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Geometric Consideration of Extension Problem

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Abstract

In the present paper, we investigate the problem of extending bounded holomorphic functions from one-codimensional subvarieties to ambient spaces.

The problem concerns deeply the $\bar{\partial}$-analysis of the theory of functions of several complex variables and we investigate it numerically too.

1 Introduction.

In the Summer Seminar at Tateyama on Several Complex Variables, 18th July 1994, Professor T. Ohsawa posed the following problem, which was the starting point of the present work:

Let Ω be a bounded pseudoconvex domain in C^n and H be a one codimensional complex linear subspace of C^n. For any bounded holomorphic function f on $\Omega \cap H$, does there exist a bounded holomorphic function F on Ω such that the restriction $F|\Omega \cap H$ of F to $\Omega \cap H$ coincides with f on $\Omega \cap H$?

H. Alexander[4] considered the problem in case that H is a Rudin variety in the unit polydisk Δ^N of C^N.

M. Henkin-P. L. Polyakov[14] and P. L. Polyakov[27] gave the theories on the extension problem in case that H is an analytic curve in general position in a polydisc in C^n.

G. M. Henkin[12] investigated the problem in case that Ω is a strictly pseudoconvex domain and H is an analytic closed submanifold in general position in Ω.

K. Adachi[2] proved that Henkin's results are still valid when Ω is a pseudoconvex domain with smooth boundary and H is a subvariety where $\partial H \cap \Omega$ consists of strictly pseudoconvex boundary points of Ω.

The present paper consists of 2 parts.

In section 2, we give counter examples for the Ohsawa's problem.

Section 3 concerns Numerical Analysis on $\bar{\partial}$-problem and we aim to solve the inhomogeneous $\bar{\partial}$-equation numerically firstly, using the integral formula by Hörmander and the finite element method. Secondly we apply it to the extension problem.

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2 Counterexamples.

At first, we give a counterexample for the Ohsawa's problem in case that Ω is an unbounded weakly pseudoconvex domain, the boundary of which is not smooth.

Theorem 2.1 (M. Tsuji[29]) *We consider a domain*

$$\Omega = \{(z, w) \in C^2; |z^2| < |w|\}$$

in C^2, and let $F(z, w)$ be a bounded holomorphic function in the domain Ω.

Then there exists a holomorphic function $h(\xi)$ in the unit disk in C such that we have $F(z, w) = h(\frac{z^2}{w})$ in Ω.

Corollary 2.1 (M. Tsuji[29]) *Consider a hyperplane in C^2*

$$H = \{(z, w) \in C^2; w = 1\}$$

and let $f(z)$ be the holomorphic function on $\Omega \cap H$ defined by

$$f(z) = z \quad ((z, 1) \in \Omega \cap H).$$

Then there is no bounded holomorphic function $F(z, w)$ on Ω such that the restriction $F|\Omega \cap H$ of F to $\Omega \cap H$ coincides with f on $\Omega \cap H$.

Next, we give a counterexample in case that Ω is bounded but has not smooth boundary, using Sibony's domain.

Let $\Delta(z, r)$ be the disk with center z and semiradius r in the complex plane. The unit disk $\Delta(0, 1)$ is denoted by Δ.

Lemma 2.1 (Sibony[28]) *Let $\{a_\nu\}_{\nu=1}^{\infty}$ be a sequence of points without cluster point in Δ such that each point of the unit circle $\partial\Delta$ is the nontangential limit of a subsequence of $\{a_\nu\}_{\nu=1}^{\infty}$. We define a function $\lambda : \Delta \to R \cup \{-\infty\}$ by*

$$\lambda(z) = \sum_{\nu=2} \epsilon_\nu \log |\frac{z - a_\nu}{2}|$$

where $\epsilon_\nu \searrow 0$ rapidly so that $\lambda \not\equiv -\infty$ and is subharmonic on Δ. Further let $\psi : \Delta \to [0, 1)$ be the subharmonic function $\psi(z) = \exp(\lambda(z))$.

Define a pseudoconvex domain $U \subset \Delta^2$ by

$$U = \{(z, w) \in \Delta^2; |w| < e^{-\psi(z)}\}.$$

Then the domain U is a proper subdomain of Δ^2 and all bounded holomorphic functions on U is extended holomorphically to Δ^2.

Moreover, he noted that there exist $0 < \eta, \zeta < 1$ so that if (z, w) satisfies $|z| < \eta$, then $|w| < \zeta$.

