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Remarks on a Reaction-Diffusion Equation with Degenerate p-Laplacian

Shingo Takeuchi (竹内 慎吾)
Department of Mathematics, Waseda University
3-4-1 Ohkubo, Shinjuku-ku, Tokyo 169, JAPAN

1 Introduction

This is a joint work with Professor Yoshio Yamada at Waseda University.

Consider the following reaction-diffusion equation

$$(\mathrm{P})\begin{cases} u_t=\lambda(|u_x|^{p-2}u_x)_x+|u|^{q-2}u(1-|u|^r), & (x,t)\in(0,1)\times(0,+\infty),\\ u(0,t)=u(1,t)=0, & t\in(0,+\infty),\end{cases}$$

with initial condition

$$u(x,0)=u_0(x), \qquad x\in(0,1),$$

where $p>2,\ q\geq 2,\ r>0$ and λ is a positive parameter. The reaction term of (P) consists of a source term $|u|^{q-2}u$ and an absorption term $|u|^{q+r-2}u$, which dominates the source term.

When the p-Laplacian is replaced by the linear diffusion term, a related problem to (P) was discussed by Chafee-Infante [2]:

$$\begin{cases} u_t=\lambda u_{xx}+u(1-|u|^r), & (x,t)\in(0,1)\times(0,+\infty),\\ u(0,t)=u(1,t)=0, & t\in(0,+\infty),\end{cases} \tag{1.1}$$

which is a special case of (P) with $p=q=2$. For (1.1), they have studied the global existence of solutions, the structure of the set of all stationary solutions and the stability of each stationary solution. In the stationary problem, it is a well-known beautiful result that a pair of stationary solutions with n-zeros bifurcates from the trivial solution as λ decreases through $\lambda_n=((n+1)\pi)^{-2}$ for $n=0,1,\cdots$ (see also Henry [6]).

When the p-Laplacian is concerned, the stationary problem associated with (P) is given by

$$(\mathrm{SP})\begin{cases} \lambda(|\phi_x|^{p-2}\phi_x)_x+|\phi|^{q-2}\phi(1-|\phi|^r)=0, & x\in(0,1),\\ \phi(0)=\phi(1)=0.\end{cases}$$

As an extension of the results of [2], Guedda-Veron [5] have studied (SP) in the special case $p=q>1$ and $r>0$. Their results say that for $p=q$, (SP) has the same structure of bifurcations with respect to λ as the linear diffusion case. They have also shown that for sufficiently small $\lambda > 0$, there exist stationary solutions with flat hat, where is a flat place in its graph (see also Kamin-Veron [8]). This phenomenon is not found in the linear diffusion case.

Our purpose is to study (P) and (SP) in the general case $p > 2$, $q \geq 2$ and $r > 0$. We will mainly discuss the following three subjects:

(i) the global existence of solutions for (P),

(ii) the structure of the set of all stationary solutions,

(iii) the stability of each stationary solution.

As the first step of our analysis, we show the existence and uniqueness of global solution $u(\cdot; u_0)$ for (P) with $u(0) = u_0 \in L^2$. It is seen that the family of solution operators $\{S(t) : u_0 \mapsto u(t; u_0); t \geq 0\}$ has the semi-group property; so that (P) generates a dynamical system on L^2. In order to investigate the asymptotic behavior of solutions, we use the notion of the ω-limit set $\omega(u_0)$ associated with $\{S(t)u_0; t \geq 0\}$. We can prove inclusion relation $\omega(u_0) \subset E_\lambda$, where E_λ is the set of solutions of (SP).

According to the above observation, we are led to study E_λ. We employ the time-map method as in Aronson-Crandall-Peletier [1], Chafee-Infante [2], Fujii-Ohta [4] and Guedda-Veron [5]. We can obtain complete understanding of the structure of E_λ. A rough content of the result is as follows: if $p > q$, then E_λ is an infinite set for any $\lambda > 0$; if $p = q$, then E_λ is a finite set for large λ and an infinite set for small λ with bifurcations from the trivial solution (the same result is obtained by Guedda-Veron [5]); if $p < q$, then E_λ is a finite set for large λ and an infinite set for small λ with spontaneous bifurcations (for the notion of spontaneous bifurcation, see Smoller [11, pp.185-191]). Furthermore, there always exist solutions with flat hat for sufficiently small $\lambda > 0$. Summarizing these results, we can express E_λ as

$$E_\lambda = D \cup \bigcup_{l=1}^{\infty} \{\pm G_\lambda^l\},$$

where D is a discrete set of functions and each G_λ^l is either a continuum generated by one stationary solution with l-zeros and flat hat, or the empty set.

Finally, we study the stability of each stationary solution with use of L^∞-norm. Our method is based on the comparison theorem for (P). The time-map helps us to construct suitable comparison functions near each stationary solution in D. It is possible to show that for $p \geq q$, a maximal (or minimal) solution is attractive and the other solutions in D are unstable; for $p < q$, the situation is slightly different because there exist a pair of positive solutions (one of which is maximal) if and only if $\lambda < \lambda^*$ with a some λ^*: the trivial solution is always attractive and the maximal (or minimal) solution is attractive for $\lambda < \lambda^*$, whereas the other solutions in D are unstable. Concerning the solutions in G_λ^l, we can show that the solutions which have adjoining layers are unstable. However, it is difficult to find suitable comparison functions for the other solutions in G_λ^l, i.e., the solutions whose layers are separated. We will know a related result to the stability property of these solutions.

The plan of this paper is as follows. In Section 2, we study the global existence of solutions of (P) and their asymptotic behaviors. In Section 3, we study the structure of E_λ by the time-map method. There, we will exhibit interesting phenomena caused by the degeneracy of the p-Laplacian and the relation between p and q. In Section 4, we give some stability results for stationary solutions. In Section 5, we list some remarks and some open problems.

2 Non-Stationary Problem

In this section we study the global existence of solutions of (P) and their asymptotic behaviors. We begin with the definition of *strong solutions.*

Definition 2.1. For $u_0 \in L^2$, a function $u : [0,T] \to L^2$ is called a strong solution of (P) in $[0,T]$ with $u(0) = u_0$ if it possesses the following properties:

(i) $u \in C([0,T]; L^2) \cap L^p(0,T; W_0^{1,p})$,

(ii) $u_t \in L^2(\delta,T;L^2)$ and $(|u_x|^{p-2}u_x)_x \in L^2(\delta,T;L^2)$ for any $\delta > 0$,

(iii) u satisfies

$$\begin{cases} u_t = \lambda(|u_x|^{p-2}u_x)_x + |u|^{q-2}u(1-|u|^r) & \text{in } (0,1)\times(0,T), \\ u(0,t) = u(1,t) = 0 & \text{in } (0,T), \end{cases}$$

(iv) $u(0) = u_0$.

