数理解析研究所講究録
1058 巻 1998 年 114-128

On a WKB-theoretic approach to the Painlevé transcendents. II

Yoshitsugu TAKEI
Research Institute for Mathematical Sciences
Kyoto University
Kyoto, 606-01, JAPAN
(京大数理研　竹井 義次)

1 Introduction.

In the previous report [T1] we explained the outline of the exact WKB analysis for Painlevé equations with a large parameter. The formal power series solutions constructed in a singular-perturbative manner were mainly discussed there. However, as we pointed out at the end of that report, to complete the whole theory we have to consider "general solutions", i.e., a 2-parameter family of solutions of painlevé equations. Such a 2-parameter family of solutions was constructed in [AKT2] by using the so-called multiple-scale analysis, and we have now succeeded in establishing the basic part of the exact WKB analysis for this 2-parameter family of solutions (called "instanton-type solutions") of Painlevé equations; the purpose of this report is to give a survey for this theory. For the details we refer the reader to [KT1], [AKT2], [KT3] and [T4].

The most part of this work has been done jointly with Professor T. Kawai and Professor T. Aoki. The author should like to express his sincere gratitude to them for the stimulating discussions with them. This work is supported in part by Grant-in-Aid for Scientific Research for Encouragement of Young Scientists (No. 09740101), the Japanese Ministry of Education, Science, Sports and Culture.

2 Instanton-type solutions.

In this section we review some basic facts about the instanton-type solutions, which are central objects of this report, as well as formal power series solutions of Painlevé equations. First of all, let us list up the Painlevé equations (P_J) $(J = \mathrm{I}, \ldots, \mathrm{VI})$ with a large parameter η.

Table 1

$$(P_{\mathrm{I}})\ \frac{d^2\lambda}{dt^2} = \eta^2(6\lambda^2 + t).$$

$$(P_{\mathrm{II}})\ \frac{d^2\lambda}{dt^2} = \eta^2(2\lambda^3 + t\lambda + c).$$

$$(P_{\mathrm{III}})\ \frac{d^2\lambda}{dt^2} = \frac{1}{\lambda}\left(\frac{d\lambda}{dt}\right)^2 - \frac{1}{t}\frac{d\lambda}{dt} + \eta^2\left[16c_\infty\lambda^3 + \frac{8c'_\infty\lambda^2}{t} - \frac{8c'_0}{t} - \frac{16c_0}{\lambda}\right].$$

$$(P_{\mathrm{IV}})\ \frac{d^2\lambda}{dt^2} = \frac{1}{2\lambda}\left(\frac{d\lambda}{dt}\right)^2 - \frac{2}{\lambda} + \eta^2\left[\frac{3}{2}\lambda^3 + 4t\lambda^2 + \left(2t^2 + 8c_1\right)\lambda - \frac{8c_0}{\lambda}\right].$$

$$(P_{\mathrm{V}})\ \frac{d^2\lambda}{dt^2} = \left(\frac{1}{2\lambda} + \frac{1}{\lambda - 1}\right)\left(\frac{d\lambda}{dt}\right)^2 - \frac{1}{t}\frac{d\lambda}{dt} + \frac{(\lambda-1)^2}{t^2}\left(2\lambda - \frac{1}{2\lambda}\right)$$
$$+ \eta^2\frac{2\lambda(\lambda-1)^2}{t^2}\left[(c_0 + c_\infty) - \frac{c_0}{\lambda^2} - \frac{c_2 t}{(\lambda-1)^2} - \frac{c_1 t^2(\lambda+1)}{(\lambda-1)^3}\right].$$

$$(P_{\mathrm{VI}})\ \frac{d^2\lambda}{dt^2} = \frac{1}{2}\left(\frac{1}{\lambda} + \frac{1}{\lambda-1} + \frac{1}{\lambda-t}\right)\left(\frac{d\lambda}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{\lambda-t}\right)\frac{d\lambda}{dt}$$
$$+ \frac{2\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2}\left[1 - \frac{\lambda^2 - 2t\lambda + t}{4\lambda^2(\lambda-1)^2}\right.$$
$$\left. + \eta^2\left\{(c_0 + c_1 + c_t + c_\infty) - \frac{c_0 t}{\lambda^2} + \frac{c_1(t-1)}{(\lambda-1)^2} - \frac{c_t t(t-1)}{(\lambda-t)^2}\right\}\right].$$

(In this report we use the symbol c_0 etc. to denote the constants, which were denoted by α_0 etc. in [T1], appearing in Table 1 above so that they can be distinguished from free parameters contained in the instanton-type solutions $\lambda_J(t;\alpha,\beta)$.)

As is clear from Table 1, each Painlevé equation (P_J) has the following structure for the η-dependence in common:

$$(P_J) \qquad \frac{d^2\lambda}{dt^2} = G_J\left(\lambda, \frac{d\lambda}{dt}, t\right) + \eta^2 F_J(\lambda, t),$$

where F_J and G_J are rational functions and, furthermore, G_J is a polynomial in $d\lambda/dt$ with degree equal to or at most 2. Making use of this expression of equations, we can readily verify that (P_J) has the following formal power series (in η^{-1}) solution, which is denoted by $\lambda_J^{(0)}(t)$ here (in [T1] it was denoted simply by λ_J):

(1) $$\lambda_J^{(0)}(t) = \lambda_0(t) + \eta^{-1}\lambda_1(t) + \eta^{-2}\lambda_2(t) + \cdots,$$

where the top term $\lambda_0(t)$ satisfies

(2) $$F_J(\lambda_0(t), t) = 0$$

and the other $\lambda_j(t)$ ($j \geq 1$) can be determined in a recursive manner. In particular, $\lambda_j(t)$ identically vanishes for every odd integer j. Since this solution $\lambda_J^{(0)}(t)$ does not contain a free parameter at all, we often call it a 0-parameter solution.

