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# FINITE AND INFINITE TIME BLOWUP OF SOLUTIONS TO SOME SEMILINEAR PARABOLIC EQUATIONS

池島良
Ryo IKEHATA
Department of Mathematics, Faculty of School Education,
Hiroshima University

## 1 Introduction

In this talk I would like to introduce the recent work with Takashi Suzuki(Osaka Univ., Graduste School of Science) concerning parabolic equations; results, methods, and motivations.

Consider the following mixed problem:

$$u_t(t,x)-\Delta u(t,x)=|u(t,x)|^{p-1}u(t,x), \quad (t,x)\in(0,T)\times\Omega \tag{1}$$

$$u(0,x)=u_0(x), \quad x\in\Omega \tag{2}$$

$$u|_{\partial\Omega}=0, \quad t\in(0,T), \tag{3}$$

where $\Omega\subset R^N$ is a smooth bounded domain. It seems that in the framework of potential -well method we have little research concerning the behavior of solutions to (1)-(3) with the critical Sobolev exponent such as

$$p=\frac{N+2}{N-2}(N\geq 3). \tag{4}$$

For the other related results to the problem (1)-(3) with subcritical $p\in(1,\frac{N+2}{N-2})$, there are a few works of Ishii[8], Ôtani[12] , Payne-Sattinger[13], and Ikehata-Suzuki[6].

To begin with, we will discuss the behavior of solutions to (1)-(3) with (4) based on the following local existence theorem due to Hoshino-Yamada[5].

**Proposition 1** *Suppose (4). For each $u_0\in H_0^1(\Omega)$, there exists a positive number $T_m>0$ such that the problem (1)-(3) has a unique solution $u\in C([0,T_m);H_0^1(\Omega))$ which becomes classical on $(0,T_m)$ and if $T_m<+\infty$, then $\lim_{t\uparrow T_m}\|u(t,\cdot)\|_\infty=+\infty$.*

**Remark 1** *Generally speaking, it is difficult to show even $\limsup_{t\uparrow T_m}\|\nabla u(t,\cdot)\|_2=+\infty$ under the assumption $T_m<+\infty$. This is because the local existence time $T$ can not be estimated uniformly for the bounded $v\in H_0^1(\Omega)$ in use of a contraction mapping principle (see [5]).*

Set

$$J(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1},$$
$$I(u) = \|\nabla u\|_2^2 - \|u\|_{p+1}^{p+1},$$

and

$$d = \inf\{\sup_{\lambda \geq 0} J(\lambda u) | u \in H_0^1(\Omega) \setminus \{0\}\}.$$

It is well-known that $d > 0$ because of the Sobolev imbedding $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$. Furthermore, stable and unstable sets are defined as follows:

$$W = \{u \in H_0^1(\Omega) | J(u) < d, I(u) > 0\} \cup \{0\}$$

and

$$V = \{u \in H_0^1(\Omega) | J(u) < d, I(u) < 0\},$$

respectively (see [13]). We impose the following several assumptions:

(A.1) $u_0(x) \geq 0$(a.e.).

(A.2) $\Omega$ is star-shaped.

(A.3) $\Omega$ is convex.

(A.4) $u_t(t_0, x) > 0$ for some $t_0 \in (0, T_m)$, all $x \in \Omega$.

(A.5) $\Omega = \{x \in R^N | \ |x| < 1\}$.

(A.6) $u(t,x)$ is radial and radially decreasing with respect to $x \in \Omega (t > 0)$.

Then, our main results read as follows.

**Theorem 1** *Suppose (A.1) and (A.3). Let $u(t,x)$ be a local solution on $[0, T_m)$ as in Proposition 1. If (A.4) is further assumed, then $T_m < +\infty$ and we have $u(t_0, \cdot) \in V$ for some $t_0 \in [0, T_m)$. Moreover, in this case it holds that*

$$J(u(t,\cdot)) = O(\log(T_m - t)) \quad (t \uparrow T_m)$$

*so that*

$$\lim_{t \uparrow T_m} \|\nabla u(t,\cdot)\|_2 = +\infty.$$

In Theorem 1, in the case when $p \in (1, \frac{N+2}{N-2})$ Giga[3] has already proved that $\lim_{t \uparrow T_m} J(u(t,\cdot)) = -\infty$ provided that $T_m < +\infty$. However, his results fully depend on its subcriticalness of $p$, and so we can not apply it to our problem. By condition (A.4), we shall rely on the theory of Friedman-McLeod[2] together with [4]. Furthermore, nobody has ever obtained the blowup rate of energy $J(u(t,\cdot))$ as $t \uparrow T_m$.

