数理解析研究所講究録
1058巻 1998年 1-11

Morrey spaces and applications to $\overline{\partial}_b$ and hypoelliptic pseudodifferential equations

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Abstract

This paper is a survey on Morrey-Hölder estimates for hypoelliptic pseudodifferential equations on nilpotent Lie groups, the Kohn Laplace equation and the tangential Cauchy-Riemann equation on CR manifolds.

1 Morrey-Hölder estimates for elliptic equations.

We will begin with Morrey-Hölder esitmates for elliptic equations which give a motivation of this paper. As is well known, the following Hölder continuity of the solutions of Laplace equation on $\mathbb{R}^n$ holds true for L^p-data with $p > n$:

Theorem C1 (well known) *Suppose f and g are distributions on an open set $U \subset \mathbb{R}^n$ which satisfy $\Delta f = g$ on U. If $g \in L^p_{loc}(U)$, $n < p < \infty$, then ∇f is locally Hölder continuous of order $1 - n/p$ on U.*

A natural question is what happen when $p \leq n$. To study this question, let us recall the classical Morrey spaces $L^{p,\lambda}_{\mathrm{cl}}(\mathbb{R}^n)$:

$$\mathcal{L}^{p,\lambda}(\mathbf{R}^n) = \left\{ f \in L^p_{\mathrm{loc}}(\mathbf{R}^n) \ : \ \sup_{x \in \mathbf{R}^n, r > 0} \frac{1}{r^\lambda} \int_{\|x-y\|<r} |f(y)|^p dy < \infty \right\},$$

where $||x||$ is the Euclidean norm on $\mathbb{R}^n$. Moreover, for an open set $U \subset \mathbb{R}^n$, let

$$\mathcal{L}^{p,\lambda}_{\text{loc}}(U) = \{f \in L^p_{\text{loc}}(U) : \varphi f \in \mathcal{L}^{p,\lambda}(\mathbb{R}^n) \text{ for } \forall\varphi \in \mathcal{D}(U)\}$$

Morrey-Hölder estimates for elliptic equations are the following:

Theorem C2 (cf. [18]) *Suppose* $1 \leq p \leq n$. *Let* $U \subset \mathbb{R}^n$ *be an open set, and* P *an elliptic pseudodifferential operator in the Hörmander class* $OPS^2_{1,0}$ *on* U. *Suppose* f *is a compactly supported distribution on* U, *and* g *a distribution on* U *which satisfy* $Pf = g$ *on* U. *If* $n - p < \lambda < n$ *and* $g \in \mathcal{L}^{p,\lambda}_{\text{loc}}(U)$, *then* ∇f *is locally Hölder continuous on* U *of order* $1-(n-\lambda)/p$.

This theorem improves Theorem C1. In fact, as we will see later in more general case, Theorem C1 is a direct consequence of Theorem C2.

In this paper, we will describe Morrey-Hölder estimates for non-elliptic equations such as $\Box_b u = f$ or $\overline{\partial}_b u = f$.

2 Morrey-Hölder estimates for $\overline{\partial}_b$.

Before moving on to the main body of this paper, we mention an application of Morrey spaces to $\overline{\partial}_b$ equation.

Let M be a compact strongly pseudoconvex CR manifold, and $\rho(x, y)$ a quasi-distance associated with an approximate Heisenberg coordinate. It was introduced by Folland and Stein [7]. By using this quasi-distance ρ, Folland and Stein introduced many non-isotropic function spaces which are appropriate for estimating solutions of the $\overline{\partial}_b$ equation. To describe Morrey-Hölder estimates for $\overline{\partial}_b$, we need one of them: Let $V \times V$ be an open set in $M \times M$ on which ρ is defined. For $0 < \mu < 1$, let

$$\Gamma_\mu(V) = \left\{ f \in C(V) : ||f||_\infty + \sup_{x \neq y} \frac{|f(x) - f(y)|}{\rho(x, y)^\mu} < \infty \right\},$$

where and always $|| \cdot ||_p$ is the usual L^p norm with respect to the measure dm on V induced by the Hermitian form $\langle \ , \ \rangle$ defined in [7]. Moreover, we denote by $\Gamma_\mu(V, loc)$ the space of all $f \in C(V)$ such that $\varphi f \in \Gamma_\mu(V)$ for every compact supported C^∞ function φ on V. These are called non-isotropic Hölder spaces.

