H_p(s;D_0;h_p,e) = p^{-s+k-(n+1)/2}R(e,D_0)(1-p^{n-2k})(1+p^{n-k})\Psi_{nk}$$
$$\times(1+p^{-2s+k-1})\prod_{i=0}^{n/2-2}(1-p^{2i-n+2k-2s})(1-p^{2i+1-2s}).$$

(2.2) Let $p=2$ and $e = 2^r e_0$ with $(2,e_0)=1$. Put

$$R(e,D_0) = (-1)^{n(n-2)/8}(\frac{2^r(-1)^{n/2}}{e_0})(\frac{2^{m_0}(-1)^{n/2}(-1)^{(e-1)/2}}{d_0}) \text{ or } 0$$

according as $m_0 \leq 1$ or not, where $(\frac{*}{*})$ denotes the Jacobi symbol. Then we have

$$H_p(s;D_0;h_p,e) = 2^{ns+(-s+k-(n+1)/2)r}R(e,D_0)(1-p^{n-2k})(1+p^{n-k})\Psi_{nk}$$
$$\times(1+p^{-2s+k-1})\prod_{i=0}^{n/2-2}(1-p^{2i-n+2k-2s})(1-p^{2i+1-2s}).$$

Proof of Theorem 1. By Theorems 5.1 and 5.2 combined with Theorem 4.2, we have

$$K([f]_1^n,s) = 2^{ns}\gamma_{nk}[\frac{(-1)^{n(n-2)/8}\zeta(f;k-n/2)}{\zeta^+(f;k-1)\prod_{i=0}^{n/2-2}\zeta(2s-2i+2)\prod_{i=1}^{n/2-1}\zeta(2s-2k+n-2i+1)}$$

$$\times \sum_{D_0} D_0^{-s+k-3/2}\zeta(f;\psi_{(-1)^{n/2}D_0};k-1)$$

$$\times \prod_p \{(1+p^{-2s+k-2}\psi_{(-1)^{n/2}D_0}(p)^2)(1+p^{-2s+k-1})-a(p)\psi_{(-1)^{n/2}D_0}(p)p^{-2s+k-2}(1+p^{2-k})\}$$

$$+\frac{\zeta(f;k-1)}{\zeta^+(f;k-1)\prod_{i=0}^{n/2-2}\zeta(2s-2i-1)\prod_{i=1}^{n/2-2}\zeta(2s-2k-2i+n)}$$

$$\times \sum_{D_0} D_0^{-s+k-(n+1)/2}\zeta(f;\psi_{(-1)^{n/2}D_0};k-n/2)$$

$$\times \prod_p \{(1+p^{-2s+k-2}\psi_{(-1)^{n/2}D_0}(p)^2)(1+p^{-2s+k-1})-a(p)\psi_{(-1)^{n/2}D_0}(p)p^{-2s+k-n/2-1}(1+p^{n-k})\}].$$

We note that

$$\zeta^+([f]_1^n;2s-k+1)=\zeta^+(f,2s-k+1)\prod_{i=1}^{n-1}\zeta(2s-i)\zeta(2s-2k+i+2).$$

Thus we complete the assertion by Theorem 3.1 keeping the remark before Theorem 1 in mind.

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