数理解析研究所講究録
1060巻 1998年 1-21



# Padé approximation for words generated by certain substitutions

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0. Introduction. Let $A^*$ be a free monoid generated by a non-empty set A, i.e., $A^*$ is the set of finite words over A with the empty word $\lambda$. We put $A^\#:=A^*\cup A^{\mathbb{N}}$, where $\mathbb{N}$ is the set of non-negative integers, so that $A^{\mathbb{N}}$ is the set of infinite words over A. Any monoid morphism $\sigma:A^*\to A^*$ can be extended to a map $\sigma:A^\#\to A^\#$ by $\sigma(\varphi_0\varphi_1\varphi_2\cdots):=\sigma(\varphi_0)\sigma(\varphi_1)\sigma(\varphi_2)\cdots$ $(\varphi_n\in A)$, which is a so-called substitution over A. A word $\varphi\in A^{\mathbb{N}}$ is referred to as a fixed point of $\sigma$ if $\sigma(\varphi)=\varphi$. For a finite set $A=\{a,b,\ldots\}$ of symbols, any word $\varphi=\varphi_0\varphi_1\varphi_2\cdots\in A^{\mathbb{N}}$ gives rise to a function

$$\varphi(z):=\sum_{n\geq 0}\varphi_n z^{-n-1}, \tag{1}$$

which is an element of $\mathbb{K}((z^{-1}))$, the field of formal Laurent series over the field $\mathbb{K}=\mathbb{Q}(A):=\mathbb{Q}(a,b,\ldots)$ of rational functions of finite variables a, b, ... with rational coefficients, where the symbols a, b, ... are considered to be independent variables. By (1), we identify a word $\varphi\in A^{\mathbb{N}}$ with a formal series $\varphi(z)\in\mathbb{K}((z^{-1}))$.

The main objective of the paper is to construct the Padé approximation, which will be explained below, for the series (1) for the fixed point $\varepsilon=\varepsilon^{(k)}:=\varepsilon_0\varepsilon_1\varepsilon_2\cdots$ $(\varepsilon_n=\varepsilon_n^{(k)}\in A)$ of the substitution $\sigma$ over $A=\{a,b\}$ defined by

$$\sigma(a):=a^k b,\ \sigma(b):=a,\ k\in\mathbb{N}\setminus\{0\}, \tag{2}$$

and to give some results on Hankel determinants.

The set $A^\#$ becomes a complete metric space with respect to the metric defined by

$$d(\xi,\eta):=\exp(-\inf\{n;\ \xi_n\neq\eta_n\})\ (\xi=\xi_0\xi_1\xi_2\cdots,\ \eta=\eta_0\eta_1\eta_2\cdots\in A^\#\ (\xi_n,\eta_n\in A)).$$

The set $\mathbb{K}((z^{-1}))$ becomes a metric space induced by a non-Archimedean norm defined by

$$\|\varphi\|:=\exp(-n_0+h),\ n_0:=\inf\{n\in\mathbb{N};\ \varphi_n\neq 0\}\ (\|0\|:=0)$$

for

$$\varphi=\sum_{n\geq 0}\varphi_n z^{-n+h}\in \mathbb{K}((z^{-1})) \tag{3}$$

with $h\in\mathbb{Z}:=\{0,\pm1,\pm2,\ldots\}$. Note that $\|\varphi\|=\exp h$ holds if $\varphi_0\varphi_1\varphi_2\cdots\in A^{\mathbb{N}}$. For any given $\varphi=\varphi_0\varphi_1\varphi_2\cdots\in\mathbb{K}^{\#}(\supset A^{\#})$, we say that $(P,Q)\in\mathbb{K}[z]^2$ is an h-Padé pair of order m for $\varphi$ if

$$\|Q\varphi-P\|<\exp(-m),\quad Q\neq 0,\quad \deg Q:=\deg_z Q\leqq m \tag{4}$$

holds for $\varphi=\varphi(z)\in\mathbb{K}((z^{-1}))$ given by (3). It is known that an h-Padé pair (P,Q) of order m for $\varphi$ always exists for any $h\in\mathbb{Z}$, $m\geqq 0$, $\varphi\in\mathbb{K}^{\#}$. For h-Padé pairs (P,Q) of order m for $\varphi$, a rational function $P/Q\in\mathbb{K}(z)$ is uniquely determined for any given $h\in\mathbb{Z}$, $m\geqq 0$, $\varphi\in\mathbb{K}^{\#}$, cf. Lemma 1. The element $P/Q\in\mathbb{K}(z)$ for an h-Padé pair (P,Q) of order m for $\varphi\in\mathbb{K}^{\#}$ is referred to as the h-Padé approximant of order m for $\varphi$. A number $m\in\mathbb{N}$ is called a normal h-index for $\varphi\in\mathbb{K}^{\#}$ if (4) with $\varphi=\varphi(z)$ given by (3) implies $\deg Q=m$. A normal h-Padé pair, i.e., deg Q is a normal h-index, is said to be normalized if the leading coefficient of Q equals one. Normal (-1)-indices (resp. (-1)-Padé pairs, (-1)-Padé approximants) will be simply referred to as normal indices (resp. Padé pairs, Padé approximants). We can consider the series (3) over $\mathbb{K}=\mathbb{Q}(a,b,\ldots)$ with $a, b, \ldots\in\mathbb{C}$. In such a case, $\varphi$ defined by (3) turns out to be not only an element of $\mathbb{C}((z^{-1}))$, but also an analytic function on $\{z\in\mathbb{C}\,;\,|z|>1\}$, and the h-Padé approximant of order m for $\varphi$ pointwise converges to $\varphi$ with respect to the usual topology on $\mathbb{C}$ for each $z\in\mathbb{C}$, $|z|>1$ as m tends to infinity.

For any word $\varphi=\varphi_0\varphi_1\varphi_2\cdots\in\mathbb{K}^{\mathbb{N}}$, and $(n,m)\in\mathbb{Z}\times\mathbb{N}$, we denote by $H_{n,m}(\varphi)$ the Hankel determinant

$$H_{n,m}:=\det(\varphi_{n+i+j})_{0\leq i,j\leq m-1}\in\mathbb{Z}[a,b,\ldots]\quad (H_{n,0}:=1),$$

where $\varphi_n:=0\in\mathbb{K}$ $(n\leqq -1)$. Hankel determinants play an important role in the theory of Padé approximation, cf. [N-S], [K-T-ZYW]. It is a classical result that, for any $h\geqq -1$, m is a normal h-index if and only if $H_{h+1,m}\neq 0$, cf. Lemma 2, 3.

J.-P. Allouche, J. Peyrière, Z.-X. Wen and Z.-Y. Wen considered $H_{n,m}(\varphi)$ for the Thue-Morse sequence $\varphi=$abbabaab$\cdots$ at $(a,b)=(1,0)$, and showed that the function $H_{n,m}(\varphi)$ (mod 2) of $(n,m)\in\mathbb{N}^2$ is 2-dimensionally automatic, cf.

[A-P-ZXW-ZYW]. In [K-T-ZYW], we considered the $H_{n,m}(\varphi)$ for the Fibonacci word $\varphi$, which is the fixed point of the substitution (2) with k=1, and gave explicit formulae for $H_{n,m}(\varphi)$. Moreover, we gave a general formula†

$$H_{h+1,m}(\varphi)=(-1)^{\lfloor m/2 \rfloor} \prod_{z\in\mathbb{C}\,;\,Q(z)=0} P(z) \quad (\varphi\in\mathbb{C}^{\#},\ h\in\mathbb{Z}), \tag{5}$$

where (P,Q) is a normalized h-Padé pair of degree m for $\varphi$, $\lfloor x \rfloor$ denotes the largest integer not exceeding a real number x, and $\prod_{z\in\mathbb{C}\,;\,Q(z)=0}$ indicates a product taken over all the roots $z\in\mathbb{C}$ of Q with their multiplicity. We determined all the Padé approximants for the function (1) for the Fibonacci word $\varphi$ at (a,b)=(1,0) and (0,1).