The following theorem was proved by Folland and Stein:

Theorem FS (cf. [7]) (A) *Suppose φ and θ are locally integrable $(0,q)$-forms, $0 < q < n$, which satisfy $\Box_b \varphi = \theta$ on V. If $\theta \in L^p(V)$, $2n+2 < p \leq \infty$, then $\Xi_j \varphi \in \Gamma_\beta(V, loc)$, where $\beta = 1 - (2n+2)/p$.*

(B) *Suppose θ is $(0,q)$-form in $L^2(M)$ $(0 < q < n)$. If $\theta \in L^p(V)$, $2n+2 < p$, then the Kohn solution $\varphi = \overline{\partial_b}^* G_b \theta$ (see [10]) is in $\Gamma_\beta(V, loc)$, where $\beta = 1 - (2n+2)/p$.*

A question arising from the above theorem is what occors when $p \leq 2n+2$. Our analysis of non-isotropic Morrey spaces can give an answer to this question. We begin with defining non-isotropic Morrey spaces on V:

$$L^{p,\lambda}(V) = \left\{ f \in L^p_{loc}(V) : \sup_{x \in V, r > 0} \frac{1}{r^\lambda} \int_{\rho(x,y) < r} |f(y)|^p dm(y) < \infty \right\}$$

$(0 < p < \infty,\ 0 \leq \lambda)$. We have that

$$L^{p,\lambda}(V) = \begin{cases} L^p(V), & \lambda = 0 \\ L^\infty(V), & \lambda = 2n+2 \\ \{0\}, & \lambda > 2n+2. \end{cases} \tag{1}$$

Using Morrey spaces, we get the following estimates:

Theorem 1 ([1]) (A) *Let φ and θ be locally integrable $(0,q)$-forms, $0 < q < n$, which satisfy $\Box_b \varphi = \theta$ on V. Suppose $1 < p \leq 2n+2$. If $\theta \in L^{p,\lambda}(V)$, $2n+2-p < \lambda < 2n+2$, then $\Xi_j \varphi \in \Gamma_{1-(2n+2-\lambda)/p}(V, loc)$, for $1 \leq j \leq 2n$.*

(B) *Suppose θ is $(0,q)$-form in $L^2(M)$ $(0 < q < n)$. Let $1 < p \leq 2n+2$. If $\theta \in L^{p,\lambda}(V)$, $2n+2-p < \lambda < 2n+2$, then $\varphi = \overline{\partial_b}^* G_b \varphi$ is in $\Gamma_{1-(2n+2-\lambda)/p}(V, loc)$.*

This theorem is not only an answer to the above mentioned question, but also an improvement of Theorem FS. **Indeed Theorem 1 implies Theorem FS by the following way**: If $1 < q < p < \infty$, then $L^p(V) \subset L^{q,\lambda}(V)$, where $\lambda = (2n+2)(1-q/p)$. Hence if $2n+2 < p < \infty$ and $\theta \in L^p(V)$, then $\theta \in L^{q,\lambda}(V)$ for every $1 < q \leq 2n+2$, and thus Theorem 1 implies that the Kohn solution φ of $\overline{\partial}_b\varphi = \theta$ is in $\Gamma_{1-(2n+2-\lambda)/p}(V, loc)$, where $1-(2n+2-\lambda)/q = 1-(2n+2)/p$. Therefore Theorem FS (B) follows from Theorem 1 (B). By a similar way, Theorem 1 (A) is proved by using Theorem FS (A).

3 Dirichlet growth theorem on nilpotent Lie groups

The classical Morrey spaces were introduced by Morrey in order to prove Morrey-Hölder estimates for solution of elliptic equations. A main step of the proof is the following classical Dirichlet growth theorem by Morrey:

Theorem M (cf. [14]) *Suppose* $1 \leq p \leq n$ *and* $0 < \mu < 1$. *If*

$$f \in H_p^1(\mathbb{R}^n) \quad \textit{and} \quad |\nabla f| \in L_{cl}^{p,n-(1-\mu)p}(\mathbb{R}^n),$$

then there exists a continuous function $\tilde{f}$ *on* $\mathbb{R}^n$ *satisfying that* $f = \tilde{f}$ *almost everywhere on* $\mathbb{R}^n$, *and that*

$$\sup_{x,y\in\mathbb{R}^n, y\neq 0} \frac{|\tilde{f}(x+y)-\tilde{f}(x)|}{\|y\|^\mu} \leq C\|\nabla f\|_{L_{cl}^{1,n-(1-\mu)}} \leq C'\|\nabla f\|_{L_{cl}^{p,n-(1-\mu)p}},$$

where C *and* C' *are positive constants depending only on* n, p *and* μ.