We will state some basic properties of strong solutions of (P).

Lemma 2.1. (i) *Let u and v be strong solutions of* (P) *in* $[0,T]$ *with* $u(0) = u_0$ *and* $v(0) = v_0$*, respectively. Then the following inequality holds:*

$$\|u(t)-v(t)\|^2 + 2^{3-p}\lambda\int_0^t \|u_x(s)-v_x(s)\|_p^p ds \le e^{2C_0 t}\|u_0-v_0\|^2 \tag{2.1}$$

for $t \in [0,T]$, where $C_0 = \sup\{f'(u); u \in \mathrm{R}\} < +\infty$.

(ii) *Every strong solution u of* (P) *in* $[0,T]$ *with $u(0) = u_0$ satisfies*

$$\lambda t\|u_x(t)\|_p^p + p\int_0^t s\|u_t(s)\|^2 ds \le C_1(t+\|u_0\|^2) \tag{2.2}$$

for all $t \in [0,T]$. In particular, if $u_0 \in W_0^{1,p}$, then

$$\lambda\|u_x(t)\|_p^p + p\int_0^t \|u_t(s)\|^2 ds \le \lambda\|u_{0x}\|_p^p + C_2\|u_0\|_{q+r}^{q+r} + C_3 \tag{2.3}$$

for all $t \in [0,T]$, where C_i $(1 \le i \le 3)$ are positive constants depending only on p, q and r.

This lemma is proved in Takeuchi-Yamada [12].

Combining the abstract existence result by Ôtani [9] with Lemma 2.1, we can show

Theorem 2.1. *For any $u_0 \in L^2$, there exists a unique strong solution $u(\cdot)$ of* (P) *in* $[0,+\infty)$ *with $u(0) = u_0$ satisfying:*

$$u \in C((0,+\infty); W_0^{1,p}), \tag{2.4}$$

$$t^{1/2}u_t(t) \in L^2(0,T;L^2) \quad \text{for every } T > 0. \tag{2.5}$$

In particular, if $u_0 \in W_0^{1,p}$, then $u(\cdot;u_0)$ satisfies

$$u \in C([0,+\infty); W_0^{1,p}),$$
$$u_t \in L^2(0,T;L^2) \quad \text{for every } T > 0.$$

By Theorem 2.1, we can define a family of mappings $\{S(t) : L^2 \to L^2; t \geq 0\}$ as follows:

$$S(t) : u_0 \mapsto u(t;u_0) \qquad \text{for every } u_0 \in L^2,$$

where $u(\cdot;u_0)$ denotes the strong solution of (P) with $u(0) = u_0$ in $[0,+\infty)$. The following lemma is derived from Lemma 2.1 and Theorem 2.1. It assures that (P) generates a dynamical system on L^2.

Lemma 2.2. *The family $\{S(t) : L^2 \to L^2; t \geq 0\}$ has the following properties.*

(i) *For each $t > 0$, $R(S(t)) \subset W_0^{1,p}$.*

(ii) *For each $t \geq 0$, $S(t)$ is continuous from L^2 into L^2.*

(iii) *For each $u_0 \in L^2$, $S(\cdot)u_0$ is continuous from $[0,+\infty)$ into L^2.*

(iv) *$S(0) = I$ (=identity operator).*

(v) *$S(t)(S(\tau)u_0) = S(t+\tau)u_0$ for all $u_0 \in L^2$ and $t,\ \tau \geq 0$.*

Owing to Lemma 2.2, it is possible to define the *ω-limit set* $\omega(u_0)$ associated with $\{S(t)u_0; t \geq 0\}$:

$$\omega(u_0) = \bigcap_{t\geq 0} Cl\{S(s)u_0; s \geq t\},$$

where "Cl" stands for the closure with respect to L^2-norm. It has the following properties (for the proof, see [12]).

Theorem 2.2. *For each $u_0 \in L^2$, $\omega(u_0)$ is non-empty, compact, invariant and connected in L^2 and*

$$\mathrm{dist}(u(t;u_0);\omega(u_0)) \equiv \inf_{v\in\omega(u_0)} \|u(t;u_0) - v\| \to 0 \qquad \text{as } t \to +\infty. \tag{2.6}$$

Furthermore, it holds that

$$\omega(u_0) \subset E_\lambda \equiv \{\phi \in W_0^{1,p}; \lambda(|\phi_x|^{p-2}\phi_x)_x + |\phi|^{q-2}\phi(1-|\phi|^r) = 0 \text{ in } (0,1)\}.$$

From Theorem 2.2, it can be seen that every $u(t;u_0)$ approaches E_λ in L^2-sense as $t \to +\infty$. The structure of E_λ will be studied in the next section.

Finally we give a comparison theorem, which plays an important role in the discussion of stability properties. We will define upper- and lower solutions of (P).

Definition 2.2. Let $u : [0,T] \to L^2$ possess regularity properties (i) and (ii) in Definition 2.1. Then u is called an *upper solution* for (P) in $[0,T]$ if it satisfies

$$\begin{cases} u_t \geq \lambda(|u_x|^{p-2}u_x)_x + |u|^{q-2}u(1-|u|^r) & \text{in } (0,1)\times(0,T), \\ u(0,t) \geq 0, \ u(1,t) \geq 0 & \text{in } (0,T). \end{cases} \tag{2.7}$$

If u satisfies (2.7) with "$\geq$" replaced by "$\leq$", then it is called a *lower solution.*

We are ready to give our comparison theorem.

Theorem 2.3. *Let u (resp. v) be an upper (resp. a lower) solution for* (P) *in* $[0,T]$. *If* $u(0) \geq v(0)$, *then $u(\cdot,t) \geq v(\cdot,t)$ for every $t \in [0,T]$.*

3 Stationary Problem

In this section we will study the structure of E_λ, i.e., the set of all solutions of (SP). Any function satisfying (SP) is called a *stationary solution.* The following proposition assures regularity properties of stationary solutions. For the proof, see Ôtani [10, Proposition 2.1].