Remark 1. †) We gave the proof of the formula (5) only for h=-1 in [K-T-W]. The formula with $h\in\mathbb{Z}$ is reduced to the case h=-1, cf. Remark 2 in Section 1.

We shall give all the Padé approximants for the function (1) for the fixed point $\varepsilon=\varepsilon^{(k)}$ of the substitution (2), and some results on Hankel determinants for $\varepsilon$ together with their variants. Such results come from a continued fraction expansion introduced in Section 1. Since the continued fraction of the series $\varepsilon(z)=\varepsilon(z;a,b;k):=(\varepsilon^{(k)})(z)$ in the case $k\geqq 2$ is not valid for k=1, we give the continued fraction of the $\varepsilon(z)$ in two sections: Section 2 (k=1), Section 3 ($k\geqq 2$), cf. Theorems 1-6 in Section 2 and Theorems 7-10 in Section 3. In Sections 4-5, a and b are always considered to be complex numbers. We give some theorems related to the distribution of the zeros of the polynomials $Q(z)\in\mathbb{C}[z]$, the denominators of Padé approximants for $\varepsilon(z)$ with normal indices, in Section 4, cf. Theorems 11- 13. In Section 5, we give some results related to the uniform convergence of the Padé approximants of the $\varepsilon(z)$ as a complex function, cf. Theorems 14-15. We also give some results on certain variants of Hankel determinants for $\varepsilon$ in Section 6, cf. Theorems 16-17. We give some conjectures in Section 7. We are intending to give no proofs for any Theorems and Lemmas here;

the proofs for Theorems 1-17 will be given in a forthcoming paper [T2].

1. Basic Lemmas. We introduce a continued fraction expansion for $\varphi \in K((z^{-1}))$, cf. [N-S], [K-T-ZYW]. As far as we are concerned only with the Padé approximation and the continued fraction, we can take any field $K$. For any finite or infinite sequence $a_m(z) \in K((z^{-1}))$ $(m=0,1,2,\ldots)$, we use the notation:

$$[a_0;a_1,a_2,\ldots,a_n]:=a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\cdots+\cfrac{1}{a_n}}} \in K((z^{-1}))\cup\{\infty\} \quad (n\geqq 0),$$

where we mean by $\alpha/0$ (resp. $\beta+\infty$, $\beta/\infty$) the element $\infty$ (resp. $\infty$, 0) in the set $K((z^{-1}))\cup\{\infty\}$ for $\alpha \in K((z^{-1}))^{\times}:=K((z^{-1}))\backslash\{0\}$, $\beta \in K((z^{-1}))$. An infinite continued fraction is defined by

$$[a_0;a_1,a_2,\ldots]:=\lim_{n\to\infty}[a_0;a_1,a_2,\ldots,a_n] \tag{6}$$

provided that the limit exists in $K((z^{-1}))$, where the limit is taken with respect to the metric induced by the non-Archimedean norm in $K((z^{-1}))$. We define $p_n$, $q_n \in K((z^{-1}))$ by

$$\begin{aligned} &p_{-1}=1,\ p_0=a_0;\ q_{-1}=0,\ q_0=1, \\ &p_n=a_np_{n-1}+p_{n-2};\ q_n=a_nq_{n-1}+q_{n-2} \quad (n\geqq 1) \end{aligned} \tag{7}$$

Then

$$p_n/q_n=[a_0;a_1,a_2,\ldots,a_n] \in K((z^{-1}))\cup\{\infty\} \quad (n\geqq 0)$$

holds. Note that (7) implies

$$p_{n-1}q_n-p_nq_{n-1}=(-1)^n, \tag{8}$$

so that $(p_n,q_n)\neq(0,0)$. The polynomial part $[\varphi]$ of $\varphi$ is defined to be a polynomial

$$[\varphi]:=\sum_{0\leqq n\leqq h}\varphi_n z^{-n+h} \in K[z]$$

for $\varphi$ given by (3). We put $\langle\varphi\rangle:=\varphi-[\varphi]$, the fractional part of $\varphi$. We define a map $T:K((z^{-1}))\to K((z^{-1}))\cup\{\infty\}$ by $T(\varphi):=1/\langle\varphi\rangle$. For any $\varphi \in K((z^{-1}))$, we can define the continued fraction expansion by the following algoritm:

$$a_n = a_n(\varphi) := [T^n(\varphi)] \quad (n \geqq 0), \tag{9}$$

where we mean by $T^n$ the n-fold iteration of the map T, $T^0$ is the identity map on $\mathbb{K}((z^{-1}))$. If $T^n(\varphi) \neq \infty$ for all $0 \leqq n \leqq N$ and $T^{N+1}(\varphi) = \infty$ for some $N \geqq 0$, then the algorithm terminates at n=N. The algorithm terminates if and only if $\varphi \in \mathbb{K}(z)$. If the algorithm terminates at n=N, then $\varphi = [a_0; a_1, a_2, \ldots, a_N]$. In general, the continued fraction (6) converges to an element in $\mathbb{K}((z^{-1}))$ if

$$a_n \in \mathbb{K}[z] \quad (n \geqq 0), \quad \deg a_n \geqq 1 \quad (n \geqq 1). \tag{10}$$

Using this fact together with (7), (8), one can see that if the algorithm does not terminate, then the continued fraction (6) converges to $\varphi$, i.e.,

$$\varphi = [a_0; a_1, a_2, \ldots], \quad \varphi \in \mathbb{K}((z^{-1})) \backslash \mathbb{K}(z)$$

as far as $a_n$ $(n \geqq 0)$ are given by the continued fraction expansion algorithm (9). We say a continued fraction is admissible if the continued fraction is obtained by applying the algorithm for an element $\varphi \in \mathbb{K}((z^{-1}))$.

Lemma 1. (1) For any $(h, m) \in \mathbb{Z} \times \mathbb{N}$, an h-Padé pair (P, Q) of order m exists for any $\varphi \in \mathbb{K}^{\#}$; for each $(h, m) \in \mathbb{Z} \times \mathbb{N}$, $P/Q = (P/Q)(z; m; \varphi; h) \in \mathbb{K}(z)$ is uniquely determined for such h-Padé pairs (P, Q).

We denote by $\Lambda_h(\varphi)$ the set of all normal h-indices for $\varphi \in \mathbb{K}^{\#}$. We put $\Lambda(\varphi) := \Lambda_{-1}(\varphi)$

Lemma 2. The set $\Pi$ of all h-Padé approximants P/Q for $\varphi \in \mathbb{K}^{\#}$ coincides with the set of all convergents $p_n/q_n$ $(n \geqq 0)$ of the continued fraction expansion $[a_0; a_1, a_2, a_3, \ldots]$ of $\varphi(z) \in \mathbb{K}((z^{-1}))$ given by (3). The set $\Pi$ is an infinite set if and only if $\varphi(z) \in \mathbb{K}((z^{-1})) \backslash \mathbb{K}(z)$, and the set $\Lambda_h(\varphi)$ coincides with the set $\{\deg q_n;\ n \in \mathbb{N}\} = \{\sum_{0 \leq m \leq n} \deg a_m;\ n \in \mathbb{N}\}$ for any $h \in \mathbb{Z}$.

