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# A FUNDAMENTAL BUT UNEXPLOITED PARTITION INVARIANT

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## 1 Introduction

Since the time of Euler who founded the theory of partitions, the subject has undergone several stages of development using combinatorial tools, $q-$theoretic identities, analytic methods, Lie algebras and the theory of modular forms. Very often, in combinatorial proofs, the conjugate of a partition is studied. More precisely, given a partition $\pi$ whose parts $b_1 \geq b_2 \geq \ldots \geq b_\nu$ are written in decreasing order, its Ferrers graph is an array of nodes equally spaced with $b_i$ nodes in the $i-$th row such that the left-most node of each row will lie on a common vertical line. If we read the nodes of this graph column wise, we get the conjugate partition $\pi^*$. For example, if $\pi$ is the partition 7+7+5+4+2+2, then its conjugate $\pi^*$ is 6+6+4+4+3+2+2.

Let $\lambda(\pi)$ denote the largest part of $\pi$, and $\nu(\pi)$, the number of parts of $\pi$. Clearly,

$$\lambda(\pi) = \nu(\pi^*) \text{ and } \nu(\pi) = \lambda(\pi^*). \tag{1.1}$$

and so $\lambda(\pi) + \nu(\pi)$ is invariant under conjugation. Another invariant is $D(\pi)$, the Durfee square of $\pi$. This the largest square of nodes starting from the upper left hand corner of the Ferrers graph. The relation (1.1) and the invariance of $D(\pi)$ have been used extensively [5]. But surprisingly one fundamental invariant has remained totally unexploited. This is $\nu_d(\pi)$, the number of different parts of $\pi$. That, is, for all partitions $\pi$, we have

$$\nu_d(\pi) = \nu_d(\pi^*). \tag{1.2}$$

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Recently we have undertaken the study of this invariant and utilized it to prove a variety of partition identities, some new, and some of which are extensions of known identities. Here we shall briefly describe (without proof) some of the identities we have obtained using (1.2). In order to do this, we need some notation.

## 2 Notation and partition interpretation

Given a complex number $a$ and a positive integer $n$, define

$$(a)_n = (a;q)_n = \prod_{j=0}^{n-1}(1-aq^j).$$

Next let

$$(a)_\infty = \lim_{n\to\infty}(a)_n = \prod_{j=0}^{\infty}(1-aq^j), \text{ for } |q|<1.$$

The expression

$$\frac{(aq)_n}{(bq)_n} = \frac{(1-aq)(1-aq^2)\dots(1-aq^n)}{(1-bq)(1-bq^2)\dots(1-bq^n)} \tag{2.1}$$

occurs quite often in the theory of basic hyper-geometric series. Fine [6] discusses in detail many transformation properties of the function $F(a,b;q)$ formed by summing the expression in (2.1) over $n \geq 0$. The standard combinatorial interpretation of (2.1) is that it is the generating function of vector partitions $(\pi_1;\pi_2)$ into parts $\leq n$, where the parts of $\pi_2$ cannot repeat. Instead of the expression in (2.1) we consider instead

$$\frac{(abq)_n}{(bq)_n}$$

and interpret it as the generating function of partitions into parts $\leq n$, such that the power of $b$ is $\nu(\pi)$ and the power of $(1-a)$ is $\nu_d(\pi)$. That is

$$\frac{(abq)_n}{(bq)_n} = \sum_{\lambda(\pi)\leq n}(1-a)^{\nu_d(\pi)}b^{\nu(\pi)}q^{\sigma(\pi)}, \tag{2.2}$$

where $\sigma(\pi)$ is the sum of the parts of $\pi$. With this different interpretation we have the following results:

# 3 Results

1. **Cauchy's identity**: The $q$–binomial theorem or Cauchy's identity is

$$\sum_{n=0}^{\infty} \frac{(a)_n t^n}{(q)_n} = \frac{(at)_\infty}{(t)_\infty}. \tag{3.1}$$

Several proofs of (3.1) are known (see Andrews [5]). Our new proof goes as follows:

First consider the three parameter generating function of all partitions, namely,

$$f(a,b,c;q) = \sum_{\pi} (1-a)^{\nu_d(\pi)} b^{\nu(\pi)} c^{\lambda(\pi)} q^{\sigma(\pi)}. \tag{3.2}$$

Using (2.2) it follows that

$$f(a,b,c;q) = 1 + \sum_{n=1}^{\infty} \frac{(1-a)(abq)_{n-1} bc^n q^n}{(bq)_n}. \tag{3.3}$$

Using (3.3) and with $n \to \infty$ in (2.2) we observe that

$$f(a,b,1;q) = 1 + \sum_{n=1}^{\infty} \frac{(1-a)(abq)_{n-1} bq^n}{(bq)_n} = \frac{(abq)_\infty}{(bq)_\infty}. \tag{3.4}$$

We call (3.4) as a *variant of Cauchy's identity.*

Next observe that (1.1) and (1.2) imply that

$$f(a,b,c;q) = f(a,c,b;q), \tag{3.5}$$

Thus from (3.4) and (3.5) we get

$$\sum_{n=0}^{\infty} \frac{(a)_n c^n q^n}{(q)_n} = f(a,1,c;q) = f(a,c,1;q) = \frac{(acq)_\infty}{(cq)_\infty} \tag{3.6}$$

which is equivalent to Cauchy's identity (3.1).