Proposition 3.1. *Let $\phi \in E_\lambda$. Then ϕ belongs to $C^1([0,1]) \cap C^{\langle q \rangle}([0,1] \setminus Z)$, where $Z = \{x \in [0,1]; \phi_x(x) = 0\}$ and*

$$\langle x \rangle = \begin{cases} +\infty & \text{if } x \text{ is an even integer}, \\ \min\{n; n \geq x, \ n \text{ is integer}\}, & \text{otherwise}. \end{cases}$$

Remark 3.1. We can also see that $|\phi_x|^{p-2}\phi_x \in C^1([0,1])$. Therefore, ϕ satisfies (SP) in the classical sense.

Remark 3.2. Suppose that $\phi \not\equiv 0$ is twice differentiable for all $x \in (0,1)$, then ϕ has to attain 1 or -1. However, the converse is not necessarily true. We can see a concrete example in Takeuchi-Yamada [12].

We use the time-map method to look for solutions of (SP). Consider the following initial value problem:

$$\text{(IP)} \begin{cases} \lambda\psi_x + f(\phi) = 0, & x \in (0,+\infty), \\ \phi(0) = 0, \ \psi(0) = \alpha, \end{cases}$$

where $\psi = |\phi_x|^{p-2}\phi_x$, $f(\phi) = |\phi|^{q-2}\phi(1-|\phi|^r)$ and α is a parameter. If one can find a solution of (IP) satisfying $\phi(1) = 0$ by varying α, then it becomes a solution of (SP).

Owing to Proposition 3.1 and Remark 3.1, we can study (IP) in $\phi\psi$-phase plane. We should note that any solution of (IP) satisfies

$$F(\phi) + \frac{\lambda(p-1)}{p}|\psi|^{p/(p-1)} = \frac{\lambda(p-1)}{p}|\alpha|^{p/(p-1)}, \tag{3.1}$$

where $F(\phi) = \int_0^\phi f(s)ds$. The orbit of each solution of (IP) corresponds to one of level curves indicated in Fig. 1. Define α_0 and ϕ_α as follows:

$$\begin{aligned} F(1) &= \frac{\lambda(p-1)}{p}|\alpha_0|^{p/(p-1)}, \\ F(\phi_\alpha) &= \frac{\lambda(p-1)}{p}|\alpha|^{p/(p-1)} \quad \text{with } 0 < \phi_\alpha \leq 1. \end{aligned} \tag{3.2}$$

If ϕ is a solution of (SP), then so is $-\phi$ by the symmetry of f. Moreover, if $\alpha = 0$, then (3.1) implies that (IP) has no solutions other than the trivial solution. So it is sufficient to study solutions of (IP) with $\phi_x(0)^{p-1} = \alpha > 0$. Since we seek solutions of (IP) satisfying $\phi(1) = 0$, we have to take $\alpha \leq \alpha_0$. (Indeed, if $\alpha > \alpha_0$ in Fig. 1, then the orbit starting from P has no intersection points with ϕ-axis.) Hereafter, we set $\alpha \in (0, \alpha_0]$.

Let $\phi(x;\alpha)$ be a solution of (IP). Define

$$X(\alpha) = X(\alpha;\lambda) = \inf\{x \in (0,+\infty); \phi_x(x;\alpha) = 0\}.$$

If we regard x as a 'time' variable, then $X(\alpha)$ means the time when the orbit starting from P arrives at Q on the curve Γ in Fig. 1. Taking account of (3.1) and (3.2) one can show

$$X(\alpha) = \left\{\frac{\lambda(p-1)}{p}\right\}^{1/p} \int_0^{\phi_\alpha} (F(\phi_\alpha) - F(\phi))^{-1/p} d\phi, \tag{3.3}$$

which is a function from $(0, \alpha_0]$ into $(0, +\infty]$. Here we should note that the function $\alpha \mapsto \phi_\alpha$ is a strictly increasing function of class C^1 in $(0, \alpha_0)$. Therefore, it is convenient to study

$$I(a) = \int_0^a (F(a) - F(\phi))^{-1/p} d\phi \qquad \text{for } a \in (0,1], \tag{3.4}$$

instead of $X(\alpha)$. The following lemma, whose proof can be found in [12], gives us important information on $I(a)$.

Lemma 3.1. *For any $p > 2$ and $q \geq 2$, $I(\cdot)$ is continuous in $(0,1]$. In particular, $I(1) = \lim_{a\to 1-0} I(a)$ is finite. Furthermore, $I(\cdot)$ has the following properties.*

(i) *For $p > q$, $I(\cdot)$ is strictly monotone increasing and*

$$\lim_{a\to 0+} I(a) = 0.$$

(ii) *For $p = q$, $I(\cdot)$ is strictly monotone increasing and*

$$\lim_{a\to 0+} I(a) = I_0 \equiv p^{1/p} \int_0^1 (1 - t^p)^{-1/p} dt.$$

(iii) *For $p < q$, there exists $a^* \in (0,1)$ such that $I(\cdot)$ is strictly monotone decreasing in $(0, a^*)$ and strictly monotone increasing in $(a^*, 1)$. Moreover, $I(\cdot)$ satisfies*

$$\lim_{a \to 0+} I(a) = +\infty.$$

In Lemma 3.1, $I(1) < +\infty$ is equivalent to $X(\alpha_0) < +\infty$. Therefore, in $\phi\psi$-phase plane, the orbit starting from $P(0, \alpha_0)$ reaches $Q(1,0)$, which is an equilibrium point, in a finite time $X(\alpha_0)$. (In the linear diffusion case $p = 2$, the orbit for $\alpha = \alpha_0$ does not reach $Q(1,0)$ in a finite time since $I(1) = +\infty$. See Chafee-Infante [2].) After arriving at $Q(1,0)$, the orbit can stay there for any finite time before it begins to leave Q for $R(0, -\alpha_0)$. In other words, after attaining 1, $\phi(\cdot\, ; \alpha_0)$ can be constant ($= 1$) for any finite time before it begins to decrease to zero. So we should note that there are infinite number of solutions of (IP) for $\alpha = \alpha_0$.