Lemma 3. $H_{h+1,m}(\varphi) \neq 0$ if and only if $m \in \Lambda_h(\varphi)$ for all $h \in \mathbb{Z}$.

For the proofs of Lemmas 1-3, see [N-S], [K-T-W].

Remark 2. The proofs of Lemmas 1, 3 are given only for $h=-1$ in [N-S], [K-T-W]; the lemmas can be easily reduced to those in the case $h=-1$ by using that $(P+Q\cdot[\varphi],Q)$ is an $h$-Padé pair for $\varphi$ if and only if $(P,Q)$ is a $(-1)$-Padé pair for $\langle\varphi\rangle$.

Lemma 4. A finite or infinite continued fraction $[a_0;a_1,a_2,\ldots]$ is admissible if and only if (10) holds. If there exist sequences $\{\xi_n\}_{n\ge 1}$ $(\xi_n\in K((z^{-1})))$, and $\{a_n\}_{n\ge 0}$ $(a_n\in K[z])$ satisfying (10) such that

$$\varphi=[a_0;a_1,a_2,\ldots,a_n+\xi_n],\quad \|\xi_n\|<1$$

holds for all $n\geqq t$ for an integer $t\geqq 0$, then the continued fraction $[a_0;a_1,a_2,\ldots]$ converges to $\varphi$.

Lemma 5. Let $M$ be a matrix of size $m\times m$ with entries consisting of two variables $a$, $b$. Then

$$\det M=(a-b)^{m-1}(pa+qb)\in\mathbb{Z}[a,b],$$

where $p$, $q$ are integers defined by

$$p=\det M\,|_{(a,b)=(1,0)},\quad q=\det M\,|_{(a,b)=(0,1)}.$$

In Sections 2, 3, $K=\mathbb{Q}(a,b)$ is the field of rational functions with two independent variables $a$, $b$ unless otherwise mentioned. In what follows, $\varepsilon(z)=\varepsilon(z;a,b)=\varepsilon(z;a,b;k)$ denotes an element of $K((z^{-1}))$ given by (1) with $\varepsilon=\varepsilon_0\varepsilon_1\varepsilon_2\cdots$ in place of $\varphi=\varphi_0\varphi_1\varphi_2\cdots$, where $\varepsilon=\varepsilon^{(k)}=\varepsilon_0\varepsilon_1\varepsilon_2\cdots$ $(\varepsilon_n=\varepsilon_n^{(k)}\in\{a,b\})$ is the unique fixed point of the substitution $\sigma$ defined by (2) with $k\geqq 1$. In Sections 4, 5, $a$, $b$ are cosidered to be complex numbers, so that $\varepsilon(z)\in\mathbb{C}((z^{-1}))$, which becomes an analytic function on $\{z\in\mathbb{C}\,;\,|z|>1\}$.

We denote by $\{f_n\}_{n\ge -2}=\{f_n^{(k)}\}_{n\ge -2}$, and $\{g_n\}_{n\ge -2}=\{g_n^{(k)}\}_{n\ge -2}$ the linear recurrence sequences with $f(x):=x^2-kx-1$ as their common characteristic polynomial satisfying

$$f_{-2}=1-k,\quad f_{-1}:=1;\quad g_{-2}=1,\quad g_{-1}:=0.$$

We put

$$h_n:=g_n a+g_{n-1}b.$$

By $|w|$, we denote the length of a finite word w, by $|w|_x$ the number of occurrences of a sybol x appearing in a finte word w. Note that

$$f_n=|\sigma^n(a)|,\quad h_n=|\sigma^n(a)|_a\cdot a+|\sigma^n(a)|_b\cdot b,\quad f_n=g_n+g_{n-1}\quad (n\geqq 0)$$

in particular, if $k=1$, $g_n=f_{n-1}$, $h_n=f_{n-1}a+f_{n-2}b$ $(n\geqq 0)$.

In [T1], the auther gave the Jacobi-Perron-Parusnikov expansion for a vector consisting of Laurent series with coefficients given by a fixed point of certain substitutions, which contains the following identity (11) as its special case:

<u>Lemma 6</u>. The equality

$$(z-1)\varepsilon(z;1,0;k)=[0;b_{-2},b_{-1},b_0,b_1,b_2,\ldots]\in\mathbb{Q}((z^{-1})) \tag{11}$$

holds for any $k\geqq 1$, where

$$b_n=:z^{f_n}\sum_{0\leqq m\leqq k-1} z^{mf_{n+1}}\quad (n\geqq -1),\quad b_{-2}:=1, \tag{12}$$

and $f_n=f_n^{(k)}:=|\sigma^n(a)|$ becomes a polynomial in k of the form:

$$f_{2n}=\sum_{0\leqq i\leqq n}({}_{n+i}C_{2i}k^{2i}+{}_{n+i}C_{2i+1}k^{2i+1}),$$

$$f_{2n+1}=\sum_{0\leqq i\leqq n}({}_{n+i}C_{2i}k^{2i}+{}_{n+i+1}C_{2i+1}k^{2i+1}).$$

<u>Remark 3</u>. The continued fraction (11) is not an admissible one in $\mathbb{Q}((z^{-1}))$ in the sense of the algorithm of the continued fraction expansion given by (9), while (11) with $z\in\mathbb{Z}$, $z\geqq 2$ turns out to be admissible in $\mathbb{R}$ in the sense of the algorithm of the simple continued fraction expansion for any $k\geqq 1$. Note that the continued fractions in the theorems in Sections 2, 3, 6 are admissible in $\mathbb{K}((z^{-1}))$.

2. The Padé approximation for the $\varepsilon(z;a,b;1)$. In this section, we suppose k=1. We set $\varepsilon(z;a,b):=\varepsilon(z;a,b;1)$. Taking k=1 in Lemma 6, we have the following

Lemma 7. $(z-1)\varepsilon(z;1,0)=[0;z^{f_{-2}},z^{f_{-1}},z^{f_0},z^{f_1},z^{f_2},\ldots]\in\mathbb{Q}((z^{-1}))$ holds.

We use the notation

$$[a_0;a_1,a_2,\ldots,a_h,c_m,d_m]_{m=l}^{\infty}:=[a_0;a_1,a_2,\ldots,a_h,c_l,d_l,c_{l+1},d_{l+2},\ldots].$$

Theorem 1. The continued fraction expansion of the $\varepsilon(z)=\varepsilon(z;a,b)$ as an element of $\mathbb{K}((z^{-1}))$ is given by $\varepsilon(z)=[0;a_1,a_2,a_3,c_m,d_m]_{m=1}^{\infty}$ with

$$a_1=a^{-2}(az-b),$$
$$a_2=-(a-b)^{-1}h_1^{-2}(a^3h_1z-a^2(a^2-ab-b^2)),$$
$$a_3=-a^{-4}(a-b)h_1^2h_2^{-1}(h_1z+a)),$$
$$c_m=(-1)^{m-1}a^4(a-b)^{-1}h_1^{-4}h_{m+1}^2(z^{f_m-1}+z^{f_m-2}+\cdots+1),$$
$$d_m=(-1)^{m-1}a^{-4}(a-b)h_1^4h_{m+1}^{-1}h_{m+2}^{-1}(z-1).$$

If $(a,b)\in\mathbb{C}^2$, then Theorem 1 is valid under the condition

$$a\neq b,\ h_n(=f_{n-1}^{(1)}a+f_{n-2}^{(1)}b)\neq 0 \text{ for all } n\geqq 0. \tag{13}$$

In Theorems 2-5 below, we give the continued fraction expansion for $\varepsilon(z)=\varepsilon(z;a,b)\in\mathbb{C}((z^{-1}))$ with $(a,b)\in\mathbb{C}^2$, which does not satisfy (13). Note that $\varepsilon(z;a,a)=[0;a^{-1}(z-1)]$ $(a\in\mathbb{C}^{\times},\ \varepsilon(z;0,0)=0)$.

