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Pricing Options with Transaction Costs with the Method of Lines

Gunter H. Meyer
School of Mathematics
Georgia Institute of Technology
Atlanta, GA 30332

1. Introduction

Extensive calculations are performed routinely by financial institutions to price options and to hedge portfolios. Invariably, these computations involve the solution of the Black Scholes partial differential equation subject to the various boundary data which describe the multitude of options which currently are traded. As is well known, the Black-Scholes equation is essentially the heat equation under a natural change of variables (see, e.g., [9]), and many popular numerical techniques such as the binomial method specifically exploit this structure of the problem. In fact, for European options the solution of the problem is given in closed form in terms of error functions, the famous Black-Scholes pricing formulas [9].

There is, however, a growing interest in solving models for pricing options which leave the narrow framework of the Black-Scholes equation. If differential equations are used to model the value of the option then the resulting problem no longer is formulated for a constant coefficient scalar (heat) equation but for a one- or multidimensional nonlinear parabolic equation subject to data on fixed and free boundaries. In developing numerical methods for this broader class of problems it appears to be a natural requirement to ask for an algorithm which is tied very little to the specific structure of the problem at hand, which can reliably solve it, but which remains competitive when applied to the standard Black-Scholes formulation which, after all, is the workhorse of the industry.

It was suggested in [6] and [7] that the so-called method of lines (also called Rothe's method) based on time discrete boundary value problems is a flexible and easily applied method for the general option valuation described by an optimal stopping problem for a scalar diffusion equation. But while applicable in principle to a broad class of problems this method is not immune to the difficulties in the model brought about by degenerate diffusion coefficients, infinite domains or dominant convection terms.

It is the purpose of this paper to examine the influence of those features of the model which are known to complicate its numerical solution, and to illustrate its numerical performance by pricing portfolios of options with transaction costs.

2. The Method of Lines

The method of lines describes the approximation of a partial differential equation by a system or sequence of ordinary differential equations to which the techniques common in the theory of ordinary differential equations are applied. Here we shall treat the following (free) boundary value problem for a general diffusion equation from this point of view:

$$A(x,t)u_{xx} + B(x,t)u_x - C(x,t)u - D(x,t)u_t = F(x,t) \tag{2.1a}$$

$$\begin{gathered} u(X,t) = \alpha(t)u_x(X,t) + \beta(t) \\ G_i(u(S(t),t), u_x(S(t),t), S(t), t) = 0, \qquad i = 1,2 \\ u(x,0) = u_0(x) \\ S(0) = s_0, \end{gathered} \tag{2.1b}$$

where $X > S(t)$ or $X < S(t)$ depending on the application, and where x lies between X and $S(t)$.

As we shall see, the equation and the boundary data allow the treatment of European, American, and some path dependent options with deterministic but not necessarily constant volatility, interest and dividend rates. Specifically, all coefficients and the source term in (2.1) are allowed to be functions of the independent variables x and t. However, they do not depend on the solution u and its derivatives. There are models where the coefficients do depend on u and its derivatives. For example, when transaction costs are included in the pricing model then the diffusion equation (2.1a) may well depend nonlinearly on u_{xx}

(see, e.g., [9] and the comments below). However, a solution algorithm for such a problem in general is iterative so that it solves a problem for (2.1) at each step. Hence an algorithm for (2.1) subject to appropriate initial and boundary data will be sufficiently general for many financial applications.

When we apply the method of lines (in the Rothe sense) then the equation (2.1) is approximated by a time implicit ordinary differential equation of the form

$$\begin{aligned} Lu \equiv A(x,t_n)u_n'' + B(x,t_n)u_n' - C(x,t_n)u_n \\ - D(x,t_n)\frac{u_n}{\Delta t} = -D(x,t_n)\frac{u_{n-1}}{\Delta t} + F(x,t_n) \end{aligned} \tag{2.2}$$

or

$$\begin{aligned} Lu \equiv A(x,t_n)u_n'' + B(x,t_n)u_n' - C(x,t_n)u_n \\ - D(x,t_n)\frac{3u_n}{2\Delta t} = -D(x,t_n)\left[\frac{3u_{n-1}}{\Delta t} - \frac{u_{n-1}-u_{n-2}}{2\Delta t}\right] + F(x,t_n), \end{aligned} \tag{2.3}$$

where $\{u_n, s_n\}$ is the approximate solution at time level $t_n = n\Delta t$, and where for convenience Δt is held constant. The boundary data at time t_n are (usually) the natural time discretization of (2.1b)

$$\begin{aligned} u_n(X) = \alpha(t_n)u_n'(X) + \beta(t_n) \\ G_i(u_n(S_n), u_n'(S_n), S_n, t_n) = 0 \end{aligned} \tag{2.4}$$

