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Some results on eigenvalues of the Cartan matrices for finite groups

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G : a finite group
F : an algebraically closed field of characteristic $p>0$
B : a block of the group algebra FG with defect group D of order p^d
$C_B=(c_{ij})$: the Cartan matrix of B i.e. c_{ij} is the multiplicity of an irreducible FG-module S_j in a projective cover P_i of S_i as a composition factor, where S_j and P_i belong to B.

The following are well known properties of the Cartan matrix C_B.

- nonnegative (integral) indecomposable symmetric
- positive definite
- all elementary divisors are a power of p, the largest one is $p^d=|D|$ and the others are smaller than p^d

$\rho(B)$: the Perron-Frobenius (i.e. the largest) eigenvalue of C_B

We note the following.

- eigenvalues and elementary divisors are not equal in general
- $G=A_5$ (the alternating group of degree 5), $p=2$, $B=B_0$ (the principal block) $\Longrightarrow \rho(B)=(7+\sqrt{33})/2>|D|=4$

1. Known properties of $\rho(B)$

The following are known about lower and upper bounds for $\rho(B)$ in [K-W].

(1) $|O_p(G)|\leq\rho(B)\leq u$ *for any block* B *of* FG, *where* $u:=\dim_F P(F_G)$ *and* $P(F_G)$ *is a projective cover of the trivial* FG*-module* F_G.

(2) *If G is p-solvable, then $\rho(B) \leq |D|$, and the equality holds if and only if the height of $\varphi = 0$ for all $\varphi \in IBr(B)$.*

(3) *If D is cyclic, then* $\dfrac{|D|}{p} + 1 \leq \rho(B) \leq |D|$.

(4) *If $D \lhd G$, then $\rho(B) = |D|$.*

We have a lower bound and an upper bound of $\rho(B)$ in (1) in terms of G, but it should be given in terms of B for any block B and any group G. In this talk we showed a lower bound of $\rho(B)$ in terms of B.

2. A lower bound of $\rho(B)$

$\mathrm{Irr}(B)$:= the set of all ordinary (complex) irreducible characters in B,

$\mathrm{IBr}(B)$:= the set of all irreducible Brauer characters in B,

$k(B) := |\mathrm{Irr}(B)|$, $\ l(B) := |\mathrm{IBr}(B)|$.

Let σ be a permutation on $\{1, 2, \dots, l\}$, where $l = l(B)$. Then we have the following:

Theorem 1([W1]). *Let $C_B = (c_{ij})$ be the Cartan matrix of any block B of FG for any finite group G. For $l = l(B)$, we set $l \setminus t := \{1, 2, \dots, l\} - \{t\}$ for $1 \leq t \leq l$. Then we have*

$$k(B) \leq \sum_{i=1}^{l} c_{ii} \ - \sum_{j \in l \setminus t} c_{j\sigma(j)}$$

for any cycle σ of length l and any choice of $1 \leq t \leq l$.

Proof. By the fact $C_B = {}^tD_B D_B$ for the decomposition matrix D_B of B, we write the right hand side of the above inequality by using decomposition numbers for B and we can show a contribution for it of any $\chi \in \mathrm{Irr}(B)$ is larger than or equal to 1.

Corollary 2. *Let B be a block of FG with defect group D. Then $k(B) \leq \rho(B)l(B)$, and the equality holds if and only if $l(B) = 1$ and $k(B) = |D|$.*

Proof. It is clear that $k(B) \leq \sum_{i=1}^{l(B)} c_{ii}$ even if we do not use Theorem 1. Combine it with the fact that $c_{ij} \leq \rho(B)$ for any i, j.

Question 1. There must be sharper inequalities than Corollary 2. For example, does it hold that $k(B) \leq \rho(B)$?

The answer is no. Let $G = \mathrm{SL}(2,p), p$ an odd prime, and B be any one of blocks of defect 1. Then $l(B) = (p-1)/2$, $k(B) = l(B) + 2$ and

$$C_B = \begin{pmatrix} 2 & 1 & 0 & \dots & 0 \\ 1 & 2 & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 1 & 2 & 1 \\ 0 & \dots & 0 & 1 & 3 \end{pmatrix}.$$

Therefore $3 < \rho(C_B) < 4$ by Lemma 3.1 in [K-W], but $k(B) \geq 4$ if $p \geq 5$.

Question 2. Does it hold that $k(B) \leq \rho(B)$, in p-solvable groups?