Theorem 2. Let $(a,b)\in\mathbb{C}^2$ with $a=0$, $b\neq 0$. Then $\varepsilon(z)=[0;a_1,a_2,c_m,d_m]_{m=1}^{\infty}$ with

$$a_1=b^{-1}z^2,\quad a_2=-bz,$$
$$c_m=(-1)^{m-1}b^{-1}f_{m-1}^2(z^{f_m-1}+z^{f_m-2}+\cdots+1),$$
$$d_m=(-1)^{m-1}bf_{m-1}^{-1}f_m^{-1}(z-1).$$

Theorem 3. Let $(a,b)\in\mathbb{C}^2$, $a\neq 0$, $h_1=0$. Then $\varepsilon(z)=\varepsilon(z;a,-a)=$

$[0;a_1,a_2,a_3,c_m,d_m]_{m=1}^{\infty}$ with

$$a_1=a^{-1}(z+1),\ a_2=-2^{-1}a(z^2-z-1),\ a_3=-2a^{-1}(z+1),$$
$$c_m=(-1)^m 2^{-1}af_{m-2}^{-1}f_{m-1}^{-1}(z-1),$$
$$d_m=(-1)^{m-1}2a^{-1}f_{m-1}^{2}(z^{f_{m+1}-1}+z^{f_{m+1}-2}+\cdots+1).$$

Theorem 4. Let $(a,b)\in\mathbb{C}^2$, $a\neq0$, $h_2=0$. Then $\varepsilon(z)=\varepsilon(z;a,-2a)=$ $[0;a_1,a_2,a_3,a_4,c_m,d_m]_{m=1}^{\infty}$ with

$$a_1=a^{-1}(z+2),\ a_2=3^{-1}a(z-1),\ a_3=-3a^{-1}(z^3-z^2+1),\ a_4=-3^{-1}a(z^2+z+1),$$
$$c_m=(-1)^m 3a^{-1}f_{m-2}^{-1}f_{m-1}^{-1}(z-1),$$
$$d_m=(-1)^{m+1}3^{-1}af_{m-1}^{2}(z^{f_{m+2}-1}+z^{f_{m+2}-2}+\cdots+1).$$

Theorem 5. Let $(a,b)\in\mathbb{C}^2$, $a\neq0$, $h_{t+2}=0$ $(t\geq1)$. Then $\varepsilon(z)=\varepsilon(z;a,-f_t^{-1}f_{t+1}a)=[0;a_1,a_2,a_3,c_1,d_1,\ldots,c_{t-1},d_{t-1},c_t,e_1,e_2,i_m,j_m]_{m=1}^{\infty}$ with

$$a_1=a^{-2}(az-b),$$
$$a_2=-(a-b)^{-1}h_1^{-2}(a^3h_1z-a^2(a^2-ab-b^2)),$$
$$a_3=-a^{-4}(a-b)h_1^{2}h_2^{-1}(h_1z+a)),$$
$$c_m=(-1)^{m-1}a^4(a-b)^{-1}h_1^{-4}h_{m+1}^{2}(z^{f_m-1}+z^{f_m-2}+\cdots+1),$$
$$d_m=(-1)^{m-1}a^{-4}(a-b)h_1^{4}h_{m+1}^{-1}h_{m+2}^{-1}(z-1),$$
$$e_1=(-1)^{t-1}a^{-4}(a-b)h_1^{4}h_{t+1}^{-2}z^{f_{t+1}}(z-1),$$
$$e_2=(-1)^{t-1}a^4(a-b)^{-1}h_1^{-4}h_{t+1}^{2}(z^{f_{t+2}-1}+z^{f_{t+2}-2}+\cdots+1),$$
$$i_m=(-1)^{m+t}a^{-4}(a-b)h_1^{4}h_{t+1}^{-2}f_{m-2}^{-1}f_{m-1}^{-1}(z-1),$$
$$j_m=(-1)^{m+t+1}a^4(a-b)^{-1}h_1^{-4}h_{t+1}^{2}f_{m-1}^{2}(z^{f_{m+t+2}-1}+z^{f_{m+t+2}-2}+\cdots+1).$$

In view of Theorem 1 together with Lemmas 2, 3, we get

Corollary 1. The Hankel determinant $H(\varepsilon)_{0,m}(\in\mathbb{Z}[a,b])$ is not zero if and only if $m\in\{0,f_0=f_1-1,f_1=f_2-1,f_2,f_3-1,f_3,\ldots\}$.

Noting that $f_{n-1}$ and $f_{n-2}$ are coprime, in view of Theorems 3-5, Corollary 1 together with Lemmas 2, 3, 5, we obtain

Corollary 2. $H_{0,f_n}(\varepsilon)=r_n(f_{n-1}a+f_{n-2}b)(a-b)^{f_n-1}$ $(n\geqq 1)$,

$H_{0,f_{n+1}-1}(\varepsilon)=s_n(f_{n-1}a+f_{n-2}b)(a-b)^{f_{n+1}-2}$ $(n\geqq 2)$,

where $r_n\neq 0$, $s_n\neq 0$ are integers independent of a, b.

Theorem 6. The numerator $p_n$ and the denominator $q_n$ of the n-th convergent of the continued fraction expansion for $\varepsilon(z;a,b)$ is given by

$$p_0=0,\ q_0=1;\ p_1=1,\ q_1=a^{-2}(az-b);$$

$$p_2=-(a-b)^{-1}h_1^{-2}(a^3h_1z+a^2(a^2-ab-b^2)),$$

$$q_2=-(a-b)^{-1}h_1^{-2}(a^2h_1z^2-a^3z-a^3);$$

$$p_{2n-1}=a^{-2}h_1^{2}h_n^{-1}(\varepsilon_0z^{f_n-1}+\varepsilon_1z^{f_n-2}+\cdots+\varepsilon_{f_n-1}),$$

$$q_{2n-1}=a^{-2}h_1^{2}h_n^{-1}(z^{f_n}-1);$$

$$p_{2n}=(-1)^na^2(a-b)^{-1}h_1^{-2}\{h_n(\varepsilon_0z^{f_{n+1}-1}+\varepsilon_1z^{f_{n+1}-2}+\cdots+\varepsilon_{f_{n+1}-1})$$
$$-h_{n+1}(\varepsilon_0z^{f_n-1}+\varepsilon_1z^{f_n-2}+\cdots+\varepsilon_{f_n-1})\}/(z-1),$$

$$q_{2n}=(-1)^na^2(a-b)^{-1}h_1^{-2}\{h_n(z^{f_{n+1}-1}+z^{f_{n+1}-2}+\cdots+1)$$
$$-h_{n+1}(z^{f_n-1}+z^{f_n-2}+\cdots+1)\},\ n\geqq 2.$$

Note that the numerator of the right-hand side of $p_{2n}$ is divisible by z-1, so that $p_{2n}\in K[z]$.