Next, we shall investigate the behavior of a global solution $u(t,x)$ as $t \to +\infty$. In the following paragraph, we always assume $T_m = +\infty$ in Proposition 1. Let

$$C_0 = \frac{2(p+1)}{p-1} \lim_{t \to +\infty} J(u(t,\cdot)).$$

It is easy to check $C_0 \geq 0$ by using Theorems stated after (see section 2). $C_0$ means the least energy level which global solution attains. Then, we have

**Theorem 2** *Assume (A.2) and $T_m = +\infty$ in Proposition 1. Then the following are equivalent to each other.*

$$(1) C_0 > 0. \qquad (2) u(t,\cdot) \notin (W \cup V) \ on [0,+\infty).$$
$$(3) J(u(t,\cdot)) \geq d \ on [0,+\infty). \qquad (4) \lim_{t\to+\infty} \|u(t,\cdot)\|_\infty = +\infty.$$

*In these cases, we have*

$$C_0 \geq \frac{2(p+1)}{p-1} > 0.$$

As a result, even in the case when $\Omega$ is not necessarily star-shaped we have

**Corollary 1** *Suppose $T_m = +\infty$. Then, $C_0 = 0$ if and only if $\lim_{t\to+\infty} \|u(t,\cdot)\|_\infty = +\infty$.*

Under the assumption $T_m = +\infty$, we find that the asymptotic behavior of a global solution $u(t,x)$ can be classified into the following two types:

(1) $\lim_{t\to+\infty} \|u(t,\cdot)\|_\infty = 0 = \lim_{t\to+\infty} \|\nabla u(t,\cdot)\|_2$ if $u(t_0,\cdot) \in W$ for some $t_0 \geq 0$.

(2) $0 < \liminf_{t\to+\infty} \|\nabla u(t,\cdot)\|_2 < +\infty$ if $u(t,\cdot) \notin (W \cup V)$ for all $t \geq 0$.

**Remark 2** *In an above argument, it is still open to show that $\limsup_{t\to+\infty} \|\nabla u(t,\cdot)\|_2 < +\infty$ holds true as shown by Ôtani in the subcritical $p \in (1, \frac{N+2}{N-2})$. Furthermore, we do not necessarily have to impose the conditions (A.1) or (A.2).*

By restricting our assumption to the framework of radial symmetry, we further meet the detailed properties of global solution which never intersect $W$ nor $V$. We have

**Theorem 3** *Assume (A.1), (A.5) and (A.6). If $T_m = +\infty$ in Proposition 1, then there exists a sequence $\{t_j\}$ satisfying $t_j \to +\infty$ as $j \to \infty$ such that*

$$|\nabla u(t_j,x)|^2 dx \rightharpoonup C_0\delta (j \to +\infty),$$
$$u(t_j,x)^{p+1} dx \rightharpoonup C_0\delta (j \to +\infty),$$

*in the sense of measure, where $\delta$ means the usual Dirac measure having an unit mass at the origin.*

**Corollary 2** *Under the same assumptions as in Theorem 3, it holds that*

$$(\frac{1}{2}|\nabla u(t,x)|^2 dx - \frac{1}{p+1} u(t,x)^{p+1}) dx \rightharpoonup C_0\delta \quad (t \to +\infty).$$

In the case when $\Omega$ is star-shaped, note that (from the arguments by Lacey-Tzanetiz) the behavior of solutions to the problem (1)-(3) can be classified into the following three types:

(1) $T_m = +\infty$ and $\lim_{t\to+\infty} \|u(t,\cdot)\|_\infty = 0$.

(2) $T_m = +\infty$ and $\lim_{t\to+\infty} \|u(t,\cdot)\|_\infty = +\infty$.

(3) $T_m < +\infty$ and $\lim_{t\uparrow T_m} \|u(t,\cdot)\|_\infty = +\infty$.