Hence the original parabolic problem is solved as a sequence of elliptic problems at discrete time levels.

The Rothe semi discrete approximation leads to boundary value problems and differs from the usual method of lines approximation for (2.1) where space is discretized. Then the resulting system of ordinary differential equations is continuous in time and reflects the evolutionary character of the process. For a finite element based application of this discretization to a Stefan problem see [8]. Since optimal stopping problems usually involve free boundaries which strongly determine the solution we consider Rothe's method an appropriate approximation of the option pricing model.

We shall make the assumption that $A(x,t)$ is continuous and positive on the open interval bounded by X and $S(t)$ so that we can rewrite equation (2.2) or (2.3) and the boundary data in the form

$$Lu \equiv u''(x) - d(x)u' - c(x)u = f(x) \tag{2.5}$$

$$u(X) = \alpha u'(X) + \beta$$

$$G_i(u, u', S) = 0, \qquad i = 1 \text{ or } 2$$

(for example, $i = 1$ for European options where S is given, or $i = 1, 2$ for American options where S is to be determined). The functions d, c, f are found by comparing with (2.2) or (2.3). For simplicity we have dropped the subscript n denoting the time level in (2.5), but it should be understood that the boundary value problem (2.5) changes with n since $f(x)$ always depends on the the solution at the preceding time level(s).

The solution of (2.5) is handled with an implicit shooting method which employs the Riccati transformation. If p is the shooting parameter and $\{u(x,p), u'(x,p)\}$ denotes the solution of

$$Lu = f(x) \tag{2.6}$$

subject to

$$u'(X,p) = p$$

$$u(X,p) = \alpha u'(X,p) + \beta = \alpha p + \beta$$

then we would need to find a p such that $\{u(x,p), u'(x,p)\}$ satisfy the boundary conditions at S. It follows from the implicit function theorem that it is possible to find the inverse $p = p(x, u')$ of $u' = u'(x,p)$, at least when $|x - X|$ sufficiently small. If we substitute this inverse function into $u = u(x,p)$ then we see that

$$u = u(x, p(x, u')) = u(x, u'). \tag{2.7}$$

Since L is a linear operator it is straightforward to verify that the expression (2.7) is actually an affine transformation, the so-called Riccati transformation,

$$u = R(x)u' + w(x) \tag{2.8}$$

where

$$R' = 1 - d(x)R - c(x)R^2, \qquad R(X) = \alpha \tag{2.9}$$

$$w' = -c(x)R(x)w - R(x)g(x), \qquad w(X) = \beta \tag{2.10}$$

If (2.9) and (2.10) are integrated (forward if $S > X$, backward if $S < X$) then at the boundary $x = S$ we need to satisfy

$$G_i(R(S)u'(S) + w(S), u'(S), S) = 0 \tag{2.11}$$

where $i = 1$ if S is given or $i = 1, 2$ if $u'(S)$ and S are both unknown. If S is given then it is assumed that (2.11) can be solved for $u'(S)$. If S is unknown then it is assumed that either G_1 or G_2 can be solved explicitly for $u'(S)$. For definiteness, let us assume that

$$G_1(R(S)u'(S) + w(S), u'(S), S) = 0$$

can be rewritten as

$$u'(S) = h(R(S), w(S), S) \tag{2.12}$$

then the second boundary condition becomes

$$\phi(S) \equiv G_2(R(S)h(R(S), w(S), S) + w(S), h(R(S), w(S), S), S) = 0.$$

Hence the free boundary S must be a root of

$$\phi(x) = 0. \tag{2.13}$$

Conversely, it can be shown that every root of $\phi(x) = 0$ defines a solution of (2.5).