Corollary 3. The normalized Padé pairs $(p_n^*,q_n^*)\in K[z]^2$ for $\varepsilon(z;a,b)\in K((z^{-1}))$ is given by

$$p_0^*=0,\ q_0^*=1;\ p_1^*=a,\ q_1^*=z-a^{-1}b;$$

$$p_2^*=az+a^{-1}(a^2-ab-b^2),\ q_2^*=z^2-a(a+b)^{-1}z-a(a+b)^{-1};$$

$$p_{2n-1}^*=\varepsilon_0z^{f_n-1}+\varepsilon_1z^{f_n-2}+\cdots+\varepsilon_{f_n-1},$$

$$q_{2n-1}^*=z^{f_n}-1;$$

$$p_{2n}^*=\{\varepsilon_0 z^{f_{n+1}-1}+\varepsilon_1 z^{f_{n+1}-2}+\cdots+\varepsilon_{f_{n+1}-1}$$
$$-h_n^{-1}h_{n+1}(\varepsilon_0 z^{f_n-1}+\varepsilon_1 z^{f_n-2}+\cdots+\varepsilon_{f_n-1})\}/(z-1),$$
$$q_{2n}^*=\{z^{f_{n+1}}-1-h_n^{-1}h_{n+1}(z^{f_n}-1)\}/(z-1),\ n\geqq 2.$$

3. <u>The Padé approximation for the $\varepsilon(z;a,b;k)$ with $k\geqq 2$</u>. In this section, we suppose $k\geqq 2$.

<u>Lemma 8</u>. Let $b_n\in\mathbb{Z}[z]$ be as in Lemma 6. Then

$$b_n=(z-1)b_n^*+k\ (n\geqq -1)$$

holds, where

$$b_n^*=b_n(z;k):=z^{f_n}\sum_{0\leqq i\leqq f_{n+1}-1} z^i\cdot\sum_{1\leqq j\leqq k-1}(k-j)z^{(j-1)f_{n+1}}+k\sum_{0\leqq i\leqq f_n-1} z^i$$

<u>Theorem 7</u>. The continued fraction expansion of the $\varepsilon(z)=\varepsilon(z;a,b;k)$ as an element of $\mathbb{K}((z^{-1}))$ is given by

$$\varepsilon(z)=[0;a^{-1}(z-1),(-1)^m(a-b)^{-1}h_m^2 b_{m-1}^*,(-1)^m(a-b)h_m^{-1}h_{m+1}^{-1}(z-1)]_{m=0}^{\infty}.$$

If $(a,b)\in\mathbb{C}^2$, then Theorem 7 is valid under the condition

$$a\neq b,\ h_n(=g_n a+g_{n-1}b)\neq 0 \text{ for all } n\geqq 0. \qquad (14)$$

In Theorems 8-10 below, we give the continued fraction expansion for $\varepsilon(z;a,b;k)\in\mathbb{C}((z^{-1}))$ with $(a,b)\in\mathbb{C}^2$, which does not satisfy (14). Note that $\varepsilon(z;a,a;k)=[0;a^{-1}(z-1)]$ $(a\in\mathbb{C}^\times,\ \varepsilon(z;0,0)=0)$.

<u>Theorem 8</u>. Let $(a,b)\in\mathbb{C}^2$ with $a=0\neq b$. Then

$$\varepsilon(z)=\varepsilon(z;0,b;k)=[0;b^{-1}z(z^k-1),(-1)^m b g_m b_m^*,(-1)^m b^{-1}g_m^{-1}g_{m+1}^{-1}(z-1)]_{m=0}^{\infty}$$

<u>Theorem 9</u>. Let $(a,b)\in\mathbb{C}^2$ with $h_t=0,\ t\geqq 0$. Then

$$\varepsilon(z)=\varepsilon(z;a,-g_{t-1}^{-1}g_t a)=[0;a_1,d_{-1},c_0,d_0,\ldots,c_{t-2},d_{t-2},e_1,e_2,i_m,j_m]_{m=0}^{\infty},$$

where

$$a_1=a^{-1}(z-1),$$

$$c_m=(-1)^m(a-b)h_m^{-1}h_{m+1}^{-1}(z-1),$$

$$d_m=(-1)^{m+1}(a-b)^{-1}h_{m+1}^{2}b_m^*,$$

$$e_1=(-1)^{l-1}(a-b)h_{l-1}^{-2}(z-1)b_{l-1},$$

$$e_2=(-1)^{l-1}(a-b)^{-1}h_{l-1}^{2}b_l^*,$$

$$i_m=(-1)^{m+l-1}(a-b)h_{l-1}^{-2}g_m^{-1}g_{m+1}^{-1}(z-1),$$

$$j_m-(-1)^{m+l}(a-b)^{-1}h_{l-1}^{2}g_{m+1}^{2}b_{m+l+1}^*.$$

<u>Theorem 10</u>. The numerator $p_n$ and the denominator $q_n$ of the n-th convergent of the continued fraction expansion for $\varepsilon(z;a,b;k)$ is given by

$$p_0=0,\ q_0=1;$$

$$p_{2n+1}=h_n^{-1}(\varepsilon_0 z^{f_n-1}+\varepsilon_1 z^{f_n-2}+\cdots+\varepsilon_{f_n-1}),$$

$$q_{2n+1}=h_n^{-1}(z^{f_n}-1);$$

$$p_{2n+2}=(-1)^n(a-b)^{-1}\{h_n(\varepsilon_0 z^{f_{n+1}-1}+\varepsilon_1 z^{f_{n+1}-2}+\cdots+\varepsilon_{f_{n+1}-1}) -h_{n+1}(\varepsilon_0 z^{f_n-1}+\varepsilon_1 z^{f_n-2}+\cdots+\varepsilon_{f_n-1})\}/(z-1),$$

$$q_{2n+2}=(-1)^n(a-b)^{-1}\{h_n(z^{f_{n+1}-1}+z^{f_{n+1}-2}+\cdots+1) -h_{n+1}(z^{f_n-1}+z^{f_n-2}+\cdots+1)\},\ n\geqq 0.$$

<u>Corollary 4</u>. The normalized Padé pairs $(p_n^*,q_n^*)\in K[z]^2$ for $\varepsilon(z;a,b;k)$ $\in K((z^{-1}))$ is given by

$$p_0^*=0,\ q_0^*=1;$$

$$p_{2n+1}^*=\varepsilon_0 z^{f_n-1}+\varepsilon_1 z^{f_n-2}+\cdots+\varepsilon_{f_n-1},$$

$$q_{2n+1}^*=z^{f_n}-1;$$

$$p_{2n+2}^*=\{\varepsilon_0 z^{f_{n+1}-1}+\varepsilon_1 z^{f_{n+1}-2}+\cdots+\varepsilon_{f_{n+1}-1} -h_n^{-1}h_{n+1}(\varepsilon_0 z^{f_n-1}+\varepsilon_1 z^{f_n-2}+\cdots+\varepsilon_{f_n-1})\}/(z-1),$$

$$q_{2n+2}^*=\{z^{f_{n+1}}-1-h_n^{-1}h_{n+1}(z^{f_n-1}-1)\}/(z-1),\ n\geqq 0.$$

In view of Corollary 4 together with Lemmas 2, 3, we get

Corollary 5. The Hankel determinant $H(\varepsilon^{(k)})_{0,m}(\in \mathbb{Z}[a,b])$ is not zero if and only if $m \in \{0=f_0-1, f_0, f_1-1, f_1, f_2-1, f_2, \ldots\}$.