However, since partial differential equations we will study are not elliptic, Theorem M is not appropriate to our aim. For this reason we prove an analogue of the Dirichlet growth theorem to stratified Lie groups. As we will describe later, our analogue is not only a generalization of the classical

Dirichlet growth theorem to stratified Lie groups, but also a refinement of it even if G is the Euclidean group.

In what follows, let G be a stratified Lie group equipped with the following stratification for the Lie algebera $\mathfrak{G}$ of G:

$$\mathfrak{G} = V_1 \oplus \cdots \oplus V_m,\ [V_1, V_j] = V_{j+1} \text{ when } 1 \leq j \leq m-1,\ [V_1, V_m] = \{0\}.$$

Let $Q = \sum_{j=1}^{m} j\dim(V_j) > 2$. Denote by $x \cdot y$ the multiplication of $x, y \in G$, and by x^{-1} the inverse element of $x \in G$. Let 0 be the unit of G. Denote by $\{\delta_r\}$ the family of dilation on G associated with the stratification of G, that is, if $x = \exp(L) \in G$ for $L = L_1 + \cdots + L_m \in V_1 \oplus \cdots \oplus V_m$, then $\delta_r(x) = \exp(rL_1 + r^2 L_2 + \cdots + r^m L_m)$. In this paper we choose once and for all a homogeneous norm $|\cdot|$ by

$$\left|\exp\left(\sum_{j=1}^{m} L_j\right)\right| = \left(\sum_{j=1}^{m} ||L_j||^{2m!/j}\right)^{1/2m!},$$

where $||\cdot||$ is a Euclidean norm on $\mathfrak{G}$ with respect to which the V_j's are mutually orthogonal. Denote by dx the Haar measure on G. Let $d(x, y) = |x \cdot y^{-1}|$, $(x, y \in G)$.

In the following, we fix a sub-laplacian $\mathcal{L} = -\sum_{j=1}^{N} X_j^2$ of G, where X_j's are left-invariant vector fields which form a basis of V_1. For $1 < p < \infty$, we denote by $\mathcal{L}_p^\alpha$ the α-th power of the smallest closed extension $\mathcal{L}_p$ of $\mathcal{L}|C_0^\infty(G)$ in $L^p(G)$. For $1 < p < \infty$ and $\alpha \geq 0$, Folland [5] defined the non-isotropic Sobolev space S_α^p as the domain of $\mathcal{L}_p^{\alpha/2}$ equipped with the norm

$$||f||_{S_\alpha^p} := ||f||_p + ||\mathcal{L}_p^{\alpha/2} f||_p.$$

We will use also the non-isotropic Hölder semi-norm of order $\mu \in (0, 1)$ defined by

$$|f|_\mu := \sup_{x,y \in G, y \neq 0} \frac{|f(x \cdot y) - f(x)|}{|y|^\mu}$$

for continuous functions f on G. For details of Sobolev spaces S_α^p and the semi-norm $|f|_\mu$, we refer Folland [5].

Morrey spaces $L^{p,\lambda}(G)$ on G are defined by

$$\left\{ f \in L^p_{loc}(G) : \|f\|_{p,\lambda} = \sup_{x\in G, r>0} \left(\frac{1}{r^\lambda} \int_{|x\cdot y^{-1}|<r} |f(y)|^p dy \right)^{1/p} < \infty \right\}$$

$(1 \leq p < \infty, 0 \leq \lambda < Q)$.