Lemma 3.1 helps us to decide the structure of E_λ completely. Let $Y(\alpha)$ be the distance between two adjacent zero points of $\phi(\cdot; \alpha)$. Recalling (3.3) and (3.4) we have

$$\begin{aligned} Y(\alpha) = 2X(\alpha) &= 2\left\{\frac{\lambda(p-1)}{p}\right\}^{1/p} I(\phi_\alpha) \\ &\equiv 2C_{\lambda,p} I(\phi_\alpha) \qquad \text{with } C_{\lambda,p} = \left\{\frac{\lambda(p-1)}{p}\right\}^{1/p} \end{aligned}$$

for $\alpha \in (0, \alpha_0)$, where α_0 and ϕ_α are defined by (3.2). For $\alpha = \alpha_0$, $\phi(\cdot; \alpha_0)$ is expressed as

$$\begin{cases} \phi(x; \alpha_0) = 0 & \text{for } x = 0 \text{ and } x = 2X(\alpha_0) + b, \\ 0 < \phi(x; \alpha_0) < 1 & \text{for } x \in (0, X(\alpha_0)) \cup (X(\alpha_0) + b, 2X(\alpha_0) + b), \\ \phi(x; \alpha_0) = 1 & \text{for } x \in [X(\alpha_0), X(\alpha_0) + b], \end{cases}$$

where b is any positive number. Therefore, it is natural to understand that $Y(\cdot)$ is multi-valued for $\alpha = \alpha_0$ with

$$Y(\alpha_0) = [2C_{\lambda,p} I(1), +\infty).$$

Typical graph of $Y(\cdot)$ is depicted in Fig. 2. We will look for stationary solutions by using $Y(\cdot)$.

Before stating results, we prepare

$$\begin{aligned} E_\lambda^l &\equiv \{\phi \in E_\lambda; \phi \text{ has } l\text{-zero points in } (0,1) \text{ and } \phi_x(0) > 0\}, \\ -E_\lambda^l &\equiv \{\phi \in E_\lambda; \phi \text{ has } l\text{-zero points in } (0,1) \text{ and } \phi_x(0) < 0\} \\ &= \{-\phi; \phi \in E_\lambda^l\} \end{aligned}$$

for $l = 0, 1, 2, \cdots$, and

$$\lambda_k(a) \equiv \frac{p}{p-1}(2(k+1)I(a))^{-p}, \qquad a \in (0,1], \quad k = 0, 1, \cdots.$$

Theorem 3.1. *Let $p > q$. For each $\lambda > 0$ it holds that*

$$E_\lambda = \{0\} \cup \bigcup_{l=0}^{\infty} \{\pm E_\lambda^l\},$$

where each E_λ^l satisfies the following properties.

(a) $E^0_\lambda = \{\phi^0_\lambda\}$ *for* $\lambda > 0$.

(b) *If* $\lambda \ge \lambda_l(1)$ *for* $l = 1, 2, \cdots$, *then* $E^l_\lambda = \{\phi^l_\lambda\}$.

(c) *If* $0 < \lambda < \lambda_l(1)$ *for* $l = 1, 2, \cdots$, *then* E^l_λ *is diffeomorphic to* l*-simplex* σ^l.

In particular, for every $\lambda > 0$ *there exists a unique positive solution of* (SP).

Proof. When positive solutions are concerned, it is easily seen that for $\alpha \in (0, \alpha_0]$

$$\phi(\cdot;\alpha) \in E^0_\lambda \qquad \text{if and only if } Y(\alpha) \ni 1$$

(recall that $Y(\alpha)$ is multi-valued at $\alpha = \alpha_0$). By (i) of Lemma 3.1, there exists a unique $\alpha^* \in (0, \alpha_0]$ such that $Y(\alpha^*) \ni 1$. Here we note

$$2X(\alpha_0) = 2C_{\lambda,p}I(1) = \left\{\frac{\lambda}{\lambda_0(1)}\right\}^{1/p};$$

so that $\alpha^* < \alpha_0$ if $\lambda > \lambda_0(1)$ and $\alpha^* = \alpha_0$ if $\lambda \le \lambda_0(1)$. Thus we can show that E^0_λ consists of a single element ϕ^0_λ with $\phi^0_\lambda = \phi(\cdot;\alpha^*)$. In particular, if $\lambda \le \lambda_0(1)$, then $\phi(\cdot;\alpha_0)$ must satisfy

$$\begin{cases} \phi(x;\alpha_0) = 1 & \text{for } x \in J_0 \equiv \left[\frac{1}{2}\left\{\frac{\lambda}{\lambda_0(1)}\right\}^{1/p}, 1 - \frac{1}{2}\left\{\frac{\lambda}{\lambda_0(1)}\right\}^{1/p}\right], \\ 0 < \phi(x;\alpha_0) < 1, & \text{for } x \in (0,1) \setminus J_0. \end{cases}$$

Sign-changing stationary solutions can be constructed with use of $Y(\cdot)$ in the same way as above. When we study the structure of E^l_λ with $l \ge 1$, we make use of the following result:

$$\phi(\cdot;\alpha) \in E^l_\lambda \qquad \text{if and only if } (l+1)Y(\alpha) \ni 1.$$

Since $2(l+1)X(\alpha_0) = (\lambda/\lambda_l(1))^{1/p}$, it is easy to see that E^l_λ consists of a single element ϕ^l_λ such that $|\phi^l_\lambda(x)| < 1$ in $(0,1)$ if $\lambda > \lambda_l(1)$. If $\lambda \le \lambda_l(1)$, then $(l+1)Y(\alpha_0) \ni 1$. In this case, E^l_λ consists of all functions ϕ satisfying the following properties: there exists $(l+1)$-intervals $J_i = [a_i, b_i]$ $(i = 1, 2, \cdots, l+1)$ such that

$$\begin{cases} \phi(x) \equiv 1 \text{ (or } -1) & \text{for } x \in J_i, \\ |\phi(x)| < 1 & \text{for } x \in [0,1] \setminus \bigcup_{i=1}^{l+1} J_i \end{cases} \tag{3.5}$$

and

$$\sum_{i=1}^{l+1}(b_i - a_i) = 1 - \left\{\frac{\lambda}{\lambda_l(1)}\right\}^{1/p}.$$

Summarizing the above results we can get the conclusion. □

Remark 3.3. (i) By the method of construction, it is seen that

$$\phi^0_\lambda(x) \ge |\phi(x)| \qquad \text{for any } \phi \in E_\lambda.$$

In this sense ϕ^0_λ (resp. $-\phi^0_\lambda$) is a *maximal* (resp. *minimal*) *solution in* E_λ.