In view of Theorems 8, Corollary 5 together with Lemmas 2, 3, 5, we get

Corollary 6. The following formulae hold:

$$H_{0,f_n}(\varepsilon^{(k)}) = r_n(g_n a + g_{n-1} b)(a-b)^{f_n - 1},$$

$$H_{0,f_{n+1}-1}(\varepsilon^{(k)}) = s_n(g_n a + g_{n-1} b)(a-b)^{f_{n+1}-2} \quad (n \geqq 0),$$

where $r_n \neq 0$, $s_n \neq 0$ are integers independent of a, b.

4. The distribution of the zeros of $q_n^*(z)$. As we have mentioned the relation (5) between the h-Padé approximants with normal h-indices for $\varphi \in K^\#$ with $K \subset \mathbb{C}$ and the Hankel determinants $H_{h+1,m}(\varphi)$, it will be interesting to know the distribution of the zeros of the denominators of Padé approximants with normal indices. In this section, we shall give results on the simplicity, and the asymptotic behavior of the zeros of the denominator $q_n^* \in \mathbb{C}[z]$ of the Padé approximants for the function $\varepsilon(z;a,b;k)$ $(a,b \in \mathbb{C},\ k \geqq 1)$.

First, we give the simplicity results. From Corollaries 3, 4, and Theorems 1, 7, it is clear that $q_{2n+1}^*(n \geqq 0)$ has only simple roots, if $a \neq b$, and $h_m \neq 0$ $(0 \leqq m \leqq n+1$ $(k=1)$; $0 \leqq m \leqq n$ $(k \geqq 2))$.

Theorem 11. (case k=1) Let $q_n^*$ be as in Corollary 3, and let $(a,b) \in \mathbb{C}^2$ with

$$a \neq b,\ h_m \neq 0 \text{ for all } 0 \leqq m \leqq n.$$

If $b/a \in \mathbb{C}$ is different from any algebraic number $\zeta$ satisfying the equation

$$f_{n+1}^{f_{n+1}}(f_{n-3}\zeta+f_{n-2})^{f_{n-1}}(f_{n-2}\zeta+f_{n-1})^{f_n}=f_{n-1}^{f_{n-1}}f_n^{f_n}(f_{n-1}\zeta+f_n)^{f_{n+1}},$$

then all the zeros of the polnomial $q_{2n}^*$ ($n\geqq2$) are simple.

Note that $h_0\neq0$ implies $a\neq0$, so that $b/a\in\mathbb{C}$ is well defined in Theorem 11. In view of Theorem 11, it is clear that if $b/a$ is a transcendental number, then all the roots of $q_n^*$ are simple. Using that any two numbers among $f_{n-2}$, $f_{n-1}$, $f_n$ are coprime for all $n\geqq0$, we can show the folowing

Corollary 7. (case $k=1$) All the denominators $q_n\in\mathbb{C}[z]$ of the Padé approximants with normal indices for $\varepsilon(z;a,0)$ ($a\in\mathbb{C}^\times$) have only simple roots.

Theorem 12. (case $k\geqq2$) Let $q_n^*$ be as in Corollary 4, and let $(a,b)\in\mathbb{C}^2$ with

$$a\neq b,\ h_n\neq h_{n-1},\ h_m\neq0 \text{ for all } 0\leqq m\leqq n-1.$$

Suppose that $b/a\in\mathbb{C}$ is different from any algebraic algebraic number $\zeta$ satisfying the equation

$$f_{n-1}^{f_{n-1}}(f_{n-1}-f_n)^{f_n-f_{n-1}}(g_{n-1}\zeta+g_n)^{f_n}$$
$$=f_n^{f_n}(g_{n-2}\zeta+g_{n-1})^{f_{n-1}}\{(g_{n-2}-g_{n-1})\zeta+(g_{n-1}-g_n)\}^{f_n-f_{n-1}}$$

Then all the zeros of the polnomial $q_{2n}^*$ ($n\geqq1$) are simple.

Corollary 8. Let $(p_n^*,q_n^*)$ be the normalized Padé pair for $\varepsilon(z;a,b;k)$ as before, and let $(a,b)\in\mathbb{C}^2$ be as in Theorem 10 with $k=1$, or as in Theorem 11 with $k\geqq2$. Then

$$p_n^*(z)/q_n^*(z)=\sum_{1\leqq m\leqq G_n}\frac{\lambda_m^{(n)}}{z-\zeta_m^{(n)}},$$

$$\lambda_m^{(n)}=\frac{p_n^*(\zeta_m^{(n)})}{\dfrac{dq_n^*}{dz}(\zeta_m^{(n)})}$$

holds for all $n\geqq0$, where $\zeta_m^{(n)}$ ($1\leqq m\leqq G_n:=\deg q_n^*$) are the roots of the polynomial

$q_n^*$.

Secondly, we give results on the asymptotic behavior of the zeros. It is clear that all the zeros of $q_{2n+1}^*$ $(n\geqq 1)$ are on the unit circle for all $k\geqq 1$. We can show that all the zeros of $q_{2n}^*$ tend to the unit circle as $n\to\infty$ under a minor conition:

Theorem 13. Let $q_n^*$ be the denominator of normalized Padé pair for $\varepsilon(z;a,b;k)$ as before, and let $(a,b)\in\mathbb{C}^2$ with

(case $k=1$) $a\neq b$, $h_n\neq 0$ for all $n\geqq 0$,

(case $k\geqq 2$) $a\neq b$, $h_n\neq h_{n-1}$, $h_n\neq 0$ for all $n\geqq 0$.

Let $n_0$ be an integer satisfying

$$\alpha-1/4<|h_{n+1}/h_n|<\alpha+1/4 \quad \text{for all } n\geqq n_0=n_0(a,b;k),$$

$$\alpha=\alpha(k)=(k+(k^2+4)^{1/2})/2.$$

Then

$$\{z\in\mathbb{C}\ ;\ q_{2n}^*(z)=0\}\subset\{z\in\mathbb{C}\ ;\ (3/4-1/\alpha)^{1/g_n}<|z|<(\alpha+5/4)^{1/g_n}\}\quad(n\geqq n_0)$$

holds for all $k\geqq 1$.

We remark that $h_{n+1}/h_n$ converges to $\alpha(k)$ for all $(a,b)\neq(0,0)$, so that an integer $n_0$ exists for all $k\geqq 1$.

5. Uniform convergence of the Padé approximants of $\varepsilon(z;a,b;k)$. In this section, we consider $p_n^*/q_n^*$ and $\varepsilon(z;a,b;k)$ as analytic functions on $\{z\in\mathbb{C}\ ;\ |z|>1\}$.

We need some definitions to state a lemma. We denote by $\{V_n\}_{n\geq 0}$, $\{W_n\}_{n\geq 0}$ the sequences of words $V_n=V_n^{(m)}=V_n^{(m,k)}$, $W_n=W_n^{(m)}=W_n^{(m,k)}$ over $\{0,1\}$ defined by the following locally catinative formulae for each $m\geqq -2$, $k\geqq 1$:

$$V_{n+2}=V_{n+1}^{\,k}V_n;\ W_{n+2}=W_{n+1}^{\,k}W_n,\ n\geqq 0$$

with (15)

$$V_0:=0^{f_{m+1}},\quad V_1=:10^{f_{m+2}-1};\quad W_0:=10^{f_{m+1}-1},\quad W_1:=W_0{}^k0^{f_m}(m\geqq -1;\ W_1{}^{(-2)}:=1).$$

Noting that $V_n$ (resp. $W_n$) is a prefix of $V_{n+1}$ (resp. $W_{n+1}$) for all $n\geqq 1$, we can set

$$V_n=\mu_0\mu_1\mu_2\cdots\mu_{f_{n+m+1}}\ (\mu_n\in\{0,1\}),\quad W_n=\nu_0\nu_1\nu_2\cdots\nu_{f_{n+m+1}}\ (\nu_n\in\{0,1\},\ n\geqq 1).$$

Noting also that $\mu_i=\mu_i{}^{(m)}=\mu_i{}^{(m,k)}$, $\nu_i=\nu_i{}^{(m)}=\nu_i{}^{(m,k)}$ are defined for any $i\geqq 0$ for each $m\geqq -2$, $k\geqq 1$, we can define infinite words $\mu=\mu^{(m)}=\mu^{(m,k)}$, $\nu=\nu^{(m)}=\nu^{(m,k)}$ by

$$\mu=\lim_{n\to\infty}V_n=\mu_0\mu_1\mu_2\cdots,\quad \nu=\lim_{n\to\infty}W_n=\nu_0\nu_1\nu_2\cdots\in\{0,1\}^{\mathbb{N}}, \tag{16}$$

where the limits are taken with respect to the metric d in $\{0,1\}^{\#}$.