Once S has been found then $u'(S)$ is determined by (2.12). From

$$Lu = f(x)$$

and the Riccati transformation follows that we can find $u'(x)$ from the initial value problem

$$(u')' - d(x)(u') - c(x)(R(x)(u') + w(x)) = f(x) \tag{2.14}$$

$$u'(S) = h(R(S), w(S), S).$$

Integrating (2.14) from S to X and substituting u' into (2.8) then yields the solution of the boundary value problem (2.5) at time t_n. Note that the initial value problems for R, w and for u' are always integrated in opposite directions.

In discretized parabolic problems in general and in financial applications in particular we observe that $c(x) > 0$. If $S > X$ so that we first integrate forward in x then $R(X) \geq 0$ and $c(x) > 0$ insure that $R(x)$ exists for all $x > X$ and remains positive. Similarly, $c(x) > 0$, $S < X$ and $R(X) \leq 0$ insure that R exists for $x < X$ and remains negative. In essence, $R(x)$ behaves linearly or like a hyperbolic tangent. A look at the linear equations for w and u' then shows that the homogeneous parts of these equations have fundamental solutions which are exponentially decreasing in the direction of integration. In principle, if ordinary differential equations have smooth bounded solutions then numerical methods are able to approximate them to a high degree of accuracy. Hence for analytic discussions it is held permissible to assume that the ordinary differential equations for R, w and u' are solved exactly.

While the theory of ordinary differential equations governs the existence and uniqueness of solutions of initial value problems, their numerical resolution will determine the practical value of the method of lines approach for financial problems. Here the algorithm is implemented with the trapezoidal rule for the numerical integration of all initial value problems. If a fixed, but not necessarily equidistant grid is used at all times then this method allows rapid communication of data between iterations at a given time level and from one time level to the next. Moreover, since the parabolic problem already is discretized with respect to time with a method which is at most of order Δt^2 it seems defensible to use a space discretization which is of order Δx^2. The trapezoidal rule discretization then is of the same order as the finite difference discretization proposed in [9] when the convection term is approximated by a central difference quotient. We note that as R and w are computed with the trapezoidal rule the function $\phi(x)$ can be evaluated. If $\phi(x)$ changes sign between adjacent mesh points then S will be found as the root of the cubic interpolant to ϕ through the nearest four points. S is added to the fixed mesh and is the only point which changes from time level to time level.

It is known [1] that the continuous Riccati method is in fact the closure of Gaussian elimination applied to the fully discrete approximation of the spacial problem (2.5). Hence algebraically our method is equivalent to applying Gaussian elimination to a matrix approximation of (2.5), and searching for S is equivalent to checking at every forward step of the elimination whether the point corresponding to that line of the matrix satisfies the criteria for the free boundary. It appears conceptually simpler to work with the closure of the method given by the above ordinary differential equations than with the algebraic equations which actually have to be solved to obtain practical answers.

However, the structure of the algebraic problem obtained from a finite difference approximation of (2.5) also influences the integration of the ordinary differential equations with the trapezoidal rule. Let $\{x_j\}$ denote mesh points such that $X = x_0$ and $X_N > S$. Let $\{R_j, w_j, u'_j\}$ denote the approximate solution at x_j of the ordinary differential equations. Then it follows from the trapezoidal rule that $\{R_j, w_j, u'_j\}$ are given by the formulas

$$\frac{\Delta x}{2} c_{j+1} R_{j+1}^2 + \left(1 + \frac{\Delta x}{2} d_{j+1}\right) R_{j+1} - \left(1 - \frac{\Delta x}{2} d_j\right) R_j + \frac{\Delta x}{2} c_j R_j^2 - \Delta x = 0$$

$$\left(1 + \frac{\Delta x}{2} c_{j+1} R_{j+1}\right) w_{j+1} = \left(1 - \frac{\Delta x}{2} c_j R_j) w_j - \frac{\Delta x}{2} (R_j f_j + R_{j+1} f_{j+1}\right)$$

$$\left(1 - \frac{\Delta x}{2}\,(c_{j+1}R_{j+1} + d_{j+1})\right) u'_{j+1} = \left(1 + \frac{\Delta x}{2}\,(c_j R_j + d_j)\right) u'_j + \frac{\Delta x}{2}\,(c_j w_j + c_{j+1} w_{j+1} + f_j + f_{j+1})$$

where $d_j = d(x_j)$, $c_j = c(x_j)$ and $\Delta x = x_{j+1} - x_j$.