Lemma 2.2 (H. Hamada and M. Tsuji[10]) *Let w_0 be a real number with $\zeta < w_0 < 1$. Then a bounded holomorphic function $1/(w - w_0)$ on $\{(z, w) \in C^2; z = 0\} \cap U$ can not be extended bounded holomorphically to the domain U.*

Finally, we give a counterexample for Ohsawa's Problem of a connected subvariety, all holomorphic functions on which cannot be extended to the whole domain Ω with smooth boundary. The boundary of the subvariety H consists of strictly pseudoconvex boundary points of Ω, but H is not in general position in a pseudoconvex domain Ω.

Lemma 2.3 *Let $\{\psi_k; k \geq 1\}$ be a sequence of C^∞ strictly subharmonic functions ψ_k on C with $\psi_k(z) \geq \psi_{k+1}(z)$ for each point $z \in C$ converging to a function ψ. Let*

$$U_n = \{(z,w) \in C^2; |z| < 1, \log|w| + \psi_n < 0\}.$$

If the function $1/(w - w_0)$ on $\{(z,w) \in C^2; z = 0\} \cap U$ can be extended to a bounded holomorphic function F_n on U_n, there exists a sequence $C_n; n \geq 1$ of positive numbers $C_n \nearrow \infty$ such that $|F_n(z,w)| \geq C_n$ for any $(z,w) \in U_n$.

Lemma 2.4 (Fornaess and Sibony[8]) *There exists a Reinhardt domain R in C^2 with smooth boundary satisfying the following conditions:*

1. $R = \{(z,w) \in C^2; \log|w| + \varphi(z) < 0\}$ for a smooth subharmonic function $\varphi(z) = \varphi(|z|)$ on the open unit disc Δ such that $\varphi(z) \to +\infty$ as $|z| \to 1$.

2. The Laplacian of φ vanishes precisely on a sequence $\{A_n; n \geq 1\}$ of disjoint annuli $A_n = \{z \in C; x_n - 2d_n < |z| < x_n + 2d_n\}$, where $x_n + 3d_n = 1 (n \geq 1)$ and $x_n \nearrow 1$ as $n \to \infty$.

3. There exist positive integers p_n, q_n, and real constants a_n such that we have $\varphi(z) = (p_n/q_n)\log|z| + a_n$ for any $z \in A_n$.

Fornaess and Sibony[8] constructed the following domain : Let ρ be a smooth nonnegative subharmonic function which vanishes precisely on $\overline{\Delta}(0,2)$ and which is strictly subharmonic when $|z| > 2$. For each $n \geq 1$, let V_n be an open set in C, K_n be a compact set in C such that $A_n \subset V_n \subset K_n$ and that $K_n \cap K_m = \phi$ for $1 \leq n < m$. Let $\sigma_n(z)$ be a C^∞ function on C such that $\sigma_n(z) \equiv 1$ on V_n and the support of $\sigma_n(z)$ is contained in K_n.

Let $\epsilon_n; n \geq 1$ be a sequence of positive numbers $\epsilon_n \searrow 0$. We define a Hartogs domain

$$B = \{(z,w) \in C^2; \log|w| + \varphi_1(z) < 0\},$$

where

$$\varphi_1(z) = \varphi(z) + \sum_{n=1}^{\infty} \epsilon_n \sigma_n(z) \rho(\frac{z - x_n}{d_n}).$$

For each $n \geq 1$, let M_n be a multiples of q_n and $\chi_n \geq 0$ be a C^∞ function on C with compact support such that $\chi_n(z) \geq 0$ for any $z \in C$ and that $\chi_n \equiv 1$ in a neighborhood of $\bar{\Delta}(x_n, 2d_n)$. Let

$$B' = \{(z,w) \in C^2; |z| < 1, \log|w| + \varphi_2(z) < 0\},$$

where

$$\varphi_2(z) = \varphi_1(z) + \sum_n \chi_n \psi_n(\frac{z - x_n}{d_n})/M_n.$$

We can choose the M_n's so large that B' has smooth boundary and is strictly pseudoconvex except in the set $\{(z,w) \in C^2; |z| = 1, |w| = 0\}$.