(ii) When $\lambda \leq \lambda_l(1)$, any $\phi \in E_\lambda^l$ satisfies $\phi_x(0) = \alpha_0^{1/(p-1)}$.

(iii) If $b_i > a_i$, we call J_i a *flat core* and the set $\{(x, \phi(x)) \in \mathrm{R}^2; x \in J_i\}$ a *flat hat* (see also Guedda-Veron [5], Kamin-Veron [8]). It follows from the above relation that the sum of the length of all flat cores for each $\phi \in E_\lambda^l$ is constant. So we see that E_λ^l is a continuum for $l = 1, 2, \cdots$ for $\lambda < \lambda_l(1)$. See Fig. 3.

Theorem 3.2. *Let $p = q$ and define*

$$\lambda_k = \frac{p}{p-1}(2(k+1)I_0)^{-p}, \qquad k = 0, 1, 2, \cdots.$$

Then it holds that

(i) *if $\lambda \geq \lambda_0$, then $E_\lambda = \{0\}$,*

(ii) *if $\lambda_k > \lambda \geq \lambda_{k+1}$, then $E_\lambda = \{0\} \cup \bigcup_{l=0}^{k} \{\pm E_\lambda^l\}$,*

where E_λ^l has the same properties as (a), (b) *and* (c) *in Theorem* 3.1. *In particular,* (SP) *has a unique positive solution if and only if $\lambda < \lambda_0$.*

Proof. For $p = q$, we should note

$$\inf_{\alpha \in (0, \alpha_0]} Y(\alpha) = Y(0) = \left(\frac{\lambda}{\lambda_0}\right)^{1/p},$$

so that there are no nontrivial solutions of (SP) for $\lambda \geq \lambda_0$. For $\lambda < \lambda_0$, the same reasoning as in the proof of Theorem 3.1 enables us to derive the conclusion. □

Remark 3.4. The same result as Theorem 3.2 has been obtained by Guedda-Veron [5, Theorem 2.2].

Theorem 3.3. *Let $p < q$ and let a^* be the constant given in Lemma* 3.1. *Then it holds that*

(i) *if $\lambda > \lambda_0(a^*)$, then $E_\lambda = \{0\}$,*

(ii) *if $\lambda_k(a^*) \geq \lambda > \lambda_{k+1}(a^*)$, then $E_\lambda = \{0\} \cup \bigcup_{l=0}^{k} \{\pm E_\lambda^l\}$,*

where $E_\lambda^l = \{\psi_\lambda^l\} \cup F_\lambda^l$ and F_λ^l has the following properties for each $l = 0, 1, \cdots$.

(a) *If $\lambda = \lambda_l(a^*)$, then $F_\lambda^l = \{\psi_\lambda^l\}$.*

(b) *If $\lambda_l(a^*) > \lambda \geq \lambda_l(1)$, then F_λ^l consists of a single element ϕ_λ^l satisfying $\phi_{\lambda x}^l(0) > \psi_{\lambda x}^l(0)$ and $1 > |\phi_\lambda^l(x)| > |\psi_\lambda^l(x)|$ for all $x \in (0, 1)$ except for zero points of ϕ_λ^l and ψ_λ^l.*

(c) *If $\lambda_l(1) > \lambda > 0$, then F_λ^l is diffeomorphic to σ^l and $\phi_x(0) > \psi_{\lambda x}^l(0)$ for all $\phi \in F_\lambda^l$.*

In particular, for every $\lambda < \lambda_0(a^*)$, (SP) *has exactly two positive solutions* ψ_λ^0, ϕ_λ^0 *satisfying* $\phi_{\lambda x}^0(0) > \psi_{\lambda x}^0(0)$ *and* $\phi_\lambda^0(x) > \psi_\lambda^0(x)$ *for all* $x \in (0,1)$.

Proof. The idea of the proof is almost the same as that for Theorem 3.1. It is sufficient to note that

$$\inf_{\alpha \in (0,\alpha_0]} Y(\alpha) = \left\{ \frac{\lambda}{\lambda_0(a^*)} \right\}^{1/p}$$

by (iii) of Lemma 3.1. □

Remark 3.5. For $\lambda < \lambda_0(a^*)$, we can show as in Remark 3.3 that

$$\phi_\lambda^0(x) \geq |\phi(x)| \qquad \text{for any } \phi \in E_\lambda;$$

so that ϕ_λ^0 is a maximal solution in E_λ.

As mentioned in Remark 3.4, Guedda-Veron have studied the homogeneous case $p = q$. Our results for the general case imply that bifurcations from the trivial solution are caused by the homogeneity between the diffusion term and the source term.

In Theorems 3.1–3.3 we have shown that (SP) has solutions with flat core. This is produced by the degeneracy of the p-Laplacian independently of q and r. On the other hand, solutions with *dead core* (or compact support) have been observed in different problems by Aronson-Crandall-Peletier [1] and Fujii-Ohta [4]. These phenomena are caused by the degenerate diffusion operators and are not observed in the linear diffusion case.

The appearance of flat core or dead core can be understood by the phase plane analysis. If there exists a homoclinic orbit for the origin, then we can construct stationary solutions with dead core, and if there exists an orbit which arrives at an equilibrium point on the ϕ-axis in a finite time, then we can construct stationary solutions with flat core. In our problem (our reaction term), there exist no solutions with dead core. But if our reaction term is replaced by $f(u) = u(1-u)(u-\alpha)$, $0 < \alpha < 1/2$, then we can find solutions with dead core as well as solutions with flat core.

4 Stability Problem

In this section, we study the stability of stationary solutions. Roughly speaking, Theorems 3.1–3.3 assert that for each $\lambda > 0$, E_λ consists of a discrete set of functions and some continua:

$$E_\lambda = D \cup \bigcup_{l=1}^{\infty} \{\pm G_\lambda^l\}, \tag{4.1}$$

where D is a discrete set composed of solutions of (SP) and G_λ^l is given by

$$G_\lambda^l \equiv \begin{cases} \text{empty set} & \text{if } \lambda \geq \lambda_l(1), \\ E_\lambda^l & \text{if } 0 < \lambda < \lambda_l(1) \text{ and } p \geq q, \\ F_\lambda^l & \text{if } 0 < \lambda < \lambda_l(1) \text{ and } p < q. \end{cases}$$

The following theorem, whose proof can be found in [12], implies that for each $u_0 \in L^2$, $u(t; u_0)$ (the solution of (P) with $u(0) = u_0$) converges to a stationary solution or a certain G_λ^l in a suitable sense as $t \to +\infty$.