On the other hand, by employing the so-called multiple-scale analysis we can construct another type of formal solutions containing 2 free parameters, called 2-parameter solutions or instanton-type solutions. The explicit description of this 2-parameter solution $\lambda_J(t;\alpha,\beta)$ is given as follows:

(3) $$\lambda_J(t;\alpha,\beta) = \lambda_0(t) + \eta^{-1/2}\Lambda(t,\eta)$$

where $\lambda_0(t)$ is a solution of (2) and $\Lambda(t,\eta)$ has the formal expansion (in $\eta^{-1/2}$)

(4) $$\sum_{j=0}^{\infty} \eta^{-j/2}\Lambda_{j/2}(t,\eta)$$

with

(5) $$\Lambda_0(t,\eta) = \mu_J(t)\left(\alpha_0 \exp\Phi_J + \beta_0 \exp(-\Phi_J)\right)$$

and $\Lambda_{j/2}(t,\eta)$ ($j \geq 1$) being of the following form:

(6) $$\Lambda_{j/2}(t,\eta) = \sum_{k=0}^{j+1} b_{j+1-2k}^{(j/2)}(t)\exp\left((j+1-2k)\Phi_J\right),$$

where

(i) α_0 and β_0 are arbitrary complex numbers,

(ii) $\Phi_J(t,\eta) = \eta \int^t \sqrt{\dfrac{\partial F_J}{\partial \lambda}}(\lambda_0(s), s)ds + \alpha_0\beta_0 \log(\theta_J(t)\eta^2)$,

(iii) $\theta_J(t)$ and $\mu_J(t)$ are the functions respectively tabulated in Table 2 and Table 3 below. (They are described with explicit use of $\lambda_0(t)$ and hence the presentation is not unique.)

Table 2

$$\theta_{\rm I}(t) = (12\lambda_0)^5$$
$$\theta_{\rm II}(t) = 16(6\lambda_0^2+t)^5(2\lambda_0^2+t)^{-2}$$
$$\theta_{\rm III}(t) = \frac{256t^4}{\lambda_0^2}\left(4c_\infty\lambda_0^2+\frac{3c'_\infty\lambda_0}{2t}-\frac{c'_0}{2t\lambda_0}\right)^5\left(2c_\infty c'_\infty\lambda_0^2+\frac{{c'_\infty}^2\lambda_0}{t}+2c_\infty c'_0\right)^{-2}.$$
$$\theta_{\rm IV}(t) = \frac{8}{\lambda_0^2}\left(3\lambda_0^2+6t\lambda_0+2(t^2+4c_1)\right)^5\left(\lambda_0^2+2t\lambda_0+8c_1\right)^{-2}.$$
$$\theta_{\rm V}(t) = \frac{256}{t^4\lambda_0^2(\lambda_0-1)^4}\left(c_0\frac{(\lambda_0-1)^2}{\lambda_0^2}+c_2t\frac{\lambda_0}{\lambda_0-1}+c_1t^2\frac{\lambda_0(\lambda_0+2)}{(\lambda_0-1)^2}\right)^5$$
$$\times\left(\frac{c_2^2}{(\lambda_0-1)^4}-\frac{4c_0c_1}{\lambda_0^2(\lambda_0-1)^2}+\frac{4c_1c_2t}{(\lambda_0-1)^5}+\frac{4c_1^2t^2}{(\lambda_0-1)^6}\right)^{-2}.$$
$$\theta_{\rm VI}(t) = \frac{256}{t^4(t-1)^4\lambda_0^2(\lambda_0-1)^2(\lambda_0-t)^2}$$
$$\times\left(c_0t\frac{(\lambda_0-1)(\lambda_0-t)}{\lambda_0^2}-c_1(t-1)\frac{\lambda_0(\lambda_0-t)}{(\lambda_0-1)^2}+c_tt(t-1)\frac{\lambda_0(\lambda_0-1)}{(\lambda_0-t)^2}\right)^5$$
$$\times\left(\frac{c_0^2}{\lambda_0^4}+\frac{c_1^2}{(\lambda_0-1)^4}+\frac{c_t^2}{(\lambda_0-t)^4}\right.$$
$$\left.-\frac{2c_0c_1}{\lambda_0^2(\lambda_0-1)^2}-\frac{2c_0c_t}{\lambda_0^2(\lambda_0-t)^2}-\frac{2c_1c_t}{(\lambda_0-1)^2(\lambda_0-t)^2}\right)^{-2}.$$