Now we assume G is p-solvable, then we have the following.

Proposition 3. *Let G be a p-solvable group and B a block of FG with $l(B) = 2$. Assume the p'-part f_i' of the degree f_i of two irreducible Brauer characters φ_i for $i = 1, 2$ are equal. Then $k(B) \leq \rho(C_B)$.*

Proof. The explicit form of C_B in this case is known in [N-W]. Theorem 1 shows that $k(B) \leq c_{11} + c_{22} - c_{12}$. We can verify that the right hand side of the above inequality $\leq \rho(B)$ by the form of C_B.

Remark 4. We added an assumption in the above proposition, but it is conjectured in [N-W, p.329] that $f_1' = f_2'$ for p-solvable groups. Isaacs showed this is true if G is solvable in [I], and it is also proved to be true in some cases in [N-W]. Therefore, $k(B) \leq \rho(C_B)$ for B with $l(B) = 2$ in p-solvable groups, for example, if G is solvable, B is the principal block, or B has an abelian defect group.

Remark 5. Proposition 3 does not hold in general. K. Erdmann determined the shape of the Cartan matrix of tame blocks in [E] (i.e. p=2 and a defect group D is dihedral, generalized quaternion or semidihedral). For example, it actually fails in the following cases.

Let G=PGL(2,31) and B be the principal block. Then D is a dihedral group of order

2^6, $l(B) = 2$, $C_B = \begin{pmatrix} 4 & 2 \\ 2 & 17 \end{pmatrix}$, $k(B) = 19$ (Erdmann's list D(2B)), but $\rho(C_B) < 19$ by Lemma 3.1(2) in [K-W].

We saw in the proof of Proposition 3 that Theorem 1 works well. So the diagonal entries of C_B for p-solvable groups seem to be not so extremely larger than the other entries, while it does not hold in general as is shown in the examples above.

Conjecture. *If G is p-solvable, then $k(B) \leq \rho(B)$.*

If Conjecture is true, then Brauer's $k(B)$ conjecture (that is $k(B) \leq |D|$ for any finite group) is true in p-solvable groups, because [K-W] has showed $\rho(B) \leq |D|$ in p-solvable groups. Since Brauer's $k(B)$ conjecture is not yet proved to be true even if G is a solvable group, it must be quite difficult to show directly that Conjecture is true. There sure is a possibility of the existence of a counter example for it. But we raise some more evidences for the conjecture.

(1) If G is of p-length 1, or D is abelian, then Conjecture can be reduced to the case that $D \lhd G$ by Külshammer [Kü].

(2) If B is tame, then Conjecture is true by [E-M, Kü, Ko1, B-W].

(3) If $p = 3$ and $D \simeq M(3)$ (i.e. extra special 3-group of order 27 with exponent 3), then Conjecture is true by [Ko2].

(4) Assume Brauer's $k(B)$ conjecture is true for p-solvable groups. If $k(B) = |D|$, then $k(B) = \rho(B)$ by [M].

3. The Cartan matrix of a certain class of finite solvable groups

If there exists a counter example for Conjecture, Theorem 1 seems to assert that the non diagonal entries of its Cartan matrix must be extremely smaller than the diagonal ones. So first we should find p-solvable groups (blocks) whose Cartan matrix has many zero entries and $l(B)$ is large like SL(2,p) because $\rho(B)$ is small and $k(B)$ is large. Here by making use of Ninomiya's result in [N] we give an explicit form of the Cartan matrix of a certain class of solvable groups. The author owes to Professor Tetsuro Okuyama who taught him the following type of groups whose Cartan matrix has zero entries.

$GF(p^n)$: the finite field with p^n elements

$A(p^n)$: the additive group of $GF(p^n)$
$M(p^n)$: the multiplicative group of $GF(p^n)$
$X(p^n)$: the affine group of $GF(p^n)$ i.e. $M(p^n) \ltimes A(p^n)$ by ordinary scalar multiplication, then $X(p^n)$ is a complete Frobenius group whose Frobenius kernel is a Sylow p-subgroup, and it is known that the Cartan matrix of $FX(p^n)$ is of the form $\begin{pmatrix} 2 & 1 & \dots & 1 \\ 1 & 2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \dots & 1 & 2 \end{pmatrix}$.

$<\sigma>$: the Galois group of $GF(p^n)$ over $GF(p)$ of order n
$G(p^n) := <\sigma> \ltimes X(p^n)$.