Define

$$G(z) = \prod_{n=1}^{\infty} \frac{z - x_n}{1 - zx_n}. .$$

Then, there exist positive constants c and C such that, for $z \in \Delta(x_n, 2d_n)$,

$$c\frac{|z - x_n|}{d_n} \leq |G(z)| \leq C\frac{|z - x_n|}{d_n},$$

and, for $z \notin \cup_{n\geq 1}\Delta(x_n, 2d_n)$, $|G(z)| > c$. Also we have $|G| < 1$ on Δ.

Define a variety V by

$$V = \{(z,w) \in \Delta{\times}C; wG(z) = 0\} = \cup_{n=1}^{\infty}\{(z,w) \in C^2; z = x_n\}\cup\{(z,w) \in C^2; w = 0\},$$

which is a connected subvariety, and a monomial P_n in $(z,w) \in C^2$ by

$$P_n = e^{a_n q_n} z^{p_n} w^{q_n}.$$

Since for $z \in \Delta(x_n, 2d_n)$, $\varphi_1(z) = (p_n/q_n)\log|z| + a_n$, it holds that

$$|P_n|^{M_n/q_n} < \exp(-\psi_n(\frac{z - x_n}{d_n})) \leq \exp(-\psi(\frac{z - x_n}{d_n})) \quad \text{on } \{z = x_n\} \cap B'.$$

Thus $|P_n|^{M_n/q_n} < \zeta < w_0$ on $\{z = x_n\} \cap B'$. As a result, a function on $V \cap B'$ given by

$$f(z,w) = \begin{cases} 1/(P_n^{M_n/q_n} - w_0) & \text{on} \quad \{(z,w) \in C^2; z = x_n\} \cap B' \\ -1/w_0 & \text{on} \quad \{(z,w) \in C^2; w = 0\} \cap B' \end{cases}$$

is a bounded holomorphic function on $V \cap B'$.

Theorem 2.2 *$f(z,w)$ can not be extended to bounded holomorphic function $F(z,w)$ on B'.*

Proof. Let $B^{(n)} = \{(z,w) \in B'; |z - x_n|/d_n < 1\}$. We have a proper holomorphic map $\Phi_n : B^{(n)} \to U_n$,

$$\Phi_n : (z,w) \longmapsto (\frac{z - x_n}{d_n}, P_n^{M_n/q_n}).$$

The function $1/(w - w_0)$, which is regarded as defined on the set $\{(0,w) \in U_n\}$, can not be extended to a holomorphic function on U_n, the modulus of which at a point is less than C_n by Lemma 2.3. If there is holomorphic function on B_n with norm less than C_n, then by averaging the solutions over fibers of Φ_n, we obtain a holomorphic function on U_n with norm less than C_n.

So if $f(z,w)$ were extended to a bounded holomorphic function $F(z,w)$ on B', we would have $\|F(z,w)\| \geq C_n$. Since $C_n \to +\infty$ as $n \to \infty$ and since the extended function $F(z,w)$ were bounded on B', this is a contradiction. ∎

It remains only to modify B' near the unit circle $T \times \{0\}$ so that the resulting Hartogs domain is strictly pseudoconvex everywhere except at (1,0). The following process is the same in [8]. Choose a smooth defining function $r(z,w)$ for B' so that some root $-(-r)^{1/N}$ is strictly plurisubharmonic on B'. We write

$-(-r)^{1/N} = -|\delta(z,|w|)|^{1/N}s(z,w)$, where δ is the signed distance function and $s > 0$ is smooth on a neighborhood of the boundary of B'. Then we get a new strictly plurisubharmonic function ρ by averaging:

$$\rho(z,|w|) = \frac{-1}{2\pi}\int_0^{2\pi} |\delta(z,|w|)|^{1/N}s(z,we^{i\theta})d\theta = -|\delta(z,|w|)|^{1/N}\tilde{s}(z,w),$$

where $\tilde{s}$ is smooth in a neighborhood of the boundary of B' and is > 0.

Next, let $\gamma \geq 0$ be a smooth function on C, strictly subharmonic away from 1 and vanishing only at 1. We can make γ vanish sufficiently fast to infinite order at 1 so that the perturbation Ω to B' will still be a counterexample to the Ohsawa's problem in case of variety by using the same example as for B'. Let Ω be defined by the inequality $\{(z,w) \in C^2; \rho(z,|w|) + \gamma(z) < 0\}$. The domain Ω satisfies all conditions.