These algebraic formulas impose constraints on Δx. For example, suppose that $S > X$ so that w is integrated forward and u' is integrated backward in x. In order to have decreasing exponential fundamental solutions in the direction of integration it is necessary that

$$0 < \frac{\Delta x}{2}\, c_j R_j < 1 \tag{2.15}$$

and

$$0 < \frac{\Delta x}{2}\,(c_j R_j + d_j) < 1 \tag{2.16}$$

in order to avoid spurious oscillations. In addition, it follows from the quadratic formula that we can solve for R_{j+1} only if

$$\left(1 + \frac{\Delta x}{2}\, d_{j+1}\right)^2 + 2\Delta x c_{j+1}\left(\Delta x + \left(1 - \frac{\Delta x}{2}\, d_j\right) R_j - \frac{\Delta x}{2}\, c_j R_j^2\right) \geq 0.$$

However, R_j, $c_j > 0$ and (2.16) insure that this condition holds so that no additional constraints are introduced by the numerical integration of the Riccati equation.

If $S < X$ then $R_j \leq 0$ and it is readily verified that the analogous constraints are

$$-1 < \frac{\Delta x}{2}\, c_j R_j < 0$$
$$-1 < \frac{\Delta x}{2}\,(c_j R_j + d_j) < 0.$$

In a typical application given below we illustrate the effect of violating these constraints.

3. Applications

Let us first consider how our considerations apply to the method of lines for the Black-Scholes equation with constant market parameters. The diffusion equation in this case is

$$Lu \equiv \frac{1}{2}\sigma^2 x^2 u_{xx} + (r - \rho) x u_x - ru - u_t = 0 \tag{3.1}$$

where $u = V/K$ and $x = S/K$ are the value V of the option and of the underlying asset S scaled by the strike price K. The volatility σ, the risk-free interest r and the dividend rate ρ in (3.1) are assumed constant. Real time τ is related to the independent variable t in (3.1) through $\tau = T - t$ where T is the time of expiry of the option.

If a backward Euler discretization of (3.1) is employed then it is straightforward to verify that

$$c(x) = \left(r + \frac{1}{\Delta t}\right)\frac{2}{\sigma^2 x^2}$$

and

$$d(x) = -(r - \rho)\frac{2}{\sigma^2 x}$$

If an American call is to be considered for (3.1) then for $\rho > 0$

$$\begin{aligned} &u(0,t) = 0 \\ &u(s(t),t) = s(t) - 1 \\ &u_x(s(t),t) = 1. \end{aligned} \tag{3.2}$$

where $s(t)$ is the early exercise (free) boundary. If a European call is to be treated then for $\rho \geq 0$ the corresponding boundary conditions are

$$\begin{aligned} &u(0,t) = 0 \\ &\lim_{x\to\infty} u(x,t) = \lim_{x\to\infty} [xe^{-\rho t} - e^{-rt}] \end{aligned} \tag{3.3}$$

It can be shown (see [3] or [7]) that in either case the time discrete solution u at $t = t_n$ has the representation

$$u(x) = x^\gamma \sum_{i=0}^{n} \delta_i^n (\ln x)^i, \qquad 0 \leq x \leq 1 \tag{3.4}$$

where γ is the positive root of

$$\frac{1}{2}\sigma^2\gamma(\gamma - 1) + (r - \rho)\gamma - \left(r + \frac{1}{\Delta t}\right) = 0 \tag{3.5}$$

and where the coefficients $\{\delta_i^n\}$ are determined by a recurrence relation

$$\delta_i^n = \frac{-\gamma\left[\delta_{i-1}^{n-1} + i(i+1)\frac{\sigma^2}{2}\delta_{i+1}^n \Delta t\right]}{i\left[r\Delta t + 1 + \gamma^2\frac{\sigma^2}{2}\Delta t\right]}, \qquad \delta_n^n = 0.$$