Table 3

$$\mu_{\rm I}(t) = (12\lambda_0)^{-1/4}.$$
$$\mu_{\rm II}(t) = (6\lambda_0^2+t)^{-1/4}.$$
$$\mu_{\rm III}(t) = t^{-1/2}\lambda_0\left(4c_\infty\lambda_0^2+\frac{3c'_\infty\lambda_0}{2t}-\frac{c'_0}{2t\lambda_0}\right)^{-1/4}.$$
$$\mu_{\rm IV}(t) = 2\lambda_0^{1/2}\left(6\lambda_0^2+12t\lambda_0+4(t^2+4c_1)\right)^{-1/4}.$$
$$\mu_{\rm V}(t) = \lambda_0^{1/2}(\lambda_0-1)\left(c_0\frac{(\lambda_0-1)^2}{\lambda_0^2}+c_2t\frac{\lambda_0}{\lambda_0-1}+c_1t^2\frac{\lambda_0(\lambda_0+2)}{(\lambda_0-1)^2}\right)^{-1/4}.$$
$$\mu_{\rm VI}(t) = (\lambda_0(\lambda_0-1)(\lambda_0-t))^{1/2}$$
$$\times\left(c_0t\frac{(\lambda_0-1)(\lambda_0-t)}{\lambda_0^2}-c_1(t-1)\frac{\lambda_0(\lambda_0-t)}{(\lambda_0-1)^2}+c_tt(t-1)\frac{\lambda_0(\lambda_0-1)}{(\lambda_0-t)^2}\right)^{-1/4}.$$

For the explicit scheme of the construction we refer the reader to [AKT2]. Especially for the explicit description of $\theta_J(t)$ see [T3] and [KT3]. Here we only mention the following facts without discussing how to construct $\lambda_J(t;\alpha,\beta)$: First, $b_l^{(j/2)}(t)$ is free from η, i.e., the η-dependence of $\Lambda_{j/2}(t,\eta)$ is only through the so-called l-instanton terms $\exp(l\Phi_J(t,\eta))$. Note that instanton terms are regarded to be of degree 0 in η. Second, $b_l^{(j/2)}(t)$ $(l\neq\pm1)$ is determined uniquely by $\{b_{j'+1-2k}^{(j'/2)}\}_{j'<j,0\leq k\leq j'+1}$ in a recursive manner. Third, $b_{\pm1}^{(j/2)}(t)=0$ for an odd integer j, while for a positive even integer $j=2n$ ($1\leq n$: integer) $b_{\pm1}^{(j/2)}(t)=b_{\pm1}^{(n)}(t)$ is characterized as a solution of some system of first-order linear differential equations. This implies that $b_{\pm1}^{(n)}(t)$ $(n\geq1)$ contains additional free parameters ([KT3, Section 1]) which we denote by (α_n,β_n). Thus the precise meaning of "2-parameter" is "2-infinite series", that is, the symbols α and β denoting free parameters should be considered as designating the infinite series $\sum_{n\geq0}\eta^{-n}\alpha_n$ and $\sum_{n\geq0}\eta^{-n}\beta_n$ respectively.

3 Fundamental results.

The formal power series solution $\lambda_J^{(0)}(t)$ of (P_J) does not converge in the usual sense. As $\lambda_J(t;\alpha,\beta)$ with $\alpha=\beta=0$ coincides with $\lambda_J^{(0)}(t)$, a similar problem of divergence also exists for the instanton-type solution $\lambda_J(t;\alpha,\beta)$. To overcome such divergence problems, the Borel resummation technique is applied in the exact WKB analysis. As was illustrated in [T1], in parallel with the case of WKB solutions of 1-dimensional Schrödinger equations, the Borel sum of $\lambda_J^{(0)}(t)$ is expected to define an analytic solution of (P_J) in each Stokes region, i.e., a region surrounded by Stokes curves. Here let us recall the definition of Stokes curves for Painlevé equations (cf. [KT1], [T1]).

Definition 1 *(i) A turning point for $\lambda_J^{(0)}$ is, by definition, a point r which satisfies*

$$F_J(\lambda_0(r),r)=\frac{\partial F_J}{\partial\lambda}(\lambda_0(r),r)=0. \tag{7}$$

A turning point r is said to be simple if $(\partial^2F_J/\partial\lambda^2)(\lambda_0(r),r)\neq0$.
(ii) A Stokes curve for $\lambda_J^{(0)}$ is defined by the following relation:

$$\Im\int_r^t\sqrt{\frac{\partial F_J}{\partial\lambda}(\lambda_0(s),s)}ds=0, \tag{8}$$

where r is a turning point for $\lambda_J^{(0)}$.

If we take the Borel sum of $\lambda_J^{(0)}(t)$ in a Stokes region and consider its analytic continuation to an adjacent region across a Stokes curve for $\lambda_J^{(0)}$, then in the adjacent region the resulting solution should correspond to one of the instanton-type solutions $\lambda_J(t;\alpha,\beta)$ (cf. [T1] and [T2] in the case of $J=\mathrm{I}$). This is the reason why we are required to discuss not only $\lambda_J^{(0)}(t)$ but also $\lambda_J(t;\alpha,\beta)$ and the most important issue in this approach is to determine the explicit form of the "connection formula", i.e., the relation between the Borel resummed formal solutions in two adjacent Stokes regions, for the 2-parameter solutions $\lambda_J(t;\alpha,\beta)$. Note that turning points and Stokes curves are defined only by the top degree part $\lambda_0(t)$ of $\lambda_J^{(0)}(t)$ and that the top degree part of $\lambda_J(t;\alpha,\beta)$ is the same as that of $\lambda_J^{(0)}(t)$. Although the Borel summability of $\lambda_J(t;\alpha,\beta)$ has not yet been proved at the present stage, this suggests that the Borel resummed $\lambda_J(t;\alpha,\beta)$ should define an analytic solution in each Stokes region also. As a matter of fact, concerning the determination of the connection formula for $\lambda_J(t;\alpha,\beta)$, we can prove the following two fundamental theorems.

