数理解析研究所講究録
1052 巻 1998 年 74-80

Differential Operators and Jacobi Forms

YoungJu Choie *

Department of Mathematics,

Pohang Institute of Science & Technology,

Pohang,790-784, Korea

1 Introduction

Classically, there are many connections between differential operators and the theory of elliptic modular forms and many interesting results have been explored. In particular, it has been known for some time how to obtain an elliptic modular form from the derivatives of N elliptic modular forms. The case $N=1$ has already been studied in detail by R. Rankin in 1956 [9]. For $N=2$ H.Cohen has constructed certain covariant bilinear operators which he used to obtain modular forms with interesting Fourier coefficients [6]. Later, these operators were called Rankin-Cohen operators by D. Zagier who studied their algebraic relations [10].
In this talk, we show how to obtain a Jacobi form from N Jacobi forms using heat operators. In particular, we introduce the results which relate Rankin-Cohen type bilinear operators of elliptic modular forms of half integral weight using theta-series expansion of a Jacobi form. In particular, we give an explicit description of covariant bilinear operators for Jacobi forms. Moreover, we intoruduce the results which relate Rankin-Cohen type bilinear operators on the Jacobi forms to those of half-integral weight elliptic modular forms.

2 Jacobi Forms

We first give the definition of Jacobi forms and the heat operator (as a general reference for Jacobi forms we refer to [8]). Denote by $\mathcal{H}$ the complex upper half plane and define, for holomorphic functions $f:\mathcal{H}\times\mathbb{C}\to\mathbb{C}$ and integers k and m, the slash operators

$$
\begin{aligned}
(f|_{k,m}M)(\tau,z) &= (c\tau+d)^{-k}e^{2\pi im(\frac{-cz^2}{c\tau+d})}f(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}),\\
(f|_mY)(\tau,z) &= e^{2\pi im(\lambda^2\tau+2\lambda z)}f(\tau,z+\lambda\tau+\nu)
\end{aligned}
$$

*The research was partially supported by BSRI 97-1431 and KOSEF 971-0101-007-2

where $\tau \in \mathcal{H}$, $z \in \mathbb{C}$, $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \subset \mathrm{SL}(2,\mathbb{Z})$ and $Y = (\lambda, \nu) \in \mathbb{Z}^2$. Here Γ is a subgroup of $SL(2,\mathbb{Z})$ with finite index.
Using these slash actions the definition of Jacobi forms is as follows.

Definition 2.1 *A Jacobi form of weight k and index m ($k, m \in \mathbb{N}$) is a holomorphic function $f : \mathcal{H} \times \mathbb{C} \to \mathbb{C}$ satisfying*

$$(f|_{k,m}M)(\tau, z) = f(\tau, z), \qquad (f|_m Y)(\tau, z) = f(\tau, z)$$

for all $M \in SL(2,\mathbb{Z})$ and $Y \in \mathbb{Z}^2$ and such that it has a Fourier expansion of the form

$$f(\tau, z) = \sum_{\substack{n = 0 \\ r \in \mathbb{Z}, r^2 \leq 4nm}}^{\infty} c(n,r) q^n \zeta^r,$$

where $q = e^{2\pi i \tau}$ and $\zeta = e^{2\pi i z}$. If f has a Fourier expansion of the same form but with $r^2 < 4nm$ then f is called a Jacobi cusp form of weight k and index m.

We denote by $J_{k,m}$ the (finite dimensional) vector space of all Jacobi forms of weight k and index m and by $J_{k,m}^{cusp}$ the vector space of all Jacobi cusp forms of weight k and index m.
Our main result (Theorem 3.5) involves the heat operator which has already been studied in [8] to connect Jacobi forms and elliptic modular forms and in ref. [2, 3] in the context of bilinear differential operators.

Definition 2.2 *Let $f(\tau, z)$ be a differentiable function from $\mathcal{H} \times \mathbb{C}$ to $\mathbb{C}$ with $\mathcal{H}$ the complex upper half plane. Then, for any complex number m, define a differential operator L_m by*

$$L_m(f) = \left(8\pi i m \partial_\tau - \partial_z^2\right)(f).$$

3 Covariant Differential Operators

In this section we show how to construct a Jacobi form from N Jacobi forms using heat operators. We first state the following result which shows how to construct a Jacobi form from a certain formal power series.