The recurrence relation does not define the constant of integration δ_0^n. However, differentiation shows that

$$u(x) = \frac{x}{\gamma}u'(x) - \frac{x^\gamma}{\gamma}\sum_{i=1}^{n}\delta_i^n i(\ln x)^{i-1} \tag{3.6}$$

It follows that on $[0,1]$

$$R(x) = \frac{x}{\gamma}$$

which can also be verified by direct substitution into (2.9). Moreover, this linear function also satisfies the difference equation obtained from the trapezoidal rule. Hence the conditions (2.15) and (2.16) for a call take on the concrete form

$$\frac{\Delta x}{\sigma^2 x\gamma}\left(r + \frac{1}{\Delta t}\right) < 1$$

and

$$0 < \frac{\Delta x}{\sigma^2 x\gamma}\left(r + \frac{1}{\Delta t}\right) - \frac{(r-\rho)}{\sigma^2 x}\Delta x < 1.$$

It is clear that these conditions cannot be satisfied as $x \to 0$. Hence a straightforward application of the trapezoidal rule on $(0, s(t))$ for a call will not succeed. On the other hand, the representation (3.4) suggest the replacement of $u(0,t) = 0$ by the reflection condition (3.6) at some $X > 0$. In this case $c(x)$, $R(x)$ and $d(x)$ are bounded above so that it now is possible to choose Δx small enough to satisfy the above sign constraints. Alternatively, the representation (3.4) suggests approximating the boundary condition $u(0,t) = 0$ by the barrier conditions $u(X,t) = 0$ for $X \ll 1$ since $u = O(x^\gamma)$ as $x \to 0$.

If, instead of a call, a put is to be considered then the Black-Scholes equation is subject to the boundary conditions

$$u(s(t),t) = 1 - s(t)$$

$$u_x(s(t),t) = -1$$

$$\lim_{x\to\infty} u(x,t) = 0$$

for an American put and

$$u(0,t) = e^{-rt}$$

$$\lim_{x\to\infty} u(x,t) = 0$$

for a European put. As shown in [7] the condition at infinity can be replaced by the reflection condition (3.6) for some $X \geq 1$ where now γ is the negative root of (3.5). Hence the above positivity constraints remain basically unchanged. An implementation of the method of lines for the time discretized parabolic problem with the trapezoidal rule must observe them in order to yield acceptable results. To illustrate this point we show in Fig. 1 a plot of the scaled "gamma"

$$u''(x) = c(x)(R(x)u'(x) + w(x)) + d(x)u'(x) + f$$

for an American put for two choices of a constant Δx. For the chosen parameters we find at $x = 1$:

$$\Delta x = .01, \qquad c(x)^* R(x)^* \Delta x/2 = -1.4905$$

$$\Delta x = .005, \qquad c(x)^* R(x)^* \Delta x/2 = -0.7378$$

The computed gamma remains unchanged as Δx is refined further. While the option value may still look acceptable the gamma is destroyed when the mesh constraints are violated. In general, a "nonphysical" oscillatory behavior of the method of lines solution invariably is an indication that the mesh parameters are not balanced.

To conclude this discussion of the choice of Δx we note that for an American put $s(t) > s_\infty > 0$ so that the degeneracy of the diffusion coefficient at $x = 0$ does not come into play. Consequently, the American put is a benign (and popular) test problem for numerical methods for the Black-Scholes equation.

Above we have considered only American options because they define a nonlinear obstacle problem for which no closed-form solution is known. The method of lines should