2. **A variant of the Rogers–Fine identity**: Although $f$ is symmetric in $b$ and $c$, this is not apparent from the series (3.3). A different series expansion for $f$ which renders this symmetry explicit can be derived using Durfee squares and the symmetry (1.2). This is

$$f(a,b,c;q) = 1 + \sum_{n=1}^{\infty} \frac{(1-a)b^n c^n q^{n^2} (abq)_{n-1}(acq)_{n-1}(1-abcq^{2n})}{(bq)_n (cq)_n}. \tag{3.7}$$

This is a variant of the Rogers–Fine identity which is proved as equation (14.1) in [6], using transformation properties of $F(a,b;t)$. Subsequently Andrews [4] gave a combinatorial proof of the Rogers–Fine identity, but our proof via (3.7) is simpler (this will be presented in [2]).

3. **Heine's transformation**: One of the fundamental results in the theory of basic hyper–geometric series is Heine's transformation, namely,

$$\sum_{n=0}^{\infty} \frac{(a)_n(\gamma)_n c^n}{(\alpha)_n(q)_n} = \frac{(\gamma)_\infty(ac)_\infty}{(\alpha)_\infty(c)_\infty} \sum_{n=0}^{\infty} \frac{(\alpha/\gamma)_n(c)_n\gamma^n}{(ac)_n(q)_n}. \tag{3.8}$$

In 1967 Andrews [3] gave a combinatorial proof by rewriting it in symmetric form and interpreting this in terms of certain vector partitions. We depart from Andrews by rewriting (3.8) in the form

$$\sum_{n=0}^{\infty} \frac{(a)_n(\alpha\gamma q^{n+1})_\infty c^n q^n}{(\gamma q^{n+1})_\infty(q)_n} = \sum_{m=0}^{\infty} \frac{(\alpha)_m(acq^{m+1})_\infty\gamma^m q^m}{(cq^{m+1})_\infty(q)_m} \tag{3.9}$$

and interpreting this in a different combinatorial way. Identity (3.9) is in a symmetric form

$$h(a,c,\gamma,\alpha) = h(\alpha,\gamma,c,a) \tag{3.10}$$

and follows by using (1.2) and the generating function of partitions formed by *cuts* of the Ferrers graphs (see [1] for a proof of (3.9)).

4. **A six parameter extension**: The combinatorial proof of Heine's transformation via (3.9) gives rise to the following new six parameter extension:

$$1+\sum_{t=1}^{\infty}\frac{(1-\alpha)(\alpha\gamma q)_{t-1}\gamma\beta^t q^t}{(\gamma q)_t}+\sum_{n=1}^{\infty}\frac{(1-a)(abq)_{n-1}bc^n q^n}{(bq)_n}\left(1+\sum_{t=1}^{\infty}\frac{(1-\alpha)(\alpha\gamma q^{n+1})_{t-1}\gamma\beta^t q^{n+t}}{(\gamma q^{n+1})_t}\right)=$$

$$1+\sum_{t=1}^{\infty}\frac{(1-a)(acq)_{t-1}cb^t q^t}{(cq)_t}+\sum_{m=1}^{\infty}\frac{(1-\alpha)(\alpha\beta q)_{m-1}\beta\gamma^m q^m}{(\beta q)_m}\left(1+\sum_{t=1}^{\infty}\frac{(1-a)(acq^{m+1})_{t-1}cb^t q^{m+1}}{(cq^{m+1})_t}\right) \tag{3.11}$$

For a sketch of the combinatorial proof of (3.11) see [1]. This identity is in the symmetric form

$$H(a,b,c,\gamma,\beta,\alpha) = H(\alpha,\beta,\gamma,c,b,a). \tag{3.12}$$

Setting $b=\beta=1$ in (3.11) yields the symmetric form of Heine's transformation (3.9), but both Cauchy's identity and the variant are necessary in the derivation (see [1]).

5. **An extension of Ramanujan's mock–theta identity**: In his last letter to Hardy, Ramanujan had stated the following fifth order mock theta function identity:

$$\sum_{n=0}^{\infty}\frac{q^n}{(q^{n+1})_n} = 1+\sum_{m=0}^{\infty}\frac{q^{2m+1}}{(q^{m+1})_{m+1}}. \tag{3.13}$$

In 1967 Andrews [3] gave a combinatorial proof of (3.13) using conjugation of Ferrers graphs. In view of the symmetry (1.2), we noticed that following Andrews' proof, (3.13) could be extended by introducing a free parameter as follows:

$$\sum_{n=0}^{\infty} \frac{(aq^{n+1})_n q^n}{(q^{n+1})_n} = \frac{1}{1-q} + \sum_{m=1}^{\infty} \frac{(1-a)(aq^{m+2})_{m-1} q^{2m+1}}{(q^{m+1})_{m+1}}. \tag{3.14}$$

For a proof of (3.14) see [1].

# 4 Concluding remarks

The results given above are a sample of what could be achieved using the invariance (1.2). What is amazing is that the usefulness of this invariant had completely escaped attention. In a forthcoming paper [2] we shall present many more results that can be derived using this invariant. A long term project is to go through many identities in Fine [6] systematically, and provide new combinatorial proofs using (1.2). This will also have the advantage of yielding extensions just as (3.11) extended (3.9).

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