For the proof of these Theorems, we refer the reader to Ikehata-Suzuki. Instaed, in the next section 2, we introduce the several Propositions and Theorems which will be used later in the proof of Theorems.

## 2 Structure of $H_0^1(\Omega)$

First, we define totality of stationary solutions by $E$:

$$E = \{u \in H_0^1(\Omega) |\ u \text{ satisfies } (5)\},$$

where

$$-\Delta u = |u|^{p-1}u \quad in\ \Omega, \quad u = 0\ on\ \partial\Omega. \tag{5}$$

Due to Trudinger[17], each $H^1$-solution of (5) is a classical one. It is known that $E = \{0\}$ if $p = \frac{N+2}{N-2}$ and $\Omega$ is star-shaped (c.f., Pohozaev[14]). From this fact we can prove the following.

**Proposition 2** *Suppose that* $p = \frac{N+2}{N-2}$ *and* $\Omega$ *is star-shaped. Then,*

*(1)* $W$ *is a bounded neighbourhood of* $0$ *in* $H_0^1(\Omega)$.

*(2)* $\bar{W} \cap \bar{V} = \phi$.

*(3)* $0 \notin \bar{V}$.

*Here and henceforth,* $\bar{U}$ *means the closure of* $U$ *in the strong* $H_0^1(\Omega)$*-topology.*

**Proposition 3** *Suppose that* $p \in (1, \frac{N+2}{N-2})$. *Then,*

*(1)* $W$ *is also a bounded neighbourhood of* $0$ *in* $H_0^1(\Omega)$.

*(2)* $\bar{W} \cap \bar{V} = E_0 \neq \phi$.

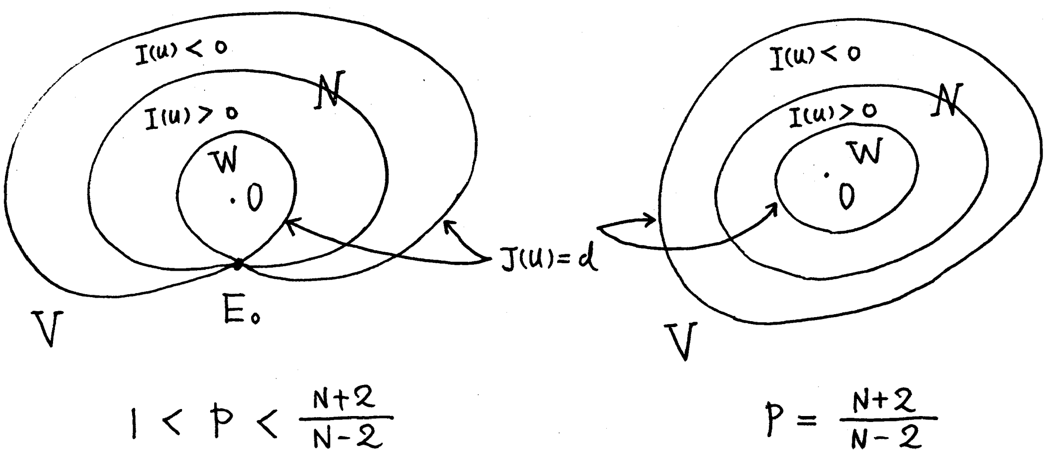
*(3)* $E \setminus \{0\} \neq \phi$.

*Here,* $E_0 = \{u \in E | J(u) = d\}$ *(*$E_0$ *represents the set of least energy steady solutions).*

Set

$$N = \{u \in H_0^1(\Omega) | I(u) = 0, u \neq 0\}.$$

It is known that $E_0 \subset N$, $E \setminus \{0\} \subset N$, in general. Then, the image of phase space $H_0^1(\Omega)$ by an energy contour line $J(u) = \alpha (\alpha \in R)$ is as follows:




The essential difference between critical and subcritical case is in the fact that $\bar{W} \cap \bar{V} = \phi$ or $\neq \phi$. In this connection, we have already proved the following Proposition (see [6]):

**Proposition 4** *Suppose* $p \in (1, \frac{N+2}{N-2})$ *and (A.3), and* $u(t,x)$ *be a local solution on* $[0, T_m)$ *as in Proposition 1. Then the following are equivalent to each other.*

*(1)* $u(t,\cdot) \notin (W \cup V)$ *on* $[0, T_m)$.