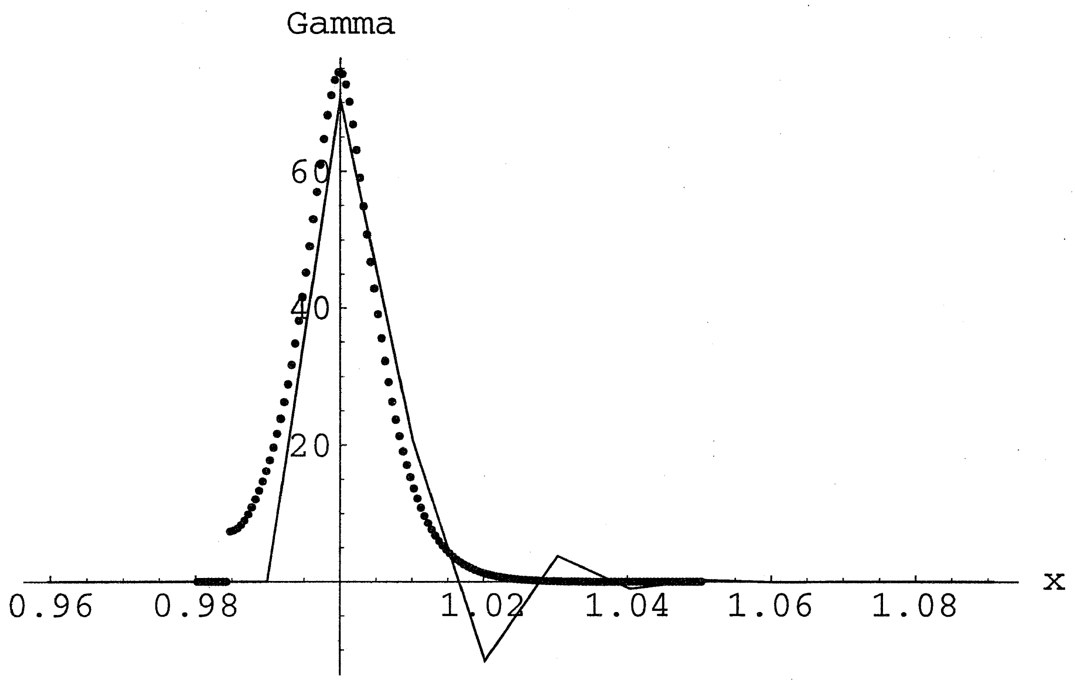

Fig. 1. Gamma for an American put near the free boundary.

$\sigma = .15, \quad r = .08, \quad \rho = 0, \quad T = .002, \quad \Delta t = .001$

solid line: $\Delta x = .01, \quad X = 2$

points: $\quad \Delta x = .005, \quad X = 2$

not be applied to the corresponding European option because they have analytic solutions. However, European options do require a numerical solution if portfolios with transaction costs are considered. There is considerable discussion in the mathematical and financial literature on how to incorporate such costs (see. e.g., [2], [4] and [9]). Here we simply note that all the modifications proposed introduce a nonlinearity into the Black-Scholes model which can be incorporated easily into an iterative solution of the problem with the method of lines. To be specific let us include transactions costs as discussed in [9]. The Black-Scholes equation (3.1) is now replaced by the nonlinear equation

$$Lu \equiv \frac{1}{2}\sigma^2 x^2 u_{xx} + (r-\rho)xu_x - ru - u_t = \alpha x^2 |u_{xx}| \tag{3.7}$$

where u and x are the unscaled values of a portfolio and of the underlying asset. The initial and boundary conditions for (3.7) are dictated by the composition of the portfolio.

Similar to the problem with jump diffusion considered in [6] we shall solve this non-linear problem iteratively. If at time level t_n the index k is an iteration count then the value u_n is found from

$$u_n(x) = \lim_{k\to\infty} u_n^k(x)$$

where u_n^k is a solution of

$$Lu_n^k = \alpha x^2 |u_{xx}^{k-1}| \tag{3.8}$$

subject to

$$u_0(x) = u(x,0), \qquad u_n^0(x) = u_{n-1}(x), \qquad u_n^k(X) = 0$$

and the appropriate boundary conditions. For each k we have precisely the problem discussed above. To illustrate the behavior of the method for this model we shall consider the two portfolios of European options solved with an explicit finite difference method in [9].