3 Numerical approach to holomorphic extension

At first, J. Kajiwara, C. L. Parihar, V. M. Raffee and M. Tsuji[19] investigated the appropriate algorithm of $\overline{\partial}-$equation, comparing algorithms. We used the Software Fortran77 and Hardware Super Computer, FACOM VP-2600.

We use the following numerical example.

Example 1 *For positive numbers c and d with $c < d$, we put*

$$\varphi_1(x) := \exp(-\frac{1}{x^2}) \quad (x > 0), \quad \varphi_2(x;c,d) := \varphi_1(x^2-c)\varphi_1(d-x^2),$$

and, for $x > 0$, we put

$$\varphi_3(x;c,d) := \frac{\int_c^x \varphi_2(t;c,d)}{\int_c^d \varphi_2(t;c,d)dt} \qquad (x > \sqrt{c}).$$

Then the function $\varphi_3(x;c,d)$ of class C^∞ satisfies $\varphi_3(x;c,d) = 0$ on $x < \sqrt{c}$, $0 < \varphi_3(x;c,d) < 1$ on $\sqrt{c} < x < \sqrt{d}$ and $\varphi_3(x;c,d) = 1$ on $x > \sqrt{d}$.

For $z \in C$, let

$$\psi(z) := 1 - \varphi_3(|z|;16,25)$$

and

$$h(z) = {}_1F_1(\frac{1}{2};\frac{1}{3},z) = \sum_{k=0}^{\infty}\frac{(\frac{1}{2})_k}{(\frac{1}{3})_k}\frac{z^k}{k!}.$$

We obtain a (0,1)-form by putting

$$g := h(z)\overline{\partial}\psi.$$

In our experiments, we have compared following algorithms in double precision computation.

Our first method is based on the integral formula by Hölmander.

Theorem 3.1 (Hörmander[15]) *Let g be a $C^{\infty}(C)$ (0,1)-form with a compact support.*

Then we have

$$u(z) := \frac{1}{2\pi i}\int\int \frac{g(\tau)}{\tau - z} d\tau d\overline{\tau}$$

as a solution of the $\overline{\partial}$-equation

$$\overline{\partial} u = g.$$

We apply double integration to τ_1 and τ_2 where $\tau = \tau_1 + i\tau_2$, using the DE formula which is effective when the function is analytic.

It takes about 46 hours to calculate $u(z)$ at 50 points of z, where $z = -5 - 5i + k(0.1 + 0.1i)$, $k = 1, \cdots, 50$. The reason why it takes long time is that when $4 < |z| < 5$, the value z becomes a singular point of calculation for single integral $\int \frac{g(\tau_1+\tau_2 i)}{\tau_1+\tau_2 i - z} d\tau_1$ with $\tau_2 = \mathrm{Im} z$, and hence many sampling points of τ near singular points of z are necessarily taken to compute $u(z)$.

However, for a small number ϵ and the maximum M of the values $|g|$ on $|\tau - z| < \epsilon$, it holds that

$$|\frac{1}{2\pi i}\int_{|\tau - z| < \epsilon} \frac{g(\tau)}{\tau - z} d\tau d\overline{\tau}| \leq M\frac{1}{2\pi}\int_0^{\epsilon}\int_0^{2\pi}(1/r) r dr d\theta = M\epsilon \to 0 \quad (\epsilon \to 0)$$

and thus the sampling points near z are not necessary to get the value $u(z)$ with high precision to some degree. Therefore, in the next method, we avoid taking sampling points near z.

For the second double integration, taking the above fact into account, in order to avoid calculating the values near the singular points for $4 < |z| < 5$ and because the DE formula is very efficient and is rather insensitive to singularities that may occur at the end points, the integral intervals for τ_1 are set on $[-5, \mathrm{Re} z - \epsilon]$ and $[\mathrm{Re} z + \epsilon, 5]$ where $\tau_2 \in [\mathrm{Im} z - \epsilon, \mathrm{Im} z + \epsilon]$ and $\epsilon = 10^{-8}$. By the second integration, we get the similar results by the first integration and it takes about 20 minutes for the same points of z.

As the third integration, after transformation of $\tau = z + t\exp(i\theta)$, we adapt the Newton-Cotes formula of 8 degree to θ and t by the adaptive automatic integration.

For the adaptive automatic integration, it is better to use the formula such that sampling points are taken on the same length mesh and hence we choose the Newton-Cotes formula not the DE formula. (See [23])

By the third method, in 18 minutes, we obtain similar results for the same points as those in the first and second integration.