The first result is the local reduction theorem for $\lambda_J(t;\alpha,\beta)$, which is a natural genaralization of the corresponding result for 0-parameter solutions $\lambda_J^{(0)}(t)$ ([KT1, Theorem 2.3], [T1, Theorem 3]):

Theorem 1 ([KT2, Theorem 2.1]) *Let $\tilde{t}_*$ be a point in a Stokes curve for $\tilde{\lambda}_J^{(0)}$ that emanates from a simple turning point $\tilde{r}$ for $\tilde{\lambda}_J^{(0)}$. Then for each 2-parameter instanton-type solution $\tilde{\lambda}_J(\tilde{t};\tilde{\alpha},\tilde{\beta})$ of (P_J) we can find a 2-parameter instanton-type solution $\lambda_{\mathrm{I}}(t;\alpha,\beta)$ of (P_{I}) for which the following holds:*

There exist a neighborhood $\tilde{V}$ of $\tilde{t}_*$ and the following formal series

$$x(\tilde{x},\tilde{t},\eta)=\sum_{j\geq 0}\eta^{-j/2}x_{j/2}(\tilde{x},\tilde{t},\eta), \tag{9}$$

$$t(\tilde{t},\eta)=\sum_{j\geq 0}\eta^{-j/2}t_{j/2}(\tilde{t},\eta) \tag{10}$$

so that the relation

$$x(\tilde{\lambda}_J(\tilde{t};\tilde{\alpha},\tilde{\beta}),\tilde{t},\eta)=\lambda_{\mathrm{I}}(t(\tilde{t},\eta);\alpha,\beta) \tag{11}$$

holds on $\tilde{V}$.

(To distinguish variables and functions relevant to (P_J) and $(P_{\rm I})$, we put the symbol $\sim$ over those relevant to (P_J).)

Remark 1 The correspondence between the parameters $(\tilde{\alpha}, \tilde{\beta})$ and (α, β), which will play an important role in determining the connection formula for $\tilde{\lambda}_J(\tilde{t}; \tilde{\alpha}, \tilde{\beta})$, is determined uniquely by some additional requirement (cf. §4 below). For example, its top degree part becomes of the following simple form:

$$\tilde{\alpha}_0 = \alpha_0 \quad \text{and} \quad \tilde{\beta}_0 = \beta_0 \tag{12}$$

(cf. [T3]). We conjecture that the complete correspondence should be given by

$$\sum_{n\geq 0} \eta^{-n} \tilde{\alpha}_n = \sum_{n\geq 0} \eta^{-n} \alpha_n \quad \text{and} \quad \sum_{n\geq 0} \eta^{-n} \tilde{\beta}_n = \sum_{n\geq 0} \eta^{-n} \beta_n \tag{13}$$

under the present normalization of $\tilde{\theta}_J(t)$.

This local reduction theorem implies that in the case of Painlevé equations the first Painlevé equation $(P_{\rm I})$ can be regarded as a canonical equation near a simple turning point (just like the Airy equation in the case of 1-dimensional Schrödinger equations). The second result is concerned with the connection formula for the canonical equation $(P_{\rm I})$. To state the result in a specific manner let us introduce some notations.

In the case of $(P_{\rm I})$ the origin $t = 0$ is the unique turning point (which is simple in the sense of Definition 1) and the Stokes curves consist of five straight lines $\{t | \arg t = \pi + 2n\pi/5\}$ (where n is an integer) emanating from the origin. Since $(P_{\rm I})$ has some symmetry and there is no essential distinction among these five Stokes curves (cf. Remark 3 below), we consider the problem on the Stokes curve $\gamma = \{t | \arg t = 3\pi/5\}$ in this report. Then the following two Stokes regions A and B have this Stokes curve γ as a common boundary:

$$\begin{aligned} &\text{Region A}: \quad \{t | \pi/5 < \arg t < 3\pi/5\}, \\ &\text{Region B}: \quad \{t | 3\pi/5 < \arg t < \pi\}. \end{aligned}$$

We define a branch of $\lambda_0(t) = \sqrt{-t/6}$ so that $\arg \lambda_0 = 4\pi/5$ holds on γ. Hence in Regions A and B $\arg \theta_{\rm I}(t)^{\alpha_0 \beta_0} = \arg(12\lambda_0)^{\alpha_0\beta_0}$ takes the following values respectively:

$$\begin{aligned} 3\pi\alpha_0\beta_0 < \arg \theta_{\rm I}(t)^{\alpha_0\beta_0} < 4\pi\alpha_0\beta_0 \quad &\text{in Region A}, \\ 4\pi\alpha_0\beta_0 < \arg \theta_{\rm I}(t)^{\alpha_0\beta_0} < 5\pi\alpha_0\beta_0 \quad &\text{in Region B}. \end{aligned}$$