Theorem 3.1 *[3] Let $\tilde{f}(\tau, z; X) = \sum_{\ell=0}^{\infty} \chi_{\ell,m}(\tau, z) X^\ell$ be a formal power series in X satisfying*

$$\tilde{f}(M\tau, \frac{z}{c\tau + d}; \frac{X}{(c\tau + d)^2}) = (c\tau + d)^k e^{2\pi i m \frac{cz^2}{c\tau + d}} e^{8\pi i m \frac{cX}{c\tau + d}} \tilde{f}(\tau, z; X),$$

for any $M = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma$. Furthermore, assume that $\chi_{\ell,m}$ is holomorphic in $\mathcal{H} \times \mathbb{C}$, satisfies

$$(\chi_{\ell,m}|_{(m)} Y)(\tau, z) = \chi_\ell(\tau, z) \text{ for all } Y \in Z^2,$$

and has a Fourier expansion of the form

$$\chi_{\ell,m}(\tau,z)=\sum_{n,r\in\mathbb{Z},r^2\le 4mn} c(n,r)e^{2\pi in\tau}e^{2\pi irz}.$$

Then ξ_ℓ, defined as

$$\xi_\ell(\tau,z)=\sum_{j=0}^{\ell}\frac{(-1)^j(\alpha+2\ell-j-2)!}{j!(\alpha+2\ell-2)!}L_m^j(\chi_{\ell-j}),$$

is a Jacobi form of weight $k+2\ell$ and index m. Here, $\alpha=k-\frac{1}{2}, x!=\Gamma(x+1)$.

(Proof) See [3].

Remark 3.2 *Originally, Eicher-Zagier has shown how to construct modular forms from Jacobi forms[8]. Theorem3.1 is generalizing the idea of Eichler-Zagier given in [[8], §I.3. pp.28−35], replacing modular forms that occur there by Jacobi forms. As a result, we introduce $\tilde{f}$, a function of three variables, in place of the two variable $\tilde{f}$ occurring in [[8], Theorem3.3, p.35]. This leads us how to construct Jacobi forms using the heat operator.*

Corollary 3.3 *[3] For $f\in J_{k,m}$, consider a formal power series*

$$\tilde{f}(\tau,z;X)=\sum_{\ell\ge 0}\frac{L_m^\ell(f)}{\ell!(\alpha+\ell-1)!}X^\ell, \alpha=k-\frac{1}{2}.$$

Then, $\tilde{f}$ satisfies a functional equation, for any $M=\begin{pmatrix} * & * \\ c & d\end{pmatrix}\in\Gamma$,

$$\tilde{f}(M\tau,\frac{z}{c\tau+d};\frac{X}{(c\tau+d)^2})=(c\tau+d)^k e^{2\pi im\frac{cz^2}{c\tau+d}}e^{8\pi im\frac{cX}{c\tau+d}}\tilde{f}(\tau,z;X).$$

(Proof) See [3].

We now state the main result which shows how to construct a Jacobi form from N Jacobi forms using the heat operator.

Theorem 3.4 *Take any $y_i\in\mathbb{C}, 1\le i\le q-1$, and any nonnegative integer ν. Define a map $[\ ]_{(y_1,y_2,..,y_{q-1}),\nu}: J_{k_1,m_1}\times...\times J_{k_q,m_q}\to\mathbb{C}$ as*

$$[f_1,f_2,\dots f_q]_{(y_1,y_2,..y_{q-1}),\nu}=$$
$$\sum_{\substack{\sum_{1\le j\le q} r_j+p=\lfloor\frac{\nu}{2}\rfloor \\ \sum_{\ell=1}^{q} u_\ell=v-2\lfloor v/2\rfloor}} C_{r_1,..,r_q,p}(k_1,..,k_q)\ D_{r_1,..,r_q,u_1,..,u_q}(m_1,..m_q;y_1,..,y_{q-1})$$
$$L_m^p\left(L_{m_1}^{r_1}(\partial_z^{u_1}f_1)L_{m_2}^{r_2}(\partial_z^{u_2}f_2)...L_{m_q}^{r_q}(\partial_z^{u_q}f_q)\right),$$