Theorem 4.1. *For every $u_0 \in L^2$, $u(t; u_0)$ satisfies one of the following conditions:*

(i) $\|u(t; u_0) - \phi\|_\infty \to 0$ *as* $t \to +\infty$ *for a stationary solution* ϕ,

(ii)

$$\mathrm{dist}(u(t; u_0); G_\lambda^l)_\infty \equiv \inf_{\phi \in G_\lambda^l} \|u(t; u_0) - \phi\|_\infty \to 0 \qquad \text{as } t \to +\infty \tag{4.2}$$

for a continuum G_λ^l, $l = 1, 2, \cdots$.

Definition 4.1. A stationary solution ϕ is called *stable* if ϕ satisfies the following property: for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\|u_0 - \phi\|_\infty < \delta$ implies $\|u(t; u_0) - \phi\|_\infty < \varepsilon$ for all $t > 0$. A stationary solution ϕ is called *attractive* if ϕ satisfies the following property: there exists $\delta > 0$ such that $\|u_0 - \phi\|_\infty < \delta$ implies $\|u(t; u_0) - \phi\|_\infty \to 0$ as $t \to +\infty$. It is said that ϕ is *asymptotically stable* if ϕ is stable and attractive. If ϕ is not stable, then we say that ϕ is *unstable*.

Since the stability of $-\phi$ coincides with that of ϕ, we have only to discuss the stability of trivial solution $\phi \equiv 0$, positive solution ϕ_λ^0 (ψ_λ^0 and ϕ_λ^0 if $p < q$), sign-changing solutions ψ_λ^l, ϕ_λ^l, $l = 1, 2, \cdots$ and $\phi \in G_\lambda^l$, $l = 1, 2, \cdots$. Among them we recall that ϕ_λ^0 is a maximal solution when it exists (see Remarks 3.3 and 3.5).

Theorem 4.2 (trivial solution). (i) *For $p > q$, the trivial solution is unstable.*

(ii) *For $p = q$, the trivial solution is asymptotically stable for $\lambda \geq \lambda_0$ and unstable for $\lambda < \lambda_0$.*

(iii) *For $p < q$, the trivial solution is asymptotically stable.*

Theorem 4.3 (positive solutions). (i) *For every $p > 2$ and $q \geq 2$, ϕ_λ^0 is asymptotically stable for $\lambda > \lambda_0(1)$ and attractive for $\lambda \leq \lambda_0(1)$.*

(ii) *For $p < q$, ψ_λ^0 is unstable for $\lambda \leq \lambda_0(a^*)$.*

The stability of sign-changing stationary solutions without flat hat is as follows:

Theorem 4.4. *Any of ψ_λ^l, ϕ_λ^l, $l = 1, 2, \cdots$ is unstable.*

Theorems 4.2–4.4 are shown by Theorem 4.1 and the comparison theorem (Theorem 2.3) with suitable functions which can be constructed with the aid of the time-map. For the detail of proofs, see [12].

We will discuss the stability of solutions in G_λ^l, $l \geq 1$, i.e., the sign-changing stationary solutions with flat hat. Recall that G_λ^l is diffeomorphic to l-simplex σ^l. By a correspondence

$$G_\lambda^l \ni \phi \longmapsto (b_1 - a_1, b_2 - a_2, \cdots, b_l - a_l) \in \sigma^l \subset \mathrm{R}^l,$$

where the a_i, b_i are defined in the proof of Theorem 3.1 (see also Fig. 3), we will identify G_λ^l with σ^l hereafter. Under this identification, '$\phi \in \partial\sigma^l$' implies that ϕ is a function in G_λ^l with adjoining layers and '$\phi \in \mathrm{Int}\,\sigma^l$' implies that ϕ is a function in G_λ^l whose layers are separated, where $\partial\sigma^l$ and $\mathrm{Int}\,\sigma^l$ are the boundary of σ^l and the interior of σ^l, respectively.

Theorem 4.5. *If $\phi \in \partial\sigma^l$, then ϕ is unstable.*

Proof. Since the proof of the other cases is essentially the same, we consider only the case $l = 2$, in particular, only $\phi^2 \in \partial\sigma^2$ satisfying (3.5) with $a_i < b_i$ ($i = 1, 3$) and $a_2 = b_2$ (see also Fig. 3). For ϕ^2, we can construct a lower solution $\phi^{2,\varepsilon} \in \partial\sigma^2$ of (P) satisfying (3.5) with $J_1 = [a_1, b_1 + \varepsilon]$, $J_3 = [a_3 + \varepsilon, b_3]$ and $J_2 = [a_2 + \varepsilon, b_2 + \varepsilon] = \{a_2 + \varepsilon\}$. Define $u_0(x) \equiv \max\{\phi^2(x), \phi^{2,\varepsilon}(x)\}$. We can make $\|u_0 - \phi^2\|_\infty$ as small as possible by taking a sufficiently small ε. Since ϕ^0_λ is a unique $\phi \in E_\lambda$ such that $\phi > \phi^{2,\varepsilon}$, we conclude $u(t; u_0) \to \phi^0_\lambda$ as $t \to +\infty$. This completes the proof. □

Concerning the stability of $\phi \in \operatorname{Int}\sigma^l$, we will see a related result in Section 5. We should note that the above proof does not hold for $\phi \in \operatorname{Int}\sigma^l$ since if $a_i < b_i$ ($i = 1, 2, 3$), then the above u_0 becomes a stationary solution of (P).

5 Remarks and Open Problems

I. ω-limit set. It is a very interesting problem to study whether $\omega(u_0)$ consists of a single element or not for every $u_0 \in L^2$. If E_λ is discrete, then $\omega(u_0)$ is a singleton since $\omega(u_0)$ is connected and included in E_λ (Theorem 2.2). However, in our problem, E_λ consists of a discrete set of functions and some continua, which makes analysis difficult.