Remark 4. V, W become fixed points of substitutions in some cases: $V^{(-2)}=\varepsilon(1,0)$, $W^{(-2)}=\varepsilon(1,1)=111\cdots$, $W^{(-1)}=\varepsilon(1,0)$, where $\varepsilon=\varepsilon(a,b)$ is the fixed point of the substitution (2).

Lemma 9. Let $P_n{}^{(m)}/Q_n{}^{(m)}$ the n-th convergent of the continued fraction $[0;b_m,b_{m+1},b_{m+2},\ldots]$, where $b_n\in\mathbb{Z}[z]$ is a polynomial (11) in Lemma 7, $P^*{}_n{}^{(m)}=P^*{}_n{}^{(m)}(z)$, $Q^*{}_n{}^{(m)}=Q^*{}_n{}^{(m)}(z)$ the functions defined by

$$P^*{}_n{}^{(m)}:=z^{-f_{n+m+1}+f_{m+2}}P_n{}^{(m)},\quad Q^*{}_n{}^{(m)}:=z^{-f_{n+m+1}+f_{m+1}}Q_n{}^{(m)}.$$

Then

$$P^*{}_n{}^{(m)}=\sum_{0\leqq i\leqq f_{n+m+1}}\mu_i z^{-i},\quad Q^*{}_n{}^{(m)}=\sum_{0\leqq i\leqq f_{n+m+1}}\nu_i z^{-i}$$

holds for all $m\geqq -2$, $n\geqq 0$, $k\geqq 1$.

Note that the function $\theta_m=\theta_m(z):=[0;b_m,b_{m+1},b_{m+2},\ldots]$ is well-defined, which is an analytic function on $\{z\in\mathbb{C};\ |z|>1\}$. In addition, for any $m\geqq -2$, $\theta_m=z^{-(f_{m+2}-f_{m+1})}(1+o(1))$ as $|z|$ tends to infinity; $\theta_m=z^{-(f_{m+2}-f_{m+1})}(c+o(1))$ as m tends to infinity for any $|z|>1$, where $c=c(z)$ is independent of m.

Theorem 14. Let a, b be complex numbers satisfying

$$a\neq b,\quad h_n\neq 0 \text{ for all } n\geqq 0.$$

Let $(p_n^*, q_n^*)$ be as in Corollary 3. Let $\rho>1$ be any fixed real number. Then the following estimates are valid for any $z\in\mathbb{C}$ with $|z|\geqq\rho$:

$$\left|\varepsilon(z;a,b,1) - \frac{p_{2n}^*}{q_{2n}^*}\right| < \frac{2\alpha|a-b|}{|z|^{2f_n-1}},$$

$$\left|\varepsilon(z;a,b,1) - \frac{p_{2n+1}^*}{q_{2n+1}^*}\right| < \frac{2|a-b|}{|z|^{f_{n+3}-1}}$$

for all $n\geqq n_0(a,b,\rho)$, where $\alpha$ is the golden ratio, and $n_0(a,b,\rho)$ is a number independent of z.

Theorem 15. Let $k\geqq 2$ be an integer, and let a, b be complex numbers satisfying

$$a\neq b,\ h_n\neq 0 \text{ for all } n\geqq 0.$$

Let $(p_n^*, q_n^*)$ be as in Corollary 4. Let $\rho>1$ be any fixed real number. Then the following estimates are valid for any $z\in\mathbb{C}$ with $|z|\geqq\rho$:

$$\left|\varepsilon(z;a,b,k) - \frac{p_{2n}^*}{q_{2n}^*}\right| < \frac{2\alpha(k)|a-b|}{|z|^{2f_n-1}},$$

$$\left|\varepsilon(z;a,b,1) - \frac{p_{2n+1}^*}{q_{2n+1}^*}\right| < \frac{2\alpha(k)|a-b|}{|z|^{f_{n+1}+f_n-1}}$$

for all $n\geqq n_0(a,b,k,\rho)$, where $\alpha=(k+(k^2+4)^{1/2})/2$, and $n_0(a,b,k,\rho)$ is a number independent of z.

In view of Theorems 14, 15, we have the following

Remark 5. The function $E(z):=\varepsilon(z^{-1};a,b;k)$ $(E(0):=a)$ is analytic on the unit disc $\{z\in\mathbb{C}\ ;\ |z|<1\}$, and its Padé approximant $p_n^*/q_n^*$ uniformly converges to $E(z)$ on any compact set in the unit disc.

## 6. Some variants of Hankel determinants $H_{n,m}(\varepsilon)$.

Theorem 16. Let $b_n \in \mathbb{Z}[z]$ be as in Lemma 6. Then we have the equality

$$(z-1)\varepsilon(z;a,b;k)=[a;(-a+b)^{-1}(z^k+z^{k-1}+\cdots+1),(-a+b)^{(-1)^n}b_n]_n^\infty \in K((z^{-1})).$$

For $\varphi=\varphi_0\varphi_1\varphi_2\cdots \in K^{\mathbb{N}}$, we denote by $M_{n,m}(\varphi)$ the Hankel matrix

$$M_{n,m}(\varphi):=(\varphi_{n+i+j})_{0\leq i,j\leq m-1},\quad (n,m)\in\mathbb{Z}\times(\mathbb{N}\setminus\{0\})$$

with $\varphi_n:=0$ $(n\leqq -1)$, so that $H_{n,m}(\varphi)=\det M_{n,m}(\varphi)$.

Corollary 9. For any $m\geqq 0$, $k\geqq 1$

$$\det(M_{1,m}(\varepsilon^{(k)})-M_{0,m}(\varepsilon^{(k)}))=\begin{vmatrix} 1 & \varepsilon_0 & \varepsilon_1 & \cdots & \varepsilon_{m-1} \\ 1 & \varepsilon_1 & \varepsilon_2 & \cdots & \varepsilon_m \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \varepsilon_m & \varepsilon_{m+1} & \cdots & \varepsilon_{2m-1} \end{vmatrix}=t_m(a-b)^m,\quad t_m\in\mathbb{Z},$$

holds, and $t_m\neq 0$ if and only if $m\in\{f_n-1;\ n\geqq 0\}$, where $t_m=t(m,k)$ is independent of $a$, $b$.

