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DIOPHANTINE ANALYSIS OF CUBIC FORMS

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Let $s \geq 2$ be fixed. For given coefficients $\lambda_i \in \mathbb{N}$ we shall be concerned with the diagonal cubic equation

$$\lambda_1 x_1^3 + \lambda_2 x_2^3 + \ldots + \lambda_s x_s^3 = 0. \tag{1}$$

When $s \leq 6$ it may happen that (1) admits only the trivial integer solution $x_1 = x_2 = \ldots = x_s = 0$. A simple example is given by

$$F(\underline{x}) = x_1^3 - A x_2^3 + p(x_3^3 - A x_4^3) + p^2(x_5^3 - A x_6^3) \tag{2}$$

where $p \equiv 1 \pmod 3$ is a prime, and A is a cubic non-residue (mod p). For $s \geq 7$, however, it is known that (1) has non-trivial solutions, and one may ask for the size of the smallest solution. In this spirit, Pitman and Ridout (1966) showed that for $s = 9$, there are solutions of (1) satisfying

$$0 < \sum_{i=1}^{9} |\lambda_i x_i^3| \ll (\lambda_1 \lambda_2 \cdots \lambda_9)^{\frac{3}{2}+\varepsilon},$$

for any set of coefficients $\lambda_1, \ldots, \lambda_9$. To simplify the statement of such results, we define, for $s \geq 7$,

$$B(s) := \inf_{g > 0} \left\{ \text{For any } \lambda_1, \ldots, \lambda_s \in \mathbb{N}, \text{ there are solutions of (1) with } 0 < \sum_{i=1}^{s} |\lambda_i x_i^3| \ll (\lambda_1 \cdots \lambda_s)^g \right\}$$

The result of Pitman and Ridout can then be restated as $B(9) \leq \frac{3}{2}$.

Later work implies that $B(s)$ is finite if and only if $s \geq 7$. The following explicit inequalities were obtained:

Leung (1983)		$B(8) \leq \frac{35}{8}$	
Baker (1989)	$B(7) \leq 61$,	$B(8) \leq 2 + \frac{5}{149}$,	$B(9) \leq \frac{7}{6}$
Brüdern (1992)		$B(8) \leq \frac{5}{3}$	
Li Hongze (1997)	$B(7) \leq 14$		

It may be of interest to compare this with known lower bounds. If $F(x_1, \dots, x_6)$ is the form given by (2), we may consider the equation

$$F(x_1, \dots, x_6) + \lambda\,(x_7^3 + \dots + x_s^3) = 0 \qquad (\lambda \in \mathbb{N} \text{ fixed})$$

For any non-trivial solution, we must have $x_j \neq 0$ for at least one $7 \leq j \leq s$ (since $F(\underline{x}) = 0$ implies $\underline{x} = 0$) whence $|\lambda(|x_7^3| + \dots + |x_s^3|)| \geq \lambda$.

This easily yields
$$B(s) \geq \frac{1}{s-6}. \tag{3}$$

Maybe this holds with equality. We can now improve the upper estimates.

THEOREM 1. We have $B(7) \leq 5 + \frac{18}{35}$, $B(8) \leq \frac{7}{5}$.

As all earlier writers, we apply the circle method to obtain the Theorem, but there is an important difference. To describe this in more detail, we briefly describe the approach of Pitman-Ridout. Let

$$\Lambda := \lambda_1\lambda_2\dots\lambda_s\,; \qquad P^3 = \Lambda^g\,; \qquad P_j := P\lambda_j^{-1/3}$$

and

$$f_j(\alpha) := \sum_{|x|\le P_j} e(\lambda_j\alpha x^3)$$

The integral

$$R(\underline{\lambda}) := \int_0^1 f_1(\alpha)f_2(\alpha)\dots f_s(\alpha)\,d\alpha$$

counts solutions of (1) with $|x_j| \le P_j$, that is, $\lambda_j |x_j^3| \le \Lambda^g$. Hence, if we can show that $R(\underline{\lambda}) \ge 2$ for any $\underline{\lambda} = (\lambda_1, \dots, \lambda_s)$, then $B(s) \le g$ follows.

Now define the "major arcs" $\mathfrak{M}$ as the union of all intervals $|q\alpha - a| \le P^{-2}$ with $1 \le a \le q \le P$, $(a,q)=1$, and the "minor arcs" as $[0,1]\setminus\mathfrak{M} = \mathfrak{m}$ (mod 1). It is easy to obtain an asymptotic formula for the contribution from $\mathfrak{M}$, this takes the shape

$$\int_{\mathfrak{M}} f_1 f_2 \dots f_s\,d\alpha \sim \mathfrak{S}(\underline{\lambda})\,P^{s-3}\Lambda^{-\frac{1}{3}}\,, \tag{4}$$

here $\mathfrak{S}(\underline{\lambda})$ is the formal singular series, and (4) will hold for $s \ge 7$, $g > 1$, and one also has $\Lambda^{-\varepsilon} \ll \mathfrak{S}(\underline{\lambda}) \ll \Lambda^{\varepsilon}$ subject to mild conditions.