For the sake of simplicity we assume that the free parameters α and β contained in $\lambda_{\mathrm{I}}(t;\alpha,\beta)$ are genuine constants (in other words, all α_n and β_n except α_0 and β_0 are equal to zero) in Theorem 2 below and, to present several formulas in a neat manner, we replace $(\alpha,\beta)=(\alpha_0,\beta_0)$ by the following (a,b):

$$a=\alpha_0 e^{4i\pi\alpha_0\beta_0}, \quad b=\beta_0 e^{-4i\pi\alpha_0\beta_0}, \tag{14}$$

so that

$$\left|\arg\frac{\alpha_0\theta_{\mathrm{I}}(t)^{\alpha_0\beta_0}}{a}\right|, \left|\arg\frac{\beta_0\theta_{\mathrm{I}}(t)^{-\alpha_0\beta_0}}{b}\right| < \pi\alpha_0\beta_0 \tag{15}$$

may be satisfied in each Region. Further, we introduce the following functions $S_j(a,b)$ and $\tilde{S}_j(\tilde{a},\tilde{b})$:

$$\begin{cases} S_1(a,b) &= ie^{-i\pi E/2}-ib\sqrt{\pi}\dfrac{2^{-E/4+1}e^{-i\pi E/2}}{\Gamma(-E/4+1)}, \\ S_2(a,b) &= -a\sqrt{\pi}\dfrac{2^{E/4+1}e^{i\pi E/4}}{\Gamma(E/4+1)}, \\ S_3(a,b) &= ib\sqrt{\pi}\dfrac{2^{-E/4+1}}{\Gamma(-E/4+1)}, \\ S_4(a,b) &= ie^{-i\pi E/2}+a\sqrt{\pi}\dfrac{2^{E/4+1}e^{-i\pi E/4}}{\Gamma(E/4+1)}, \\ S_5(a,b) &= ie^{i\pi E/2}, \end{cases} \tag{16}$$

$$\begin{cases} \tilde{S}_1(\tilde{a},\tilde{b}) &= ie^{-i\pi \tilde{E}/2}, \\ \tilde{S}_2(\tilde{a},\tilde{b}) &= ie^{i\pi \tilde{E}/2}-\tilde{a}\sqrt{\pi}\dfrac{2^{\tilde{E}/4+1}e^{i\pi \tilde{E}/4}}{\Gamma(\tilde{E}/4+1)}, \\ \tilde{S}_3(\tilde{a},\tilde{b}) &= i\tilde{b}\sqrt{\pi}\dfrac{2^{-\tilde{E}/4+1}}{\Gamma(-\tilde{E}/4+1)}, \\ \tilde{S}_4(\tilde{a},\tilde{b}) &= \tilde{a}\sqrt{\pi}\dfrac{2^{\tilde{E}/4+1}e^{-i\pi \tilde{E}/4}}{\Gamma(\tilde{E}/4+1)}, \\ \tilde{S}_5(\tilde{a},\tilde{b}) &= ie^{i\pi \tilde{E}/2}-i\tilde{b}\sqrt{\pi}\dfrac{2^{-\tilde{E}/4+1}e^{i\pi \tilde{E}/2}}{\Gamma(-\tilde{E}/4+1)}, \end{cases} \tag{17}$$

where $E=-8ab$, $\tilde{E}=-8\tilde{a}\tilde{b}$ and $\Gamma(z)$ denotes the Gamma function. We conventionally define $S_j(a,b)$ [resp., $\tilde{S}_j(\tilde{a},\tilde{b})$] for every integer j by requiring $S_{j+5}(a,b)=S_j(a,b)$ [resp., $\tilde{S}_{j+5}(\tilde{a},\tilde{b})=\tilde{S}_j(\tilde{a},\tilde{b})$]. Under these notations we can state our second result in the following way:

Theorem 2 *Let the Stokes curve γ and the Stokes regions A and B for $\lambda_{\mathrm{I}}^{(0)}$ be chosen as above. For any 2-parameter solution $\lambda_{\mathrm{I}}(t;\alpha,\beta)$ of (P_{I}) in Region A, let $\lambda_{\mathrm{I}}(t;\tilde{\alpha},\tilde{\beta})$ denote the corresponding solution in Region B which can be obtained as the analytic continuation of $\lambda_{\mathrm{I}}(t;\alpha,\beta)$ across γ. Then among the parameters (α,β) and $(\tilde{\alpha},\tilde{\beta})$ the following relations hold:*

$$S_j(a,b) = \tilde{S}_j(\tilde{a},\tilde{b}), \tag{18}$$

that is,

$$S_j(\alpha e^{4i\pi\alpha\beta},\beta e^{-4i\pi\alpha\beta}) = \tilde{S}_j(\tilde{\alpha}e^{4i\pi\tilde{\alpha}\tilde{\beta}},\tilde{\beta}e^{-4i\pi\tilde{\alpha}\tilde{\beta}}) \tag{19}$$

($j = 1,2,3,4,5$). In particular, the Borel resummed $\lambda_{\mathrm{I}}^{(0)} = \lambda_{\mathrm{I}}(t;0,0)$ in Region A corresponds to $\lambda_{\mathrm{I}}(t;\frac{i}{2\sqrt{\pi}},0)$ in region B after the analytic continuation across γ.