Let $\mu=\mu^{(m)}=\mu_0\mu_1\mu_2\cdots:=\lim_{n\to\infty} V_n$ , $\nu=\nu^{(m)}=\nu_0\nu_1\nu_2\cdots:=\lim_{n\to\infty} W_n$ be the infinite words (16) over $\{0,1\}$ with $V_n$, $W_n$ defined by (15). Let $\gamma_m(z)$ be a function defined by

$$\gamma_m(z)=z^{-(f_{m+2}-f_{m+1})}\sum_{n\geq 0}\mu_n z^{-n}\Big/\sum_{n\geq 0}\nu_n z^{-n}. \tag{17}$$

Then $\gamma_m(z)$ becomes an analytic function on $\{z\in\mathbb{C};\ |z|>1\}$. Since $\mu_0=\nu_0=1$, we can define rational numbers $\gamma_n^{(m)}$ $(n\geqq 0)$ by

$$\gamma_m(z)=\sum_{n\geq 0}\gamma_n^{(m)}z^{-n-1}.$$

Theorem 17. Let $b_n\in\mathbb{Z}[z]$ be as in Lemma 6. Then the equality

$$\gamma_m(z)=[0;b_m,b_{m+1},b_{m+2},\ldots]\in\mathbb{Q}((z^{-1}))$$

holds for all $m\geqq -2$.

Remark 6. $\gamma_{-2}(z)=(z-1)\varepsilon(z;1,0;k)$, $\gamma_{-1}(z)=\varepsilon(z;0,1;k)/\varepsilon(z;1,0;k)$ $(k\geqq 1)$.

Corollary 10. Let $\gamma^{(i)}=\gamma_0^{(i)}\gamma_1^{(i)}\gamma_2^{(i)}\cdots(\in Q^N)$, $n\geqq-2$. Then, for any $i\geqq-2$, $H_{0,m}(\gamma^{(i)})\neq0$ if and only if $m\in\{f_n-f_{i+1}: n\geqq i+1\}$.

7. Problems. In view of (17) together with (15), (16), the function $\gamma_m(z)$ is represented by a ratio $\mu_m^*(z)/\nu_m^*(z)$ with $\mu_m^*(z)$, $\nu_m^*(z)\in\mathbb{Z}[[z^{-1}]]$ with bounded coefficients. In fact, the coefficients of $\mu_m^*(z)$, $\nu_m^*(z)$ are 0, or 1 that come from the locally catinative formulae (15). On the other hand, it seems very likely that the sequence $\{\gamma_n^{(m)}\}_{n\geq0}$ is unbounded both from above, and from below for all $m\geqq-1$; while, it is clear from Remark 6 that the sequence is bounded for $m=-2$. For example,

$\{\gamma_n^{(-1)}\}_{n\geq0}=1,0,-1,0,1,1,-1,-2,0,3,1,-3,-3,3,5,-1,-8,-2,10,8,-11,\ldots,$ $\gamma_{24}^{(-1)}=-29$, $\gamma_{31}^{(-1)}=93$, $\gamma_{34}^{(-1)}=-125$, $\gamma_{45}^{(-1)}=506$, $\gamma_{47}^{(-1)}=-796$, etc.

(Conjecture i) The sequence $\{\gamma_n^{(m)}\}_{n\geq0}$ is unbounded from above, and from below for all $m\geqq-1$.

We have already shown that the set $S_n$ of all the roots of the denominator $q_n^*$ of the Padé approximant for $\varepsilon^{(k)}$ with a normal index N(n) “uniformly coverges” to the unit circle as n tends to infinity, cf. Theorem 13. On the other hand, we have difficulty to have a result related to the distribution of the zeros of the numerators $p_n^*$ of the Padé approximants with normal indices. If $N=f_n$, then N is a normal index and the polynomial

$$P^*(z):=\varepsilon_0z^{N-1}+\varepsilon_1z^{N-2}+\cdots+\varepsilon_{N-1}$$

is one of the numerators of the Padé approximants with respect to the normalized Padé pairs for $\varepsilon(z;a,b;k)$. Noting,

$$z^{N-1}P^*(z^{-1})=\sum_{0\leqq n\leqq|\sigma^n(a)|}\varepsilon_nz^n,$$

it will be interesting to investigate the distribution of the zero points of the polnomials

$$\Theta_n(z;\varphi)=\Theta_n(z;\varphi;a_1,\ldots,a_s):=\sum_{0\leqq m\leqq n}\varphi_m z^m$$

with $\varphi_n=\varphi_n(a_1,\ldots,a_s)$ ($a_1, \ldots, a_s\in\mathbb{C}$) for a fixed point $\varphi=\varphi_0\varphi_1\varphi_2\cdots\in A^{\mathbb{N}}$ of a substitution $\tau$ over $A=\{a_1,\ldots,a_s\}$. We put

$$\Theta(z;\varphi)=\Theta(z;\varphi;a_1,\ldots,a_s):=\sum_{n\geqq 0}\varphi_n z^n.$$

(Conjecture ii) Let $\Theta_n(z;\varphi;a_1,\ldots,a_s)$ be as above with any fixed point $\varphi$ of any substitution $\tau$ over $A=\{a_1,\ldots,a_s\}$ having some fixed points. Let $Z_n$ be the set defined by

$$Z(n)=Z(n;\varphi;a_1,\ldots,a_s):=\{z\in\mathbb{C}\,;\ \Theta_n(z;\varphi;a_1,\ldots,a_s)=0\}$$

for $a_1, \ldots, a_s\in\mathbb{C}$. If $\Theta(z;\varphi;a_1,\ldots,a_s)\notin\mathbb{C}[z]$, then

$$\bigcap_{n=0}^{\infty}\left(\overline{\bigcup_{m=n}^{\infty}Z(m)}\right)\supset S,$$

where $S:=\{z\in\mathbb{C}\,;\ |z|=1\}$ is the unit circle, and $\bar{M}$ denotes the closure of a set $M\subset\mathbb{C}$ with respect to the usual topology in $\mathbb{C}$.

It is clear that Conjecture ii holds if $\varphi$ is a periodic word. In the same notation as in Conjecture ii, we have the following conjectures. Conjecture iii is stronger than Conjecture ii.

(Conjecture iii) If $\Theta=\Theta(z;\varphi;a_1,\ldots,a_s)\notin\mathbb{C}[z]$, then

$$\left(\bigcap_{n=0}^{\infty}\left(\overline{\bigcup_{m=n}^{\infty}Z(m)}\right)\right)\cap D = S\cup F,$$

where $D:=\{z\in\mathbb{C}\,;\ |z|\leqq 1\}$ is the unit disc, and $F$ is a finite set depending on $\varphi$, and $a_1, \ldots, a_s$. In addition $F$, coincides with the set of all zero points of the function $\Theta$ if $\Theta\notin\mathbb{C}(z)$.

(Conjecture iv) Let $\tau(a_1)=a_1W$ ($W\in A^*\backslash\{\lambda\}$), $\lim_{n\to\infty}|\tau^n(W)|=\infty$, and let $\varphi:=\lim_{n\to\infty}\tau^n(W)$, i.e., $\varphi$ is the fixed point prefixed by $a_1$. If $\Theta=\Theta(z;\varphi;a_1,\ldots,a_s)\notin\mathbb{C}(z)$, then

$$\bigcap_{n=0}^{\infty} \left( \overline{\bigcup_{m=n}^{\infty} Z(|\tau^m(a_1|)} \right) = S \cup G,$$

where G is a finite set depending on $\varphi$, and $a_1$, ..., $a_s$. In addition, G is not empty if $\Theta \notin \mathbb{C}(z)$.

Acknowledgement. The auther would like to thank Maki Fourcado and Vuillon Laurent for their help in making calculation of the zero points of $\Theta(z;\varphi;a_1,...,a_s)$ in various cases by computers.

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