II. Stability of $\phi \in$ Int σ^l. The stability of solutions whose layers are separated has been considered in different problems by Chen-Elliott [3] and Ito [7]. Chen-Elliott have shown in a double obstacle problem with Neumann boundary conditions that there are solutions with flat hat and that the solutions whose layers are separated are stable. They have proved, by comparing with a lower solution attaining the flat hat of stationary solution in a finite time, that the non-stationary solution touches there in a finite time. This fact plays an essential role to prove that these solutions are stable. Ito has applied their idea to Allen-Cahn equation for the p-Laplacian with constraints and has obtained the same result as theirs. The flat hat in studies of these authors is produced by constraints. On the other hand, in (P), the equation itself produces the flat hat of stationary solutions by the degeneracy of diffusion term.

Recently, the author obtained an result, which will throw light on the stability property of $\phi \in \operatorname{Int}\sigma^l$. Roughly speaking, the solutions of (P) do *not* touch the flat hat of stationary solutions in a finite time.

Proposition 5.1. *Let $\phi \in E_\lambda$ be a stationary solution with flat hat and suppose that $u_0 \in W^{1,p}_0$ satisfies one of the following conditions:*

(i) $\|u_0\|_\infty < 1$,

(ii) $u_0(x) > \phi(x)$ *or* $u_0(x) < \phi(x)$ *for all* $x \in (0, 1)$.

Then for any $t \in [0, +\infty)$,

$$u(x, t; u_0) \neq \phi(x), \qquad x \in \operatorname{Int}\operatorname{Flat}(\phi),$$

where Flat(ϕ) *is the flat core of* ϕ.

III. Forming flat hat. Proposition 5.1 gives us other problems. For general $u_0 \in L^2$, whether the solution $u(t; u_0)$ has flat hat in a finite time or not? Moreover, for u_0 with flat hat, whether $u(t; u_0)$ has flat hat for all $t \in [0 + \infty)$ or not? They will lead us to a free boundary problem for the flat core.

IV. Porous medium equation. Consider the following stationary problem associated with the porous medium equation:

$$\begin{cases} \lambda(|\phi|^{m-1}\phi)_{xx} + f(\phi) = 0, & x \in (0,1), \\ \phi(0) = \phi(1) = 0, \end{cases} \tag{5.1}$$

where $f(\phi) = |\phi|^{q-2}\phi(1 - |\phi|^r)$ and $m > 1,\ q \geq 2,\ r > 0$. Setting $\psi = |\phi|^{m-1}\phi$, we have

$$\begin{cases} \lambda\psi_{xx} + g(\psi) = 0, & x \in (0,1), \\ \psi(0) = \psi(1) = 0, \end{cases} \tag{5.2}$$

where $g(\psi) = f(|\psi|^{1/m-1}\psi)$. Carrying out the same analysis as Section 3, we can show the following lemma for (5.2), which is slightly different from Lemma 3.1. Define

$$I(a) = \int_0^a (G(a) - G(\psi))^{-1/2} d\psi \qquad \text{for } a \in (0,1],$$

where $G(\psi) = \int_0^\psi g(s)ds$, then

Lemma 5.1. *For any $m > 1$ and $q \geq 2$, $I(\cdot)$ is continuous in $(0,1)$. In particular, $\lim_{a \to 1-0} I(a) = +\infty$. Furthermore, $I(\cdot)$ has the following properties.*

(i) *For $m + 1 > q$, $I(\cdot)$ is strictly monotone increasing and*

$$\lim_{a \to 0+} I(a) = 0.$$

(ii) *For $m + 1 = q$, $I(\cdot)$ is strictly monotone increasing and*

$$\lim_{a \to 0+} I(a) = \sqrt{2}\int_0^1 (1 - t^2)^{-1/2} dt = \frac{\pi}{\sqrt{2}}.$$

(iii) *For $m + 1 < q$, there exists $a^* \in (0,1)$ such that $I(\cdot)$ is strictly monotone decreasing in $(0, a^*)$ and strictly monotone increasing in $(a^*, 1)$. Moreover, $I(\cdot)$ satisfies*

$$\lim_{a \to 0+} I(a) = +\infty.$$

From this lemma we can see that all solutions of (5.1) have neither flat core nor dead core, and that the set of all solutions for (5.1) is at most countable set for every $\lambda > 0$. This is a different phenomenon from Aronson-Crandall-Peletier [1], in which the solutions with dead core appear under the condition $f(\phi) = \phi(1 - \phi)(\phi - \alpha)$. (See also the last paragraph in Section 3.)

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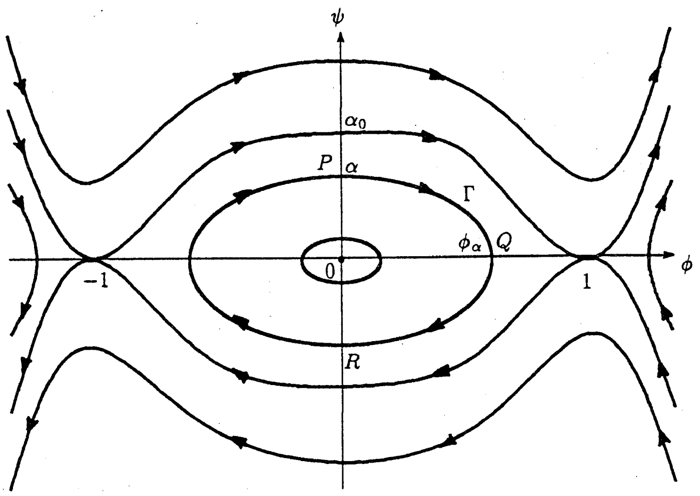
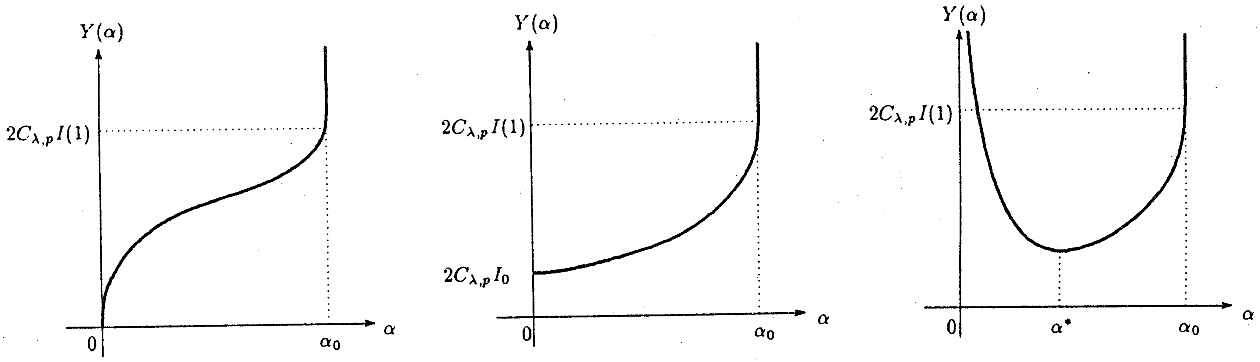


Fig. 1. $\phi\psi$-phase plane with $\psi = |\phi_x|^{p-2}\phi_x$.