The following theorem is an analogue of the Dirichlet growth theorem:

Theorem 2 ([1]) *Suppose* $1 \leq p < \infty$, $0 < \mu < 1$, *and* $\mu < \alpha < \min\{\mu + (Q/p), Q\}$. *Let* $1 < q < Q/\alpha$. *If*

$$f \in S^q_\alpha \quad \textit{and} \quad \mathcal{L}_q^{\alpha/2} f \in L^{p, Q-(\alpha-\mu)p}(G),$$

then there exists a continuous function $\tilde{f}$ *on* G *satisfying that* $f = \tilde{f}$ *almost everywhere on* G, *and that*

$$|\tilde{f}|_\mu \leq C\|\mathcal{L}_q^{\alpha/2} f\|_{1,Q-(\alpha-\mu)} \leq C'\|\mathcal{L}_q^{\alpha/2} f\|_{p,Q-(\alpha-\mu)p},$$

where C *and* C' *are positive constants depending only on* G, p, μ *and* α.

As a consequence of Theorem 2 we have a version of Theorem M to the group G: As usual, a multi-index $I = (i_1, \cdots, i_k)$ is a k-tuple with k arbitrary and $1 \leq i_j \leq N$ for $j = 1, \cdots, k$, and we set $|I| = k$. Then we define X_I to be $X_{i_1} X_{i_2} \cdots X_{i_k}$.

Corollary 3 ([1]) *Suppose* $1 < p \leq Q$ *and* $0 < \mu < 1$. *Let* k *be an integer with* $1 \leq k < \min\{\mu + (Q/p), Q\}$. *Let* $1 < q < Q/k$. *If*

$$f \in S^q_k \quad \textit{and} \quad \sum_{|I|=k} |X_I f| \in L^{p,Q-(k-\mu)p}(G),$$

then there is a continuous function $\tilde{f}$ *on* G *so that* $f = \tilde{f}$ *almost everywhere on* G, *and that*

$$|\tilde{f}|_\mu \leq C \sum_{|I|=k} \|X_I f\|_{p,Q-(k-\mu)p},$$

where C *is a positive constant depending only on* G, L, p *and* μ.

Let us compare Theorem 2 and Corollary 3 with Thoerem M: In the classical case, Morrey's theorem show us that the Hölder seminorm of a function f is estmated from above by some Morrey space norm of gradient ∇f of f. However, our results assert that the non-isotropic Hölder norm of f is estimated by $X_j f$ for only $X_1, \cdots, X_N \in V_1$, which generate never the tangent bundle TG of G except when G is euclidean. In addition, Theorem 2 concerns with not only ∇f but also fractional derivative $\mathcal{L}_p^{\alpha/2} f$ of f.

4 Morrey spaces and pseudodifferential equations on Lie groups

In this section we apply what we have obtained to pseudodifferential operators on a stratified Lie group G which were introduced in Christ, Geller, Głowacki and Polin [4]. Let us recall the definition of their pseudodifferential operators. Denote by $\mathcal{S}$ the usual Schwartz space on G. For $f \in \mathcal{S}$, $t > 0$, we write $f_t(x) = t^{-Q} f(\delta_{1/t} x)$. A distribution $K \in \mathcal{S}'$ is said to be homogenous of degree k if $K(f_t) = t^k K(f)$ for all $t > 0$ and $f \in \mathcal{S}$. Let $Rhom_k$ be the set of all regular homogeneous distributions of degree k on G, and let $\mathbb{K}^k = Rhom_k$ when $k \notin \{0, 1, 2, \cdots\}$, and

$$\begin{aligned} \mathbb{K}^k = \{&K' + p(x) \log|x| : K' \in Rhom_k, \\ &p(x) \text{ a homogeneous polynomial of degree } k\}, when\ k \in \{0, 1, 2, \cdots\}. \end{aligned}$$

Definition 2 ([4]). Suppose $j \in \mathbb{C}$ and $U \subset G$ is open. Let $\mathcal{U} = \{(x, y) : x \in U, x \cdot y^{-1} \in U\}$. We define the core class $\mathcal{C}^j(U)$ to consist of the set of $K \in \mathcal{D}'(\mathcal{U})$ with the following properties (i) and (ii):

(i) There exist $K_u^m \in \mathbb{K}^{-Q-j+m}$ depending smoothly on the parameter $u \in U$ such that for each $N > 0$ there exists $M > 0$ such that

$$K - \sum_{m=0}^{M} K^m = E_M \in C^N(\mathcal{U}).$$

(ii) For some finite $R \geq 0$, $K_u(w) = K(u,w)$ vanishies identically for $|w| > R$.