*(2)* $J(u(t,\cdot)) \geq d$ *on* $[0, T_m)$.

*(3)* $T_m = +\infty$, $0 \notin \omega(u_0) \subset E \setminus \{0\} \subset N$.

*Here,* $\omega(u_0)$ *is an usual* $\omega$*-limit set in* $H_0^1(\Omega)$ *with respect to the initial data* $u_0$.

Note that the statement of Proposition 1 is also right even in the subcritical case.

Next, in order to understand the role of such stable and unstable sets we give two (comparatively) simple but important Theorems as follows:

**Theorem 4** *Let* $p = \frac{N+2}{N-2}$ *and let* $u(t,x)$ *be a local solution on* $[0, T_m)$ *as in Proposition 1. Assume* $T_m = +\infty$. *Then, there exists a time* $t_0 \in [0, +\infty)$ *such that* $u(t_0, \cdot) \in W$ *if and only if* $u(t,\cdot) \to 0$ *(as* $t \to +\infty$*) in* $H_0^1(\Omega)$. *Furthermore, in this case it holds that*

$$\|\nabla u(t,\cdot)\|_2 = O(e^{-\alpha t}) \quad (as \quad t \to +\infty)$$

*with some constant* $\alpha > 0$.

**Theorem 5** *Let* $p = \frac{N+2}{N-2}$ *and let* $u(t,x)$ *be a local solution on* $[0, T_m)$ *as in Proposition 1. Then, if there exists a time* $t_0 \in [0, T_m)$ *such that* $u(t_0, \cdot) \in V$, *it holds that* $T_m < +\infty$ *so that* $\lim_{t \uparrow T_m} \|u(t,\cdot)\|_\infty = +\infty$.

**Remark 3** *In Theorem 4, in the case of subcritical p, of course we can derive* $T_m = +\infty$ *from the assumption* $u(t_0,\cdot) \in W$. *However, in the case of critical p it is still open to show such statements (see Remark 1). Exponential decay estimates will be proven by the Komornik[9] method. On the other hand, proof of Theorem 5 essentially is derived from [12]. Furthermore, in some sense, Theorem 1 prescribe a class of initial orbits for which the assumption in Theorem 5 holds good.*

# 3 Outline of proof: Theorem 2

In this section, we shall describe an outline of proof of Theorem 2. Since other equivalence is easy to prove, we state the equivalence of (1) and (4) by restricting ourselves only to the case $N = 3$. In this connection, we need the following two lemmas.

**Lemma 1** *Assume (A.2). If* $T_m = +\infty$ *in Proposition 1, then there exists a sequence* $\{t_j\}$ *satisfying* $t_j \to +\infty$ *(as* $j \to +\infty$*) such that*

$$\lim_{j \to +\infty} \|\nabla u(t_j, \cdot)\|_2^2 = \lim_{j \to +\infty} \|u(t_j, \cdot)\|_{p+1}^{p+1} = C_0$$

The following lemma plays a key role in our argument.

**Lemma 2** *Assume (A.2) and* $T_m = +\infty$ *in Proposition 1. If* $C_0 = 0$, *then*

$$\int_T^{+\infty} \|\Delta u(t,\cdot)\|_2^2 dt < +\infty$$

*for some* $T > 0$.

*Proof.* Let $A = -\Delta$ in $L^2(\Omega)$ with the Dirichlet null condition. By the Sobolev imbedding $D(A^\beta) \hookrightarrow L^{2p}(\Omega)$ with $\beta = \frac{N}{N+2}$ and the moment inequality (see Tanabe[16])

$$\|A^\beta u\|_2 \le C\|Au\|_2^{2\beta-1}\|A^{\frac{1}{2}}u\|_2^{2-2\beta}$$

we have

$$\|u^p\|_2 \le C\|\Delta u\|_2\|\nabla u\|_2^{\frac{4}{N-2}} \tag{6}$$

for $u \in H^2(\Omega)$. If $C_0 = 0$, then Lemma 1 and (1) of Proposition 2 imply $u(t_j, \cdot) \in W$ for some $j$ and from Theorem 4 we have

$$\|\nabla u(t,\cdot)\|_2 = O(e^{-\alpha t}) \quad (t \to +\infty). \tag{7}$$

On the other hand, by (1) and (6) we have

$$\|\Delta u(t,\cdot)\|_2 \le \|u_t(t,\cdot)\|_2 + C\|\Delta u(t,\cdot)\|_2\|\nabla u(t,\cdot)\|_2^{\frac{4}{N-2}}. \tag{8}$$