where $C_{r_1,..,r_q,p}(k_1,..,k_q) = \frac{(-1)^p(\gamma+2v-p-2)!}{p!(\beta+2\gamma-2)!}\prod_{j=1}^q \frac{1}{r_j!(\alpha_j+r_j-1)!}$,
$D_{r_1,..,r_q,u_1,..,u_q}(m_1,..m_q; y_1,..,y_{q-1})$
$= \prod_{j=1}^q \left(-\sum_{i=1}^{j-1} m_i + \sum_{i=j+1}^q m_i\right)^{u_j} \left(1 - \sum_{i=1}^{j-1} m_i y_i + \sum_{i=j+1}^q m_i y_j\right)^{r_j}$,
$\gamma = (\sum_{j=1}^q k_j) - \frac{1}{2}$, $\alpha_j = k_j - \frac{1}{2}, m = \sum_{j=1}^q m_j$, *and* $1 \le j \le q$.
Then $[f_1, f_2, \dots f_q]_{(y_1,..,y_{q-1}),\nu} \in J_{k_1+\dots+k_q, m_1+m_2+\dots+m_1}$.

(Proof of Theorem3.4) Let, for $f_j \in J_{k_j,m_j}, 1 \le j \le q$,

$$\tilde{f}_j(\tau, z; X) = \sum_{\ell \ge 0} \frac{L^\ell_{m_j}(f_j)}{\ell!(\alpha_j + \ell - 1)!} X^\ell.$$

One can see, from Corollary3.3, that

$$\begin{aligned} h(\tau, z; X) &= \prod_{j=1}^q \tilde{f}_j(\tau, z; (1 - \sum_{i=1}^{j-1} m_i y_i + \sum_{i=j+1}^q m_i y_j)X) \\ &= \sum_{\ell=0}^\infty \left(\sum_{r_1+r_2+..+r_q=\ell} \prod_{j=1}^q \frac{L^{r_j}_{m_j}(f_j)(1 - \sum_{i=1}^{j-1} m_i y_i + \sum_{i=j+1}^q m_i y_j)^{r_j}}{r_j!(\alpha_j + r_j - 1)!} \right) X^\ell \end{aligned}$$

satisfies a functional equation

$$h(M\tau, \frac{z}{c\tau + d}; \frac{X}{(c\tau + d)^2}) = (c\tau + d)^k e^{2\pi i m \frac{cz^2}{c\tau+d}} e^{8\pi i m \frac{cX}{c\tau+d}} h(\tau, z; X),$$

for any $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, k = \sum_{j=1}^q k_j, m = \sum_{j=1}^q m_j$ and any $y_j \in \mathbb{C}$. Now, by applying Theorem3.1 to the above function $h(\tau, z; X)$ and from the fact $(L_m f)|_{(m)} Y = L_m(f|_{(m)} Y)$, $\forall Y \in \mathbb{Z}^2$. We conclude the above main result for the case when ν is even. When ν is odd, using the fact that

$$(\partial_z h)(\tau, z; X) = (c\tau + d)^{-k} e^{\frac{-2\pi i m z^2}{c\tau+d}} e^{-8\pi i m \frac{cX}{c\tau+d}} (-2\pi i m_j z h + \frac{1}{c\tau + d} \partial_z h)(M\tau, \frac{z}{c\tau + d}; \frac{X}{(c\tau + d)^2}),$$

Theorem follows.

As a special case, when q=2, the brackets $[\,,\,]_{X,\nu}$ are, up to constant factor, the Rankin-Cohen type bilinear differential operators on the space of Jacobi forms which were already studied in [4].

Theorem 3.5 *[4] Let f and f' be Jacobi forms of weight and index k, m and k', m', respectively. For any $X \in \mathbb{C}$ and any non-negative integer v define*

$$[f, f']_{X,v} = \sum_{r+s+p=v} C_{r,s,p}(k, k')\; D_{r,s,i,j}(m, m', X)\; L^p_{m+m'}\,(L^r_m(f)\, L^s_{m'}(f'))$$

where

$$\begin{aligned} D_{r,s}(m,m',X) &= (1+mX)^s(1-m'X)^r, \\ C_{r,s,p}(k,k') &= \frac{(\alpha+v-1)_{s+p}}{r!}\cdot\frac{(\beta+v-1)_{r+p}}{s!}\cdot\frac{(-(\gamma+v-1))_{r+s}}{p!} \\ &\quad (\alpha=k-1/2,\ \beta=k'-1/2,\ \gamma=k+k'-1/2+(v-2\lfloor v/2\rfloor)), \end{aligned}$$

where $(x)_m=\prod_{0\le i\le m-1}(x-i)$. *Then* $[f,f']_{X,v}$ *is a Jacobi form of weight* $k+k'+v$ *and index* $m+m'$ *and, even more, a Jacobi cusp form for* $v>1$.