Remark 2 The functions $S_j(a,b)$ and $\tilde{S}_j(\tilde{a},\tilde{b})$ defined above satisfy the following cyclic relations respectively:

$$1 + S_{j-1}(a,b)S_j(a,b) + iS_{j+2}(a,b) = 0, \tag{20}$$

$$1 + \tilde{S}_{j-1}(\tilde{a},\tilde{b})\tilde{S}_j(\tilde{a},\tilde{b}) + i\tilde{S}_{j+2}(\tilde{a},\tilde{b}) = 0, \tag{21}$$

($j = 0,\pm1,\pm2,\ldots$). These relations entail that among the five relations (18) only two of them are independent. Hence, when (α,β) is given, the relations (18) determines $(\tilde{\alpha},\tilde{\beta})$ almost completely. In this sense (18) may be considered as the connection formula for $\lambda_{\mathrm{I}}(t;\alpha,\beta)$. Concerning previous results for the connection formula for (P_{I}) see [JK1], [JK2], [K], [KK] etc.

Remark 3 The connection formula for $\lambda_{\mathrm{I}}(t;\alpha,\beta)$ on the other Stokes curves is described in a similar manner. To be more specific, let us define a transformation T (in the space of parameters (a,b)) by $T(a,b) = (-ib,-ia)$. Then the connection formula on any Stokes curve is given by the following:

$$S_j(T^k(a,b)) = \tilde{S}_j(T^k(\tilde{a},\tilde{b})) \quad (j = 1,2,3,4,5). \tag{22}$$

Here k is an appropriately chosen integer which depends only on the Stokes curve in question and the choice of the branch of $\lambda_0(t)$.

In the subsequent section we will give a sketch of proofs of Theorem 1 and Theorem 2.

In principle, combining Theorem 1 and Theorem 2, we should obtain the connection formula for $\lambda_J(t;\alpha,\beta)$ for general J. However, the local reduction (i.e., the relation (11)) holds only in the formal sense and to obtain the connection formula for $\lambda_J(t;\alpha,\beta)$ we need some analytic interpretation of (11). We believe that to clarify the precise analytic meaning of (11) is one of the most important open problems in this theory.

4 Sketch of proofs.

A key idea of the proofs of Theorem 1 and Theorem 2 is the relationship between Painlevé equations and isomonodromic deformations of some relevant Schrödinger equations (cf. [JMU], [O] etc.): As is well-known, (P_J) is equivalent to the Hamiltonian system

$$(H_J)\qquad \begin{cases} \dfrac{d\lambda}{dt} &= \eta\dfrac{\partial K_J}{\partial \nu}, \\ \dfrac{d\nu}{dt} &= -\eta\dfrac{\partial K_J}{\partial \lambda}, \end{cases}$$

which arises as a condition for isomonodromic deformations of the following Schrödinger equation:

$$(SL_J)\qquad \left(-\frac{\partial^2}{\partial x^2}+\eta^2 Q_J(x,t,\eta)\right)\psi(x,t,\eta)=0.$$

To be more precise, (H_J) describes a compatibility condition of (SL_J) and the following deformation equation:

$$(D_J)\qquad \frac{\partial \psi}{\partial t}=A_J\frac{\partial \psi}{\partial x}-\frac{1}{2}\frac{\partial A_J}{\partial x}\psi,$$

and the compatibility of (SL_J) and (D_J) analytically implies that the monodromy data (i.e., the monodromy groups and the Stokes multipliers) of (SL_J) should be independent of t. For the explicit forms of K_J, Q_J and A_J see [KT1] and [KT3]. This relationship will play a crucially important role in the proofs of the fundamental results. Before explaining their outlines, let us first list up several basic properties of (SL_J) and (D_J).

Using the instanton-type solution $\lambda_J(t;\alpha,\beta)$ of (P_J), we can construct the instanton-type solution $(\lambda_J(t;\alpha,\beta),\nu_J(t;\alpha,\beta))$ of (H_J) ([KT3, Remark 1.3]). Let us substitute the instanton-type solution into (λ,ν) in the coefficients of $Q_J(x,t,\eta)$ and A_J. In what follows the resulting equations thus obtained will be denoted by the same symbols (SL_J) and (D_J). Then we have the following

Proposition 1 ([KT1, Proposition 1.3]) *The point $x = \lambda_0(t)$ is a double turning point of (SL_J), that is,*

$$Q_{J,0}(\lambda_0(t), t) = \frac{\partial Q_{J,0}}{\partial x}(\lambda_0(t), t) = 0, \tag{23}$$

where $Q_{J,0}$ is the top term of the potential Q_J of (SL_J).

In the WKB theory for Painlevé equations the double turning point $\lambda_0(t)$ of (SL_J) plays the role of "guide-post" in a sense. For example, $\lambda_0(t)$ relates the Stokes geometry of (P_J) with that of (SL_J) in the following way:

Proposition 2 ([KT1, Proposition 2.1 and Corollary 2.1]) *Let r be a simple turning point for $\lambda_J^{(0)}$. Then there exists a simple turning point $x = a(t)$ of (SL_J) which satisfies the following:*
(i) When $t = r$, $a(t)$ merges with the double turning point $x = \lambda_0(t)$.
(ii) When $t(\neq r)$ lies on a Stokes curve for $\lambda_J^{(0)}$ emanating from r, there exists a Stokes curve of (SL_J) that connects the two turning points $\lambda_0(t)$ and $a(t)$.