Now, by (7) there exists a real number $T > 0$ such that

$$C\|\nabla u(t,\cdot)\|_2^{\frac{4}{N-2}} < \frac{1}{2} \quad for \quad t \ge T. \tag{9}$$

(8) and (9) imply

$$\frac{1}{2}\|\Delta u(t,\cdot)\|_2 \le \|u_t(t,\cdot)\|_2 \quad for \quad t \ge T. \tag{10}$$

Therefore, we have

$$\int_T^\infty \|\Delta u(t,\cdot)\|_2^2 dt \le C\int_0^\infty \|u_t(t,\cdot)\|_2^2 dt = C(J(u_0) - J(u(t,\cdot))) \le C(J(u_0)).$$

This means the desired inequality. ∎

*Proof of Theorem 2 for the case $N = 3$:* The part (1) $\Rightarrow$ (4) is standard, and so we shall only to prove the part (4) $\Rightarrow$ (1). Indeed, suppose that (1) does not hold true under the assumption (4). Then, it follows from Lemma 2 that

$$\liminf_{t\to+\infty} \|\Delta u(t,\cdot)\|_2 = 0.$$

Since $N = 3$ (for the other dimension $N \ge 4$, see [7]), by the Sobolev imbedding Theorem $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$ we have

$$\liminf_{t\to+\infty} \|u(t,\cdot)\|_\infty = 0.$$

This contradicts the assumption (4). ∎

# 4 Outline of proof: Theorem 3

In this section, we shall give an outline of proof of Theorem 3. First, we prepare the following two lemmas which are used after.

**Lemma 3** *Let $p = \frac{N+2}{N-2}$ and assume (A.5). Then, there is no radially symmetric solutions $v \in C^2(\bar{\Omega} \setminus \{0\})$ to the problem:*

$$-\Delta v(x) = v(x)^p, \quad in \quad \Omega \setminus \{0\}, \tag{11}$$

$$v(x) > 0 \quad in \quad \Omega \setminus \{0\}, \tag{12}$$

$$v|_{\partial\Omega} = 0. \tag{13}$$

The following lemma is more or less known.

**Lemma 4** *Suppose (A.1), (A.5), and (A.6). If $T_m = +\infty$ in Proposition 1, then, for all $K \subset\subset \bar{B} \setminus \{0\}$, there exists a constant $C_K > 0$ such that*

$$\|u(t, \cdot)\|_{L^\infty(K)} \leq C_K$$

*for all $t \gg 1$, where $B = B_1(0)$.*

*Proof of Theorem 3.* For an arbitrarily fixed sequence $\{t_n\}$ satisfying $t_n \to +\infty$ as $n \to \infty$, it follows from Lemma 4 with $t = t_n$ and the parabolic regularity that

$$u(t_{n_j}, \cdot) \to v \quad as \quad j \to \infty,$$

locally uniformly in $\bar{B} \setminus \{0\}$, where $\{t_{n_j}\}$ is some subsequence of $\{t_n\}$. Since $v \in C^2(\bar{B} \setminus \{0\})$ satisfies (11)-(13) and is radially symmetric, it should hold that $v \equiv 0$ in $\bar{B} \setminus \{0\}$, where we have just used the standard strong minimum principle and Lemma 3. From these arguments and the parabolic regularity, we have

$$u(t, \cdot) \to 0 \quad as \quad t \to \infty,$$

$$|\nabla u(t, \cdot)|^2 \to 0 \quad as \quad t \to \infty,$$

where the convergence is locally uniform in $\bar{B} \setminus \{0\}$, respectively. By combining this with Lemma 1, we have the desired conclusion of Theorem 3. ∎

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