Futhermore, let us mention a result by Böcherer [1].

Theorem 3.6 *For fixed* v *and* k,m,k',m' *large enough the vector space of all covariant bilinear differential operators mapping* $J_{k,m}\times J_{k',m'}$ *to* $J_{k+k'+v,m+m'}$ *has dimension* $\lfloor v/2\rfloor+1$.

Remark 3.7 *We note that the Theorem3.5 describes a basis of this space explicitly. For fixed* v *and* k,m *and* k',m' *large enough the operators* $[\cdot,\cdot]_{X,v}$ $(X\in\mathbb{C})$ *span a vector space of dimension* $\lfloor\frac{v}{2}\rfloor+1$ *This shows that the space of such Rankin-Cohen operators is, in general, at least* $\lfloor\frac{v}{2}\rfloor+1$ *dimensional. A result of Böcherer [1], obtained by using Maaß operators, shows that this dimension actually equals* $\lfloor\frac{v}{2}\rfloor+1$ *in general (cf. Theorem3.6).*

4 Connection with elliptic modular forms of half integral weight

In this section, as a special case, we consider the Rankin-Cohen type bilinear differential operator which has a connection with that of elliptic modular forms. One bilinear operator for each even v has already been constructed in [2]:
More explicitley,

Theorem 4.1 *[2] Let* $f_i\in J_{k_i,m_i}$ *with* $i=1$ *or* 2. *For given any nonnegative integer* ν, *consider a linear map* $[[\,,\,]]_\nu;\ J_{k_1,m_1}\times J_{k_1,m_1}\to\mathbb{C}$ *defined by*

$$[[f_1,f_2]]_\nu=\sum_{\ell=0}^{\nu}(-1)^\ell\begin{pmatrix}\alpha_1+\nu-1\\ \nu-\ell\end{pmatrix}\begin{pmatrix}\alpha_2+\nu-1\\ \ell\end{pmatrix}m_1^{\nu-\ell}m_2^\ell L_{m_1}^\ell(f_1)L_{m_2}^{\nu-\ell}(f_2) \qquad (1)$$

Here, $\alpha_i=k_i-\frac{1}{2}$ *and* $x!=\Gamma(x+1)$.
Then, $[[f_1,f_2]]_\nu$ *is a Jacobi form of weight* $k_1+k_2+2\nu$ *and index* m_1+m_2.

Remark 4.2 *The bilinear differential operator* $[[\,,\,]]_\nu$ *is equal to*

$$\frac{(\alpha_1+\nu-1)!(\alpha_2+\nu-1)!}{\nu!}\left(\frac{d}{dX}\right)^{v/2}[f,f']_{X,\nu} \qquad (v\in 2\mathbb{N}=\{0,2\ldots\}).$$

This operator $[[\,,\,]]_\nu$ *was the first found Rankin-Cohen type differential operators on the space of Jacobi forms*

To find a relation between bilinear differential operators of Jacobi forms and those of elliptic modular forms, we recall that any Jacobi form of weight k and index m has an expansion

$$\sum_{\mu \pmod{2m}} h_\mu(\tau)\theta_{m,\mu}(\tau, z)$$

in terms of standard theta-series

$$\theta_{m,\mu}(\tau, z) = \sum_{\substack{r \in Z \\ r \equiv \mu \pmod{2m}}} q^{\frac{r^2}{4m}}\zeta^r,$$

where the h_μ are modular forms of weight $k - \frac{1}{2}$(see [8]). The following results state the theta-expansion in this sense of $[[f_1, f_2]]_\nu, f_i \in J_{k_i,m_i}$. This gives the relation between the ordinary Rankin-Cohen brackets for the half-integral weight elliptic modular forms studied given in [10] and those for the Jacobi forms.