The results indicates that for singular points of z, the Newton-Cotes formula is best algorithm, on the other hand, for no singular points of z, the DE formula is best one.

Finally, by the fourth method, we apply the finite element method to the equation $\Delta u = \frac{1}{4}\bar{\partial}\partial u = \frac{1}{4}\partial g$ and the boundary condition $u = 0$ on the boundary. We point out in [20] that it is impossible to apply the finite element method directly to $\bar{\partial} u = g$.

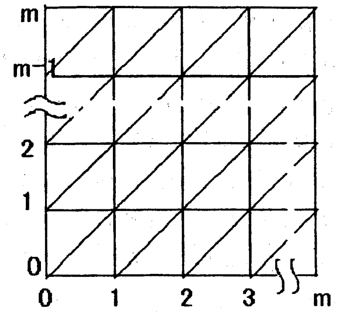
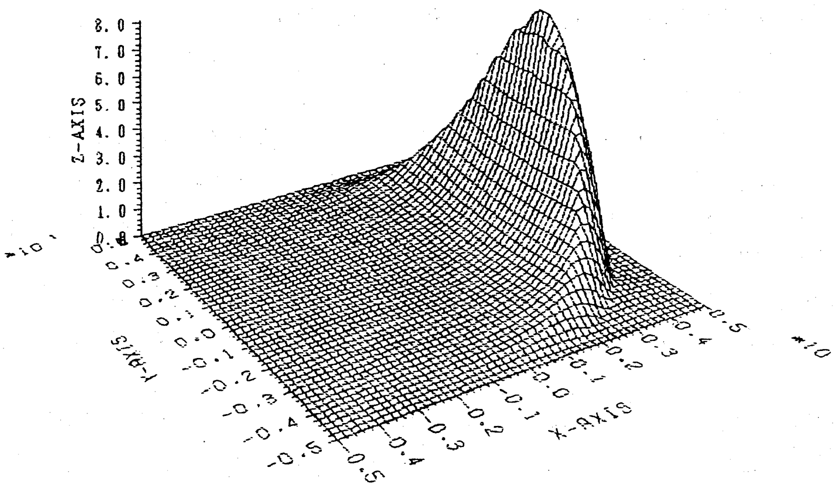
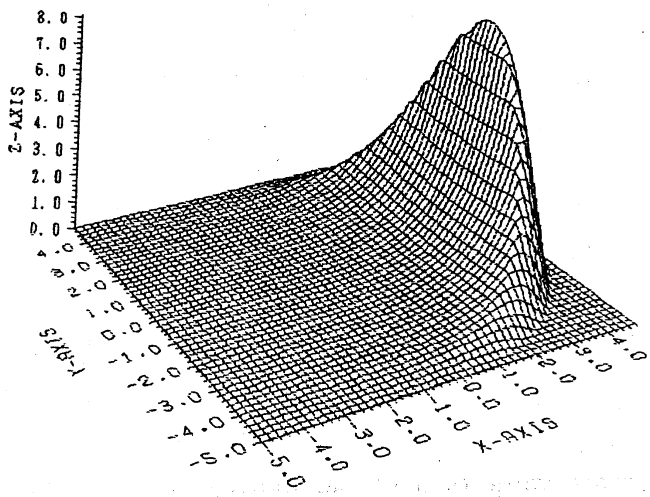


Figure 1: Figure of division of the finite element method.



Finite element method



Newton-Cotes formula

Figure 2: The graph of $|u|$ by the finite element method with m =100 division as Figure 1 and by the Newton-Cotes formula.

In Figure 3, we show the graph of $|u|$ by the finite element method with $m = 100$ division as Figure 1, compared with the graph by the Newton-Cotes formula.

In Figure 2, we show the graph of $|u|$ by the finite element method with $m = 100$ division as Figure 1, compared with the graph by the Newton-Cotes formula.

A reasonably close agreement is observed in the graph of the finite element metod with $m = 100$ division, where it takes only 6 seconds to calculate u and in the graph of the Newton-Cotes formula, where it takes about 1 hour 30 minutes.

However, the maximum error in the values computed by the finite element method are as much as about 0.38.

If the division m should be increased to get more precise values, we would need large memory capacity in hardware. It is a limit in the finite element method.