Furthermore, on a neighborhood of $\lambda_0(t)$ the equation (SL_J) together with (D_J) can be transformed to some canonical equation in the following manner: Let (Can) denote the equation

$$\left(-\frac{\partial^2}{\partial x^2} + \eta^2 Q_{\mathrm{can}}(x, t, \eta)\right) \psi(x, t, \eta) = 0 \tag{24}$$

with

$$Q_{\mathrm{can}} = 4x^2 + \eta^{-1}E + \frac{\eta^{-3/2}\rho}{x - \eta^{-1/2}\sigma} + \frac{3\eta^{-2}}{4(x - \eta^{-1/2}\sigma)^2}, \tag{25}$$

where E, ρ and σ are parameters satisfying $E = \rho^2 - 4\sigma^2$, and let (D_{can}) denote the equation

$$\frac{\partial \psi}{\partial t} = A_{\mathrm{can}} \frac{\partial \psi}{\partial x} - \frac{1}{2}\frac{\partial A_{\mathrm{can}}}{\partial x}\psi \tag{26}$$

with

$$A_{\mathrm{can}} = \frac{1}{2(x - \eta^{-1/2}\sigma)}. \tag{27}$$

We can readily verify that (Can) and (D_{can}) are compatible if ρ and σ satisfy the following Hamiltonian system

$$(H_{\text{can}}) \qquad \begin{cases} \dfrac{d\rho}{dt} = -4\eta\sigma, \\ \dfrac{d\sigma}{dt} = -\eta\rho. \end{cases}$$

Otherwise stated, (Can) can be isomonodromically deformed if (H_{can}) is satisfied. Under these notations we can prove that the simultaneous equations (SL_J) and (D_J) are transformed to the simultaneous equations (Can) and (D_{can}), that is, the following holds:

Proposition 3 ([KT3, Proposition 3.1]) *Let $\tilde{V}$ be a sufficiently small neighborhood of a point $\tilde{t}_*$ in question. Then there exist a neighborhood $\tilde{U}$ of $\tilde{x} = \tilde{\lambda}_0(\tilde{t})$ and the formal series*

$$x_J(\tilde{x},\tilde{t},\eta) = \sum_{j\geq 0} \eta^{-j/2} x_{J,j/2}(\tilde{x},\tilde{t},\eta), \tag{28}$$

$$t_J(\tilde{t},\eta) = \sum_{j\geq 0} \eta^{-j/2} t_{J,j/2}(\tilde{t},\eta), \tag{29}$$

whose coefficients $x_{J,j/2}(\tilde{x},\tilde{t},\eta)$ and $t_{J,j/2}(\tilde{t},\eta)$ are holomorphic on $\tilde{U}\times\tilde{V}$ and $\tilde{V}$ respectively, so that the following holds:

Let $\psi(x,t,\eta)$ be a WKB solution of (Can) that satisfies (D_{can}) also, and let $\tilde{\psi}(\tilde{x},\tilde{t},\eta)$ denote the following function:

$$\tilde{\psi}(\tilde{x},\tilde{t},\eta) = \left(\frac{\partial x_J}{\partial \tilde{x}}\right)^{-1/2} \psi(x_J(\tilde{x},\tilde{t},\eta), t_J(\tilde{t},\eta),\eta). \tag{30}$$

Then $\tilde{\psi}$ satisfies both (SL_J) and (D_J).

Note that $x_J(\tilde{x},\tilde{t},\eta)$ is (almost) uniquely determined, while $t_J(\tilde{t},\eta)$ is unique only modulo additive constants.

Outline of the proof of Theorem 1. Proposition 2 tells us that if $\tilde{t}_*$ is a point in a Stokes curve for $\lambda_J^{(0)}$ a Stokes curve of (SL_J) connects a simple turning point $\tilde{a}(\tilde{t}_*)$ and the double turning point $\tilde{\lambda}_0(\tilde{t}_*)$. Let $\tilde{\Gamma}$ denote the portion of the Stokes curve that begins at $\tilde{a}(\tilde{t}_*)$ and ends at $\tilde{\lambda}_0(\tilde{t}_*)$. What we have to do is to construct a "semi-global" transformation $(x(\tilde{x},\tilde{t},\eta), t(\tilde{t},\eta))$

that brings (SL_J) to (SL_{I}) in a neighborhood of $\tilde{\Gamma}$. According to Proposition 3, we have already the transformation near $\tilde{\lambda}_0(\tilde{t}_*)$ defined by

$$(31)\qquad \begin{cases} x(\tilde{x},\tilde{t},\eta) &= x_{\mathrm{I}}^{-1}(x_J(\tilde{x},\tilde{t},\eta),t_J(\tilde{t},\eta),\eta), \\ t(\tilde{t},\eta) &= t_{\mathrm{I}}^{-1}(t_J(\tilde{t},\eta),\eta). \end{cases}$$