Take $s = 9$ for a typical example of a standard treatment of the minor arcs. One starts with

$$\int_{\mathfrak{m}} \le \Big(\sup_{\alpha\in\mathfrak{m}} |f_9(\alpha)|\Big) \int_0^1 |f_1(\alpha)\dots f_8(\alpha)|\,d\alpha \tag{5}$$

Pitman and Ridout now applied Hölder's inequality to the integral in (5), followed by Hua's inequality in the form

$$\int_0^1 \Big| \sum_{|x|\le X} e(\alpha x^3)\Big|^8 d\alpha \ll X^{5+\varepsilon}. \tag{6}$$

This yields

$$\int_0^1 |f_1\cdots f_8|\, d\alpha \le \prod_{j=1}^{8} \Big(\int_0^1 |f_j|^8 d\alpha\Big)^{\frac{1}{8}} \ll (P_1P_2\cdots P_8)^{\frac{5}{8}+\varepsilon} \tag{7}$$

By Weyl's inequality, one may hope for $|f_9(\alpha)| \ll P_9^{3/4+\varepsilon}$ for $\alpha\in\mathfrak{m}$, and a comparison of (4) and (5) produces $B(9)\le\frac{3}{2}$. However, although (6) is essentially best possible (can be replaced by an asymptotic formula $\sim CX^5$ with $C>0$), the use of Hölder's inequality is highly inefficient, at least in some cases. For example, we can also use Cauchy's inequality to produce

$$\int_0^1 |f_1\cdots f_8|\, d\alpha \le \Big(\int_0^1 |f_1f_2f_3f_4|^2 d\alpha\Big)^{\frac{1}{2}} \Big(\int_0^1 |f_5\cdots f_8|^2 d\alpha\Big)^{\frac{1}{2}},$$

and we have

$$\int_0^1 |f_1\cdots f_4|^2 d\alpha = \#\Big\{ |x_i|\le P_i,\ |y_i|\le P_i : \sum_{i=1}^{4} \lambda_i (x_i^3-y_i^3)=0 \Big\}$$

This is again a counting problem of the type considered earlier, and we may expect at least an upper bound of the form

$$\int_0^1 |f_1\cdots f_4|^2 d\alpha \ll C(\lambda_1;\lambda_2;\lambda_3;\lambda_4)\, P_1^2P_2^2P_3^2P_4^2 P^{-3} + P_1P_2P_3P_4 \tag{8}$$

(the second term accounts for diagonal solutions $x_i=y_i$). For coprime values of $\lambda_1,\dots,\lambda_4$ and $\lambda_5,\dots,\lambda_8$ we get a considerable improvement of (7), namely

$$\int_0^1 |f_1 \cdots f_8|\, d\alpha \ll (P_1 \cdots P_8)\, P^{-3}$$

which, if true, is best possible. We have not attempted to verify (8), but the above discussion clearly shows that the arithmetic nature of the coefficients should help to estimate $B(s)$. In fact, the use of Hölder's inequality in (7) makes such an interplay of various coefficients impossible.

When $s = 7$ or 8, it is more adviseable to work with 6-th moments. Here it is possible to establish a partial analogue of (8).

THEOREM 2. Let $\lambda, \mu \in \mathbb{N}$ be cube-free, $X \geq 1$, $Y \geq 1$. Let $S_{\lambda,\mu}(X,Y)$ denote the number of solutions of

$$\lambda (x_1^3 - x_2^3) = \mu (y_1^3 + y_2^3 - y_3^3 - y_4^3) \tag{9}$$

subject to

$$|x_j| \leq X \;; \qquad y_j \in \mathcal{A}$$

where

$$\mathcal{A} = \{ y \leq Y : p | y \Rightarrow p \leq Y^{\eta} \}.$$

Then, for sufficiently small $\eta > 0$, one has

$$S_{\lambda,\mu}(X,Y) \ll XY^2 \left(1 + (XY)^{\frac{1}{8}} \frac{(\lambda,\mu)^{\frac{1}{2}}}{(\lambda\mu)^{\frac{1}{2}}} \right)$$

When $\lambda = \mu = 1$ and $X = Y$, this is contained in Wooley (1992). The proof follows Wooley's variant of Vaughan's new iterative method, with the simple observation that (9) implies $x_1^3 \equiv x_2^3 \pmod{\mu}$. If μ is not too small, this allows an "efficient differencing" to be performed. Theorem 1 is derived from Theorem 2 along the patterns outlined above, but the details are very complicated.