Hence if it is not necessary to get precise values, for example, if you only draw a graph, and your computer has sufficient memory, it is better to use the finite element method and you may obtain a quick result.

Next we consider extension problem, numerically.

Let H be a hypersurface given by $H := \{(z_1, z_2) \in C^2; z_1 + z_2 + \sqrt{2} = 0\}$. Then $\Delta^2 \cap H$ is expressed by $\{(z_1, z_2) \in C^2; |z_1| < 1, |z_1 + \sqrt{2}| < 1\}$.

Let $\pi : C^2 \to C^2$ be the Euclidean projection $\pi(z_1, z_2) = (z_1, -z_1 - \sqrt{2})$ and let $B = \{z \in \Delta^2; \pi(z) \notin \Delta^2 \cap H\}$.

Now, let $f(z_1, z_2)$ be a holomorphic function on $\Delta^2 \cap H$ and ψ be a function of class C^∞ on Δ^2 so that $\psi = 1$ in a relative neighborhood of $\Delta^2 \cap H$ and $\psi = 0$ on B.

Then the function $\frac{f(\pi(z))\bar{\partial}\psi}{z_1+z_2+\sqrt{2}}$ is well defined and C^∞ function, since it holds $\psi = 0$ on $B = \{z \in \Delta^2; \pi(z) \notin \Delta^2 \cap H\}$ on which $f(\pi(z))$ is not defined and $\bar{\partial}\psi = 0$ on a neighborhood of $\Delta^2 \cap H = \{(z_1, z_2) \in \Delta^2; z_1 + z_2 + \sqrt{2} = 0\}$. Moreover the function satisfies

$$\bar{\partial}\Big(\frac{f(\pi(z))\bar{\partial}\psi}{z_1 + z_2 + \sqrt{2}}\Big) = 0.$$

Theoretically, we have a solution u in Δ^2 such that

$$\bar{\partial}u = \frac{f(\pi(z))\bar{\partial}\psi}{z_1 + z_2 + \sqrt{2}}.$$

Let

$$F(z_1, z_2) := f(\pi(z))\psi - (z_1 + z_2 + \sqrt{2})u(z_1, z_2).$$

Since we have $\overline{\partial}F = 0$, F is holomorphic on the ambient bidisk Δ^2 with $F|_{\Delta^2 \cap H} = f$, that is, F is the desired holomorphic extension of f. Numerically we can get the approximate value of u by calculating the function u_1, expressed in the exact form by the following theorem.

Theorem 3.2 (J. Kajiwara, C. L. Parihar, V. M. Raffee and M. Tsuji[19]) *Let $g = g_1 d\overline{z_1} + g_2 d\overline{z_2}$ be a (0,1)-form of class C^∞ on $\{z \in C^2 : |z_1| < R, |z_2| < R\}$ such that $\overline{\partial}g = 0$.*

Then for $0 < r < R$,

$$
\begin{aligned}
u(z) &= \frac{1}{2\pi i}\int_{|\zeta|<r} g_1(\zeta, z_2)\frac{d\zeta\wedge d\bar{\zeta}}{\zeta - z_1} \\
&+\frac{1}{2\pi i}\int_{|\zeta'|<r}\left\{\frac{1}{2\pi i}\int_{|\zeta|=r} g_2(\zeta,\zeta')\frac{d\zeta}{\zeta - z_1}\right\}\frac{d\zeta'\wedge d\bar{\zeta'}}{\zeta' - z_2}
\end{aligned}
$$

satisfies

$$\bar{\partial}u = g \text{ on } \{z \in C^2 : |z_1| < r, |z_2| < r\}$$

Next we construct an example function ψ.

The set $\Delta^2 \cap H = \{(z_1, z_2) \in C^2; |z_1| < 1, |z_1 + \sqrt{2}| < 1\}$ is mapped by $v(z_1, z_2) := \sqrt{2}z_1 + 1$ to the circle-arced di-angular

$$|v - 1| < \sqrt{2}, \qquad |v + 1| < \sqrt{2}$$

which is mapped to the unit disk $|w| < 1$ by the mapping $w = V(v) := \frac{2v}{1-v^2}$.