Let $K \in C^j(U)$. For $f \in \mathcal{D}(U)$, let $\mathcal{K}f(x) = f * K_x(x)$, if the right-hand side is defined. We say that $\mathcal{K}$ is a pseudodifferential operator of order j on U with core K, and denote $\mathcal{K} = \mathcal{O}(K)$, $K = \kappa(\mathcal{K})$, and $\mathcal{O}C^j(U) = \{\mathcal{K} : K \in C^j(U)\}$. We also wirte the relation in (i) by $\mathcal{K} \sim \sum \mathcal{K}^m$.

We say that $\mathcal{K}(\sim \sum_i \mathcal{K}^i) \in \mathcal{O}C^j(U)$ has a local right parametrix at a point $x_0 \in U$, if there is an open neighborhood W of x_0 satisfying that for every open set $W_1 \subset\subset W$, there exist an operator $\mathcal{P}_1 \in \mathcal{O}C^{-j}(W)$ and a smoothing map $S : \mathcal{E}'(W) \to C^\infty(W)$ such that

$$\mathcal{P}\mathcal{P}_1 h = h + Sh \text{ on W},$$

for $h \in \mathcal{E}'(W_1)$.

The following Morrey-Hölder estimates of pseudodifferential equations are proved by using Theorem 2 and Corollary 3:

Theorem 4 **([1])** *Let k be a positive even number with $k < Q$ and $\mathcal{P} \in \mathcal{O}C^k(G)$. Suppose $\mathcal{P}$ is hypoelliptic, and has a local right parametrix at a point $x_0 \in G$. Then x_0 has an open neighborhood $W \subset G$ as follows: Suppose α, p and λ are positive numbers with $0 < \alpha < k$, $1 < p \leq Q/(k-\alpha)$ and $Q - p(k-\alpha) < \lambda < \min\{Q - p(k-\alpha) + p, Q\}$. Let $f, g \in \mathcal{D}'(W)$, and assume that $\mathcal{P}f$ is defined and*

$$\mathcal{P}f = g \quad \text{on } W.$$

(1) *If $g \in L^{p,\lambda}(W, loc)$, then for every $\varphi \in \mathcal{D}(W)$,*

$$\left|\mathcal{L}_p^{\alpha/2}(\varphi f)\right|_{k-((Q-\lambda)/p)-\alpha} < \infty.$$

(Note that $0 < k - ((Q-\lambda)/p) - \alpha < 1$.)

(2) *If in addition to the above hypotheses, α is an integer, then*

$$\sum_{|I| \leq \alpha} X_I f \in \Gamma_{k-((Q-\lambda)/p)-\alpha}(W, loc).$$

There are some sufficient conditions on pseudodifferential operators to be hypoelliptic and to have right parametrix. For them, we refer the reader to [4], and also to [17] when G is the Heisenberg group.

Using these results, in particular Theorem 2, Corollary 3 and Theorem 4, we can prove Theorem 1. We also use Morrey space boundedness of non-isotropic singular integrals which was proved in [1] or in [3].

Since $L^q(G) \subsetneq L^{p,Q(1-(p/q))}(G)$ for $1 < p < q < \infty$, Theorem 10 yields the following corollary which is an extension, to non-elliptic case, of the regularity result for second order elliptic equations of L^p data $(p > n)$:

Corollary 5 ([1]) *Let k, $\mathcal{P}$, x_0 be as in Theorem 4. Then there exists an open neighborhood $W \in G$ of x_0 as follows: Suppose α is an integer with $0 < \alpha < k$, and p a real number with $Q/(k-\alpha) < p < Q/(k-\alpha-1)$ where we can regard $Q/0$ as ∞. Let $f, g \in \mathcal{D}'(W)$, and assume that $\mathcal{P}f$ is defined and $\mathcal{P}f = g$ on W. If $g \in L^p(W, loc)$, then $\sum_{|I|\le\alpha} X_I f \in \Gamma_\ell(W, loc)$, where $\ell = k - (Q/p) - \alpha$.*

These results extend Theorems C1 and C2, the classical theorems on regularity of the Laplace equation, to certain hypoelliptic, higher order equations

Acknowledgements. The author thanks to Professor G. Komatsu for his invitation to the conference helt at RIMS, Kyoto.

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