The first portfolio consists of a long call with strike price $K_1 = 45$ and a short call with strike price $K_2 = 55$. This imposes the following initial and boundary conditions on (3.7):

$$u(x,0) = \max\{x - K_1, 0\} - \max\{x - K_2, 0\}$$

$$u(0,t) = 0$$

$$\lim_{x\to\infty} u(x,t) = (K_2 - K_1)e^{-rt}$$

The application of the method of lines to this problem is straightforward. We do point out, however, that exponentials in time dependent boundary conditions, particularly for a European put, should not simply be evaluated but should be approximated by the product formula, for example,

$$e^{-rt_n} = (1 - r\Delta t)^n, \qquad t_n = n\Delta t$$

in order not to introduce unbounded derivatives at $x = 0$ into the ordinary differential equation (3.8). In fact, it is straightforward to verify that the time discrete European call and put for (2.2) satisfy the relationship

$$P_n(x) = C_n(x) + K(1 - r\Delta t)^n - x(1 - \rho\Delta t)^n$$

From $n\Delta t = t$ and $\Delta t \to 0$ we recover the well-known call put parity relation for the Black-Scholes solution.

For the numerical simulation the boundary condition at $x = 0$ is replaced by the barrier condition

$$u(1, t) = 0$$

while the second boundary condition is enforced at X which is chosen sufficiently large so that the value of the portfolio near the strike prices is no longer affected by changes in X.

In Fig. 2 we show the value $u(50)$ of the portfolio as a function of the cost parameter α in (3.7) for the data used in [9]. Our graph shows consistency between our result and the plotted value for $u(50)$ in [9] which corresponds to a cost parameter of $\alpha = 0.032$.

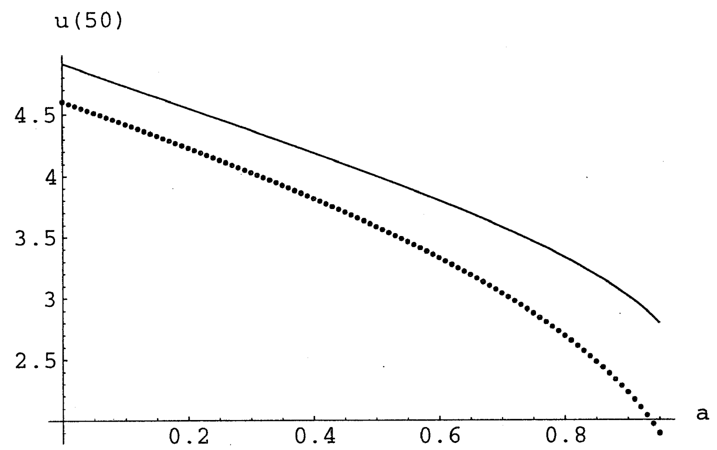

Fig. 2. Value of the unscaled portfolio at $x = 50$ as a function of the cost parameter α.

line graph - European call (cf. [9, p. 258])

point graph - European put

$\sigma = .4, \quad r = .1, \quad \rho = 0, \quad t = .5, \quad \Delta x = .05, \quad \Delta t = .5/200, \quad X = 200$

The second portfolio consists of two long calls with strike prices $K_1 = 45$ and $K_2 = 65$ and two short calls with strike price $K_3 = 55$. The initial condition and boundary conditions are now

$$u(x,0) = \max\{x - K_1, 0\} + \max\{x - K_2, 0\} - 2\max\{x - K_3, 0)$$

$$u(0,t) = 0$$

$$\lim_{x\to\infty} u(x,t) = 0$$

Fig. 3 shows the "gamma" of the second portfolio for $\alpha = .5$. The curve again is obtained by linear interpolation of the nodal values without additional smoothing.

