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Squares of Characters in Finite Groups

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This is a report of my joint paper [K,S] with Hiroshi Suzuki (Department of Mathematics, International Christian University).

Let G be a finite group and let χ be a real valued character of G. The representation diagram of G with respect to χ, denoted by $D(G,\chi)$, is a graph with Irr(G) as the vertex set such that vertices χ_i and χ_j are adjacent if and only if $(\chi\chi_i, \chi_j) > 0$. $D(G,\chi)$ is undirected as χ is real valued, but $D(G,\chi)$ may have some loops. Note that $D(G,\chi)$ is connected if and only if χ is faithful. The problem we are interested in is that if we know the graph structure of the representation diagram $D(G,\chi)$, then what can be said about the group structure of G. Here we consider the simplest case i.e. the case $D(G,\chi)$ is a path (open polygon) possibly with some loops. The following lemma is fundamental but easy to prove.

Lemma 1. Let χ be a real valued character of a finite group G. Let the representation diagram $D(G,\chi)$ be a path possibly with some loops. Then $\chi = a1_G + b\chi_1$, for some faithful real valued χ_1 in Irr(G) and for some integers $a\geqq 0$, $b>0$. In particular the diagrams $D(G,\chi)$ and $D(G,\chi_1)$ are identical modulo loops, i.e. neglecting loops.

If $D(G,\chi)$ is a path then we may assume χ is irreducible, and so we have

$$(*) \qquad \chi^2 = 1_G + a\chi + b\psi,$$

for some ψ in Irr(G) and for some integers $a\geqq 0$, $b\geqq 0$, since in the diagram χ

is adjacent to 1_G and ψ and possibly χ itself (loop). The groups with irreducible characters χ and ψ satisfying (*) are completely determined in the next theorem.

Theorem 2. Let χ and ψ be irreducible characters of a finite group G. Suppose that the equation (*) holds. If χ is faithful and real valued, then one of the following holds.

(1) $\chi(1) = 1$ and G is cyclic of order at most two.

(2) $\chi(1) = 2$ and G is the symmetric group of degree 3.

(3) $\chi(1) = 2$ and G is one of the binary polyhedral groups of order 24, 48 or 120.

(4) $\chi(1) = 3$ and G is the alternating group of degree 5.

For the proof we refer to [K,S]. By inspection of the representatipon diagram of each group listed in Theorem 2, we have the following

Corollary 3. Let χ be a real valued character of a finite group G of order at least two. Let the representation diagram $D(G,\chi)$ be a path possibly with some loops. Then G is the cyclic group of order two, or the symmetric group of degree 3.

If you are familiar with some terminology in algebraic combinatorics (for example in [B,I]), you may find that Corollary 3 is equivalent to the following

Corollary 4. Let G be a finite group of order at least two. Suppose that the group association scheme $X(G)$ is Q-polynomial. Then G is the cyclic group of order two, or the symmetric group of degree 3.

Here, we state some open problems.

Problem 5. Study the structure of finite groups G when $D(G,\chi)$ is a tree possibly with some loops.

Problem 6. Determine all finite groups whose group association scheme is P-polynomial. In other words, prove the dual statement of Corollary 4.

Problem 7. Study the structure of finite groups G with χ and ψ in Irr(G) satisfying

$$(**) \qquad \chi \overline{\chi} = 1_G + a(\chi + \overline{\chi}) + b\psi.$$

There are many interesting examples such as $GL(2, 3)$, $PSL(2, 7)$ and $PSU(4, 2^2)$.

references

[B,I] E. Bannai and T. Ito, "Algebraic Combinatorics I", Benjamin, 1984.

[K,S] M. Kiyota and H. Suzuki, Character products and Q-polynomial group association schemes, preprint.