Theorem 4.3 *For each of* $f_i \in J_{k_i,m_i}, i = 1, 2$, *let* $f_i(\tau, z) = \sum_{\mu_i \pmod{2m_i}} h_{\mu_i}\theta_{m_i,\mu_i}$ *be the theta-expansion for* f_i. *Then,*

1.

$$[[f_1, f_2]]_\nu = (8\pi i m_1 m_2)^\nu \sum_{\substack{\mu_1 \pmod{2m_1} \\ \mu_2 \pmod{2m_2}}} [h_{\mu_1}, h_{\mu_2}]_\nu \theta_{m_1,\mu_1}\theta_{m_2,\mu_2},$$

where $[h_{\mu_1}, h_{\mu_2}]_\nu$ *is the ordinary Rankin-Cohen bracket for the half-integral weight elliptic modular forms* h_{μ_i} *[10];*

$$[h_{\mu_1}, h_{\mu_2}]_\nu = \sum_{\ell=0}^{\nu}(-1)^\ell \binom{\alpha_1 + \nu - 1}{\nu - \ell}\binom{\alpha_2 + \nu - 1}{\ell} D_\tau^\ell(h_{\mu_1})D_\tau^{\nu-\ell}(h_{\mu_2})$$

Here, $\alpha_i = k_i - \frac{1}{2}, i = 1, 2$, *and* $D_\tau = \frac{d}{d\tau}$.

2.

$$\theta_{m_1,\mu_1}(\tau, z)\theta_{m_2,\mu_2}(\tau, z) = \sum_{\mu \pmod{2m}} \Theta_{\mu;\mu_1,\mu_2}(\tau)\theta_{m,\mu}(\tau, z),$$

where $m = m_1 + m_2$ *and*

$$\Theta_{\mu;\mu_1,\mu_2}(\tau) = \sum_{\substack{s \in Z \\ s = m_1\mu - m\mu_1 \pmod{2mm_1} \\ s = m\mu_2 - m_2\mu \pmod{2mm_2}}} q^{\frac{s^2}{4mm_1m_2}}.$$

3. The theta expansion of $[[f_1, f_2]]_\nu$ *is given as*

$$[[f_1, f_2]]_\nu(\tau, z) = (8\pi i m_1 m_2)^\nu \sum_{\mu \pmod{2m}} h_\mu(\tau)\theta_{m,\mu}(\tau, z),$$

where $m = m_1 + m_2$*, and*

$$h_\mu(\tau) = \sum_{\substack{\mu_1 \pmod{2m_1} \\ \mu_2 \pmod{2m_2}}} \Theta_{\mu;\mu_1,\mu_2}(\tau)[h_{\mu_1}, h_{\mu_2}]_\nu$$

(Proof of Theorem) See [2].

Acknowledgements The author would like to thank Prof. T. Yamazaki and Prof. A. Murase to invite me to visit Japan and RIMS conference.

References

[1] S. Böcherer, private communication.

[2] Y. Choie, *Jacobi Forms and the Heat Operator*, Math. Zeit., 225, No1, 95 – 101 (1997).

[3] Y. Choie, *Jacobi Forms and the Heat Operator II*, to appear in Ill. Jour. of Mahtmeatics (1998).

[4] Y. Choie and W. Eholzer, Rankin-Cohen operator and Jacobi and Siegel forms, Jour. of Number Theory, Vol.68 , No.2, 160-177, (1998).

[5] Y.Choie and H.Kim, Differential operator on Jacobi forms of Several variables, Preprint (1998).

[6] H. Cohen, *Sums Involving the Values at Negative Integers of L Functions of Quadratic Characters*, Math. Ann. 217, 271-285 (1975).

[7] P. Cohen, Y. Manin and D. Zagier, Automorphic Pseudodifferential operators, 'Algebraic aspects of integrable systems', Fokas and Gelfand, editors; Progress in Nolinear Differential Equations and their Applications, 26: Birkhäuser (1997).

[8] M. Eichler, D. Zagier, *The Theory of Jacobi Forms*, Prog. Math. Vol. 55, Birkhäuser, Boston-Basel-Stuttgart (1985).

[9] R. Rankin, *The Construction of Automorphic Forms from the Derivatives of a given Form*, J. Indian Math. Soc. 20, 103-116 (1956).

[10] D. Zagier, *Modular Forms and Differential Operators*, Proc. Indian Acad. Sci. (Math. Sci.) Vol. 104, No. 1, 57-75 (1994).