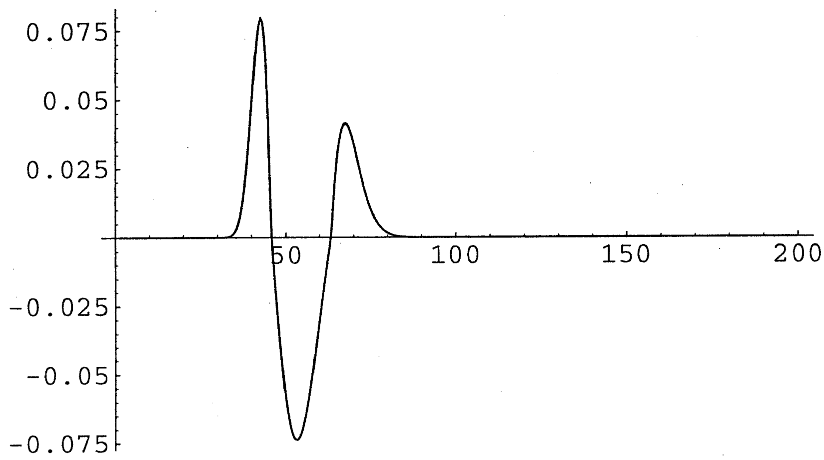

Fig. 3. "Gamma" for the "long butterfly spread" of [9, p. 259] for high transaction costs $\alpha = .5$ $(K = 2.934)$.
$T = 1$ month, $\sigma = .4$, $r = .1$, $\rho = 0$, $\Delta x = .05$, $\Delta t = 1/2400$, $X = 200$

From a computational view the nonlinearity adds little complexity when the option problem is solved iteratively at each time step provided this simple iteration converges. The numerical experiments indicate that convergence slows down as α increases. For $\alpha > .95$ the iteration failed to converge in both cases. Finally, we note that for $\alpha > \frac{1}{2}\sigma^2$ (as in the second example) the equation (3.7) becomes the backward heat equation over each interval where $u_{xx} > 0$; but as long as the iteration converged no influence on the stability of the numerical solution with respect to the mesh parameters was observed.

Comments on the fixed point iteration

The Black-Scholes model with transaction cost is fully nonlinear (in contrast to mildly or quasi nonlinear) and thus uncommon in the numerical literature. Hence it may be of interest to establish under what conditions the above fixed point iteration at a given time level can be guaranteed to converge. We use arguments similar in spirit to those employed in the model with jump diffusion considered in [6].

Let H be the completion of the inner product space of twice continuously differentiable functions on $[0, X]$ with inner product

$$\langle f, g\rangle = \int_0^X \{x^2 f'' g'' + \beta[f'g' + fg]\}dx$$

where β is a positive constant. Since X is finite it follows that $H^2[0, X]$ is a subspace of the space H. Let $\mathcal{M}$ be the set of functions in H which have continuous second derivatives on $(0, X)$ and which assume the boundary condition

$$u(0) = A, \qquad u(X) = B.$$

For $f \in \mathcal{M}$ define the operator T by

$$u = Tf$$

where u is the solution of

$$Lu = \alpha x^2 |f''(x)|$$

$$u(0) = A, \qquad u(X) = B.$$

Because L is equivalent to a constant coefficient equation on $[0, \infty)$ with real exponential solutions it follows from [5] that T well defined on $\mathcal{M}$ and maps $\mathcal{M}$ into $\mathcal{M}$. We shall show that T is a contraction. If $u = Tf$ and $v = Tg$ for f and $g \in \mathcal{M}$ then the equation

$$\int_0^X (Lu - Lv)(u - v)'' dx = \alpha \int_0^X x^2(|f''| - |g''|)(u - v)'' dx$$

and integration by parts yield the estimate

$$\int_0^X \frac{1}{2}\sigma^2 x^2 (u-v)''^2 dx + \left(\frac{r+\rho}{2} + \frac{1}{\Delta t}\right)\int_0^X (u-v)'^2 dx \le \alpha \int_0^X x^2 |f'' - g''||(u-v)''|dx$$

provided only that $r \ge \rho$. Since

$$\int_0^X (u-v)^2 dx \le \left(\frac{X}{\pi}\right)^2 \int_0^X (u-v)'^2 dx$$

it follows that one can find constants β and γ such that

$$||Tf - Tg|| \le \gamma ||f - g||$$

where $\gamma = \frac{\alpha}{\frac{1}{2}\sigma^2}$. Hence T is a contraction whenever $\alpha < \frac{1}{2}\sigma^2$.

In terms of the constant K defined in [9] for the above model we find that a contraction is guaranteed for

$$K \le \frac{1}{2}\sqrt{\frac{\pi}{2}} \cong .62$$

We remark that this estimate is quite weak in view of the numerical simulations which converged for

$$K \le 7.05.$$

Finally, we observe that the specific form of the source term in (3.7) is not important. Only the Lipschitz continuity with respect to u_{xx} is used. Hence this approach will apply, in principle, to the model developed in [2].

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