Let $G(z) = V(v(\pi(z)))$ for $z \in \Delta^2$

For positive numbers c and d with $c < d$, we put

$$\phi_1(x) := \exp(-\frac{1}{x}) \qquad (x > 0), \quad \phi_2(x; c, d) := \phi_1(x - c)\phi_1(d - x),$$

and, for $x > 0$, we put

$$\phi_3(x; c, d) := \frac{\int_c^x \phi_2(t; c, d)}{\int_c^d \phi_2(t; c, d)dt} \qquad (x > c).$$

Let $t = |z_1 + z_2 + \sqrt{2}|^2$ and $w = G(z)$.

We define one example for ψ by, for $\frac{\pi}{2} > \epsilon > 0$,

$$\rho = \phi_3\left(\epsilon - \arctan(\frac{t}{1 - |w|^2}), -\frac{\pi}{2} + \epsilon, 0\right),$$

where $0 \le \arctan x \le 2\pi$.

The function ρ satisfies the condition that $\rho = 1$ in a relative neighborhood of $\Delta^2 \cap H$ and that $\rho = 0$ on B.

Example 2 *We define f by*

$$ {}_1F_1(\frac{1}{2}; \frac{1}{3}, z) = \sum_{k=0}^{\infty}\frac{(\frac{1}{2})_k}{(\frac{1}{3})_k}\frac{z^k}{k!}.$$

By calculating the approximate value of u on $D_1 = \{(z_1, z_2) \in C^2 : |z_1| < 0.98, |z_2| < 0.98\}$ we draw the graphs of $|u(z_1 + z_2 + \sqrt{2})|$, $|f\rho|$ and $|f\rho - u(z_1 + z_2 + \sqrt{2})|$ on $\{z_1 \in C : |z_1| \le 0.97\}$ at $z_2 = 0$.

Example 3 *We define f by*

$$f(w) = \sum_{k=0}^{\infty} w^{k!},$$

which is unbounded at each boundary point of $\Delta^2 \cap H$.

As the same as Example 2, we draw the graphs of $|u(z_1 + z_2 + \sqrt{2})|$, $|f\rho|$ and $|f\rho - u(z_1 + z_2 + \sqrt{2})|$ on $\{z_1 \in C : |z_1| \le 0.97\}$ at $z_2 = 0$.

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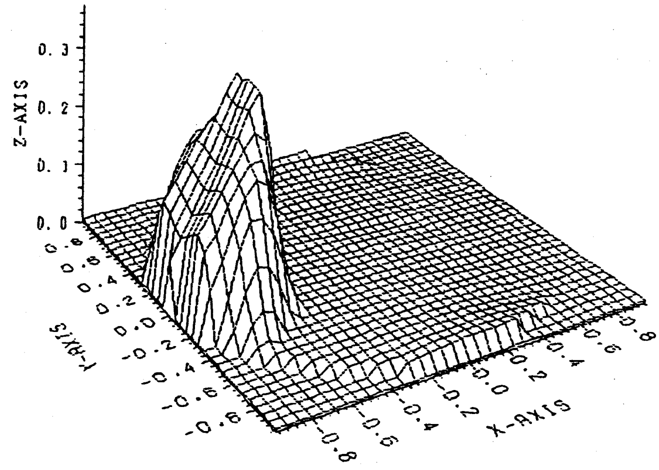
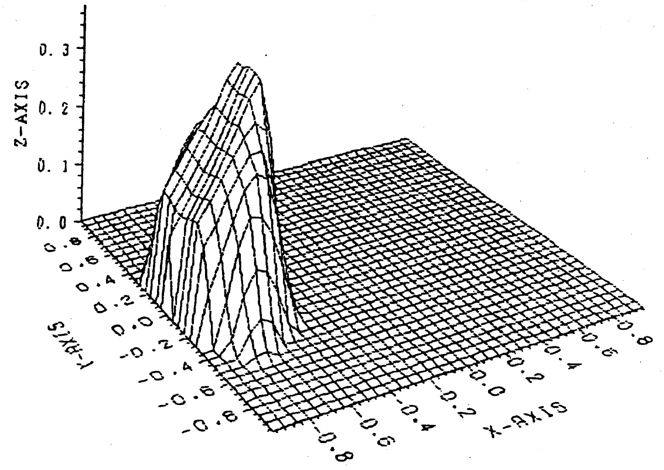
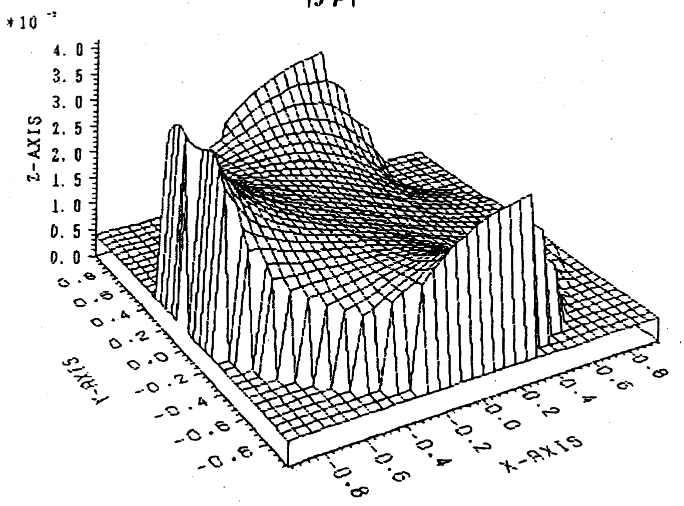