We consider the case $n = pq$, where q is a prime number different from p. Let us set $G = G(p^{pq})$, then since $O_{p'}(G)$ is trivial, G has only the principal block by a theorem of Fong and G is of p-length 2.

Theorem 6. *Under the above notation* (see [W2] for more detailed notation), *the Cartan matrix $C(G)$ of FG is the following.*

$$
\begin{array}{cccc|c|cccc|c}
\alpha_1 & \alpha_2 & \dots & \alpha_{p-1} & \beta & \gamma_1 & \gamma_2 & \dots & \gamma_n & \theta \\
\hline
2pI_q & pI_q & \dots & pI_q & & pI_q & pI_q & \dots & pI_q & \\
pI_q & 2pI_q & \ddots & \vdots & & pI_q & pI_q & \dots & pI_q & \\
\vdots & \ddots & \ddots & pI_q & pJ'_1 & \vdots & \vdots & & \vdots & pJ'_2 \\
pI_q & \dots & pI_q & 2pI_q & & pI_q & pI_q & \dots & pI_q & \\
\hline
 & & p\,{}^tJ'_1 & & B_1 & & & pJ'_3 & & pqJ'_4 \\
\hline
pI_q & pI_q & \dots & pI_q & & (p+1)I_q & pI_q & \dots & pI_q & \\
pI_q & pI_q & \dots & pI_q & & pI_q & (p+1)I_q & \ddots & \vdots & \\
\vdots & \vdots & & \vdots & p\,{}^tJ'_3 & \vdots & \ddots & \ddots & pI_q & pJ'_5 \\
pI_q & pI_q & \dots & pI_q & & pI_q & \dots & pI_q & (p+1)I_q & \\
\hline
 & & p\,{}^tJ'_2 & & pq\,{}^tJ'_4 & & & p\,{}^tJ'_5 & & B_2
\end{array}
$$

, where I_s is the unit matrix of degree s, $J'_1, J'_2, J'_3, J'_4, J'_5$ is the $(p-1)q \times m, (p-1)q \times (r-m)/p, m \times nq, m \times (r-m)/p, nq \times (r-m)/p$ matrix all of whose entries are 1,respectively. Furthermore, $B_1 = pI_m + pqJ_m$ and $B_2 = I_{\frac{r-m}{p}} + pqJ_{\frac{r-m}{p}}$, where J_s is the $s \times s$ matrix all of whose entries are 1.

It is known in general that $\sum_{i,j=1}^{l(B)} c_{ij}/l(B) \leq \rho(B)$ for any block B of FG for any finite group G, and now when $G = G(p^{pq})$ we can verify $k(FG) \leq \sum_{i,j=1}^{l(FG)} c_{ij}/l(FG)$. So we have

$k(FG) \leq \rho(FG)$. When $G = G(p^q)$ and $G(p^p)$, we have also $k(FG) \leq \rho(FG)$.

4. Eigenvalues and elementary divisors of C_B

Elementary divisors of C_B are invarant under elementary operations i.e. C_B and SC_BT for unimodular matrices S, T have the same elementary divisors, while eigenvalues of them are different in general. So elementary divisors and eigenvalues of C_B do not coincide in general. When do they coincide? We have an answer to it in p-solvable groups as follows. This is a part of joint work with A. Hanaki, M. Kiyota and M. Murai [H, K, M, W].

Theorem 7. Let G be a p-solvable group, B a block of FG with defect group D. Then the following are equivalent.

(a) Elementary divisors and eigenvalues of C_B coincide.

(b) $\rho(B) = |D|$.

(c) The height of $\varphi = 0$ for all $\varphi \in \mathrm{IBr}(B)$.

Proof. We have the following two results for p-solvable groups.

(1) Let G be a p-solvable group and η_G the character aforded by the principal indecomposable FG-module corresponding to the trivial FG-module F_G. Then $\eta_G(x)$ is a power of p for any p-regular element $x \in G$.

(2) Let G be a p-solvable group and B a block of FG of full defect. Suppose the height of $\varphi = 0$ for all $\varphi \in \mathrm{IBr}(B)$. Then elementary divisors and eigenvalues of C_B coincide.

Then Fong's two reduction theorem works well, and we have the result.

In this case Conjecture is equivalent to Brauer's $k(B)$ conjecture as $\rho(B) = |D|$.

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