(i) $p > q$ (ii) $p = q$ (iii) $p < q$

Fig. 2. Graphs of $Y(\cdot)$.

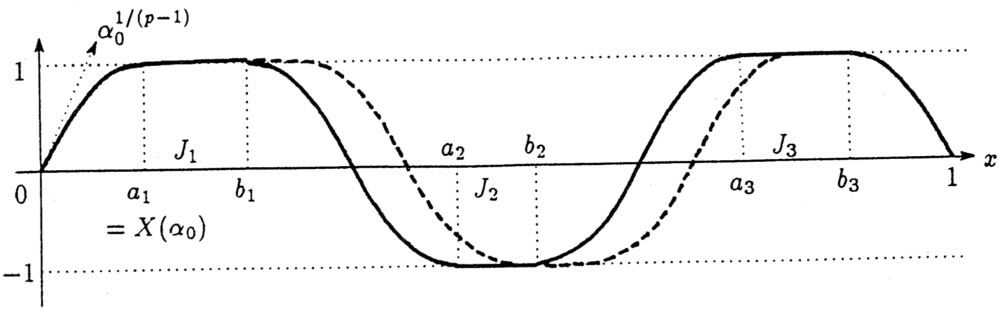


Fig. 3. A typical function in E_λ^2, which consists of all stationary solutions with flat core satisfying $\sum_{i=1}^{3}(b_i - a_i) = 1 - 6X(\alpha_0)$.

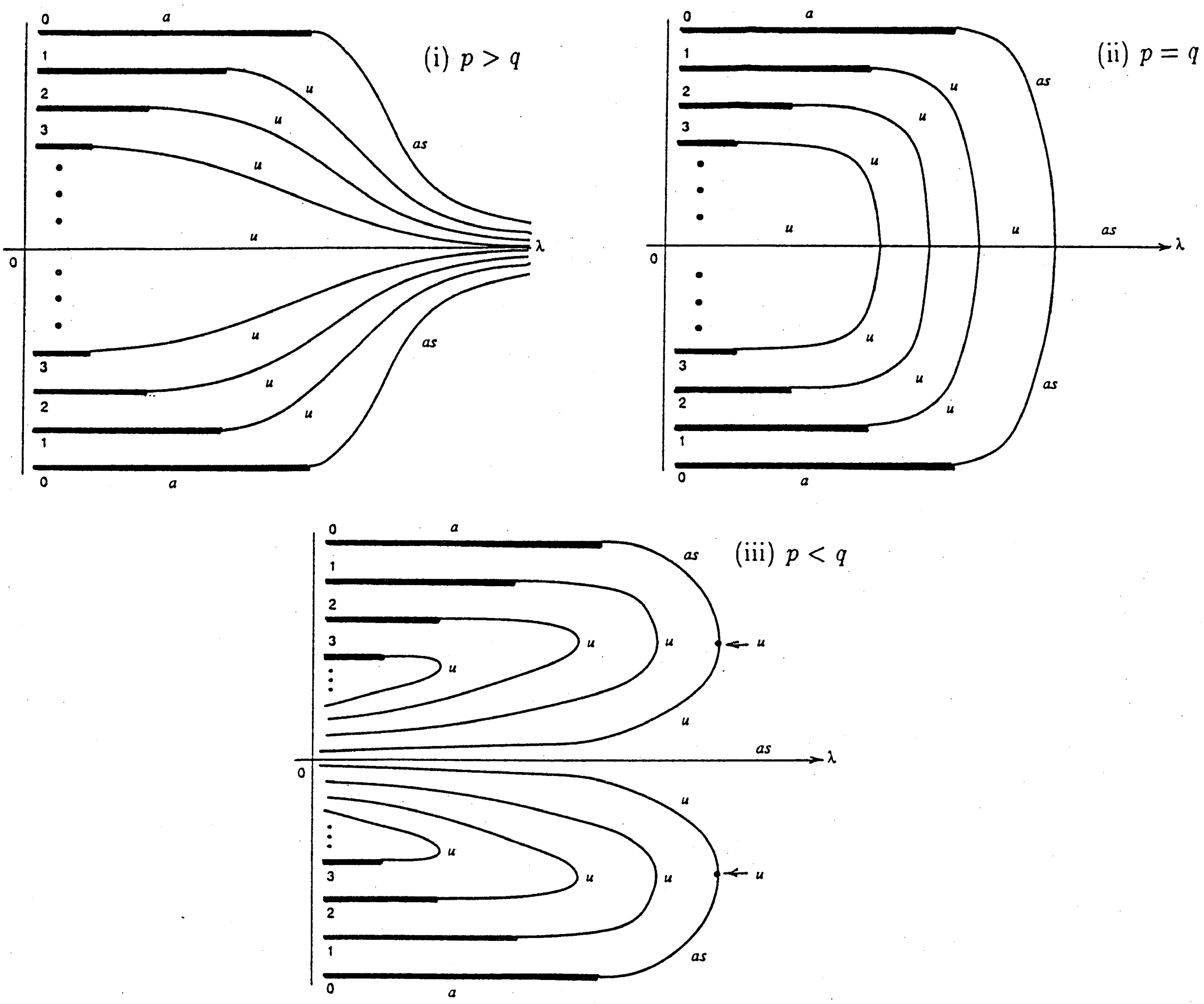


Fig. 4. The bifurcation diagram and the stability of corresponding stationary solutions (the number for each branch represents the number of zeros in $(0,1)$ of the corresponding stationary solution; each thick branch corresponds to E_λ^l, where l is equal to the number for branch; the indices a, s and u represent attractive, stable and unstable, respectively.)