To prove the holomorphy at $\tilde{a}(\tilde{t}_*)$ of the transformation thus defined, let us consider another transformation $y(\tilde{x},\tilde{t},\eta)$ which brings (SL_J) to (SL_{I}) near $\tilde{a}(\tilde{t}_*)$ and compare the two transformations $x(\tilde{x},\tilde{t},\eta)$ and $y(\tilde{x},\tilde{t},\eta)$. Making use of the deformation equation (D_J) effectively, we can verify that the difference between $x(\tilde{x},\tilde{t},\eta)$ and $y(\tilde{x},\tilde{t},\eta)$ is essentially described by some constant depending only on η and being independent of $(\tilde{x},\tilde{t})$. Hence, by choosing the free parameter contained in t_{I} correctly we can make this constant vanish. Thus the semi-global transformation $(x(\tilde{x},\tilde{t},\eta),t(\tilde{t},\eta))$ is obtained and for this transformation we can prove the relation (11). Note that in the course of the construction of the transformation near the double turning point $\tilde{\lambda}_0(\tilde{t}_*)$ one relation between parameters $(\tilde{\alpha},\tilde{\beta})$ and (α,β) contained in $\tilde{\lambda}_J(\tilde{t};\tilde{\alpha},\tilde{\beta})$ and $\lambda_{\mathrm{I}}(t;\alpha,\beta)$ respectively appears and that the above adjustment of the free parameter finally determines the correspondence of $(\tilde{\alpha},\tilde{\beta})$ and (α,β) uniquely. For the complete proof of Theorem 1 see [KT3, Section 4].

Outline of the proof of Theorem 2. The equation (Can) is known to be a variant of the classical Weber equation (cf. [T2, §4.2]). Hence the connection formula for the Weber equation and Proposition 3 provide a connection formula for WKB solutions of (SL_{I}) at the double turning point $x=\lambda_0(t)$ and by the homogeneity of (SL_{I}) and (P_{I}) we find it is exact. Using the formula thus obtained together with the well-known connection formula at a simple turning point (cf. [V], [AKT1], [DDP]), we can compute the Stokes multipliers of (SL_{I}) in an exact manner. As a matter of fact, $S_j(a,b)$ and $\tilde{S}_j(\tilde{a},\tilde{b})$ are nothing but the Stokes multipliers of (SL_{I}) in respective regions. Note that Proposition 2 implies that the consequences of the computation should be different in respective regions. (Compare (16) and (17)!) Then the relation (18) immediately follows from the isomonodromy property, that is, the fact that the Stokes multipliers of (SL_{I}) should be preserved if (P_{I}) is satisfied. For the details of the computation we refer the reader to [T4]. (The computation for the top degree part has already been done essentially in [T2].)

References

[AKT1] T. Aoki, T. Kawai and Y. Takei: The Bender-Wu analysis and the Voros theory. Special Functions, Springer-Verlag, 1991, pp. 1–29.

[AKT2] ————: WKB analysis of Painlevé transcendents with a large parameter. II. — Multiple-scale analysis of Painlevé transcendents. Structure of Solutions of Differential Equations, World Scientific, 1996, pp.1–49.

[DDP] E. Delabaere, H. Dillinger and F. Pham: Résurgence de Voros et périodes des courbes hyperelliptiques. Ann. Inst. Fourier (Grenoble), **43**(1993), 163–199.

[JK1] N. Joshi and M.D. Kruskal: An asymptotic approach to the connection problem for the first and the second Painlevé equations. Phys. Lett., **130**(1988), 129–137.

[JK2] ————: Connection results for the first Painlevé equation. Painlevé Transcendents, Plenum, 1992, pp. 61–79.

[JMU] M. Jimbo, T. Miwa and K. Ueno: Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. I. Physica D, **2**(1981), 306–352.

[K] A.A. Kapaev: Asymptotics of solutions of the Painlevé equation of the first kind. Differential Equations, **24** (1989), 1107–1115. (Translated from the Russian original (1988).)

[KK] A.A. Kapaev and A.V. Kitaev: Connection formulae for the first Painlevé transcendent in the complex domain. Lett. in Math. Phys., **27**(1993), 243–252.

[KT1] T. Kawai and Y. Takei: WKB analysis of Painlevé transcendents with a large parameter. I. Adv. in Math., **118**(1996), 1–33.

[KT2] ————: On the structure of Painlevé transcendents with a large parameter. II. Proc. Japan Acad. Ser. A, **72**(1996), 144–147.

[KT3] ————: WKB analysis of Painlevé transcendents with a large parameter. III. — Local equivalence of 2-parameter Painlevé transcendents. In prep.

[O] K. Okamoto: Isomonodromic deformation and Painlevé equations, and the Garnier systems. J. Fac. Sci. Univ. Tokyo, Sect. IA, **33**(1986), 575–618.

[T1] Y. Takei: On a WKB-theoretic approach to the Painlevé transcendents. XIth International Congress of Mathematical Physics, International Press, 1995, pp. 533–542.

[T2] ————: On the connection formula for the first Painlevé equation — from the viewpoint of the exact WKB analysis —. RIMS Kôkyûroku, No. 931, 1995, pp. 70–99.

[T3] ————: A remark on instanton-type formal solutions of Painlevé equations with a large parameter. In prep.

[T4] ————: An explicit description of the connection formula for the first Painlevé equation. In prep.

[V] A. Voros: The return of the quartic oscillator. The complex WKB method. Ann. Inst. H. Poincaré, **39**(1983), 211–338.