$|u(z_1 + z_2 + \sqrt{2})|$

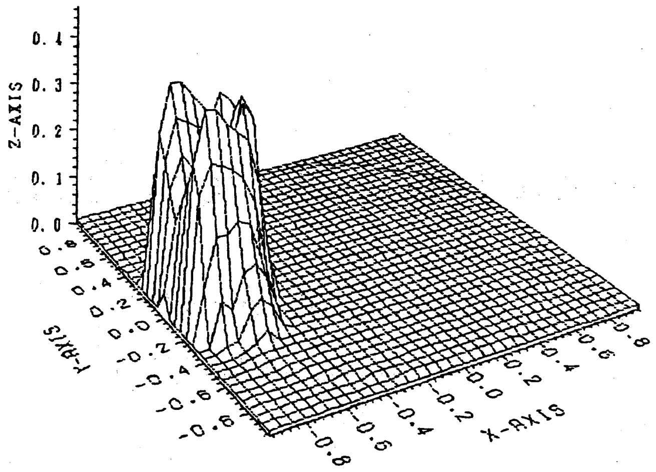


$|f\rho|$

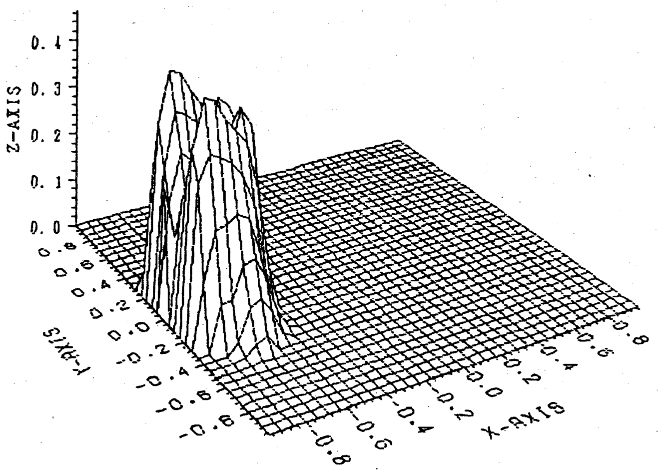


$|f\rho - u(z_1 + z_2 + \sqrt{2})|$

Figure 3: The graphs in Example 2.



$|u(z_1 + z_2 + \sqrt{2})|$



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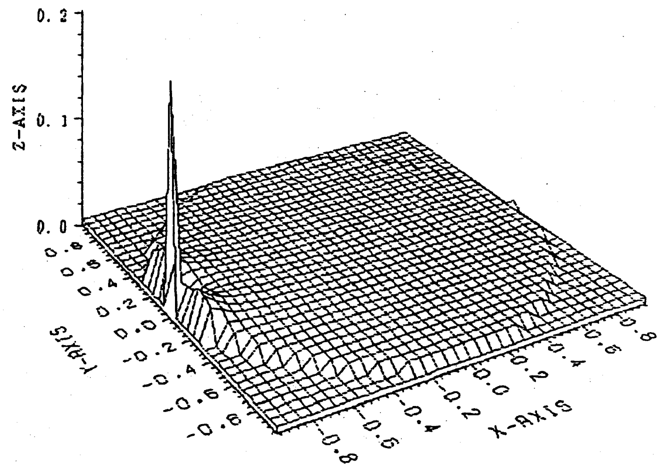

$|f\rho - u(z_1 + z_2 + \sqrt{2})|$

Figure 4: The graphs in Example 3.