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EXISTENCE OF PERIODIC SOLUTIONS FOR NONLINEAR EVOLUTION EQUATIONS IN BANACH SPACES

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1. Introduction

Let X be a Banach space, let A be an m-accretive subset of $X \times X$, let $f : \mathbb{R} \times \overline{D(A)} \to X$ be a Carathéodory mapping which is T-periodic in its first variable, and let $h \in L^1(0,T;X)$. In this paper, we study the existence of T-periodic solutions to a class of a nonlinear evolution equations of the form

$$u'(t) + Au(t) \ni f(t, u(t)) + h(t) \quad \text{for } t \in \mathbb{R}. \tag{1.1}$$

This problem has been studied by many authors; cf. [1, 3, 5, 12, 14, 15, 19, 20]. In the case when A is the subdifferential of a proper, lower semicontinuous convex function on a Hilbert space, Ôtani [14] obtained a nice result. Vrabie [20] considered the case that A is a fully nonlinear operator. He considered the case that X is a Banach space, A is an m-accretive operator and $f : \mathbb{R} \times \overline{D(A)} \to X$ is a Carathéodory mapping such that $\overline{D(A)}$ is convex, $-A$ generates a compact semigroup, f is T-periodic in its first variable and there exists $a > 0$ such that $A - aI$ is m-accretive, and

$$\lim_{r\to\infty} \frac{1}{r} \sup\{\|f(t,v)\| : t \in \mathbb{R}, v \in \overline{D(A)}, \|v\| \le r\} < a,$$

and he showed that (1.1) has a T-periodic, integral solution in the case of $h = 0$. Caşcaval and Vrabie [5] partially extended his result to the case that X is a Hilbert space, $-A$ generates a compact semigroup and $f : \mathbb{R} \times \overline{D(A)} \to X$ is a continuous mapping such that f is T-periodic in its first variable and bounded on every bounded subset in $\mathbb{R} \times \overline{D(A)}$, and there exists $r > 0$ such that $B_r \cap D(A)$ is nonempty and

$$\langle y - f(t,x), x\rangle \ge 0 \quad \text{for every } (x,y) \in A \text{ with } \|x\| = r \text{ and } t \in [0,T],$$

and they showed that (1.1) has a T-periodic, strong solution in the case of $h = 0$.

The objects of this paper are to obtain a generalization of Caşcaval and Vrabie's result by relaxing the conditions that X is a Hilbert space and f is a continuous mapping, and an existence result on the T-periodic problem (1.1) for every $h \in L^1(0,T;X)$ in the case when X is a Banach space and f is a Carathéodory mapping. The idea is inspired by [10] in which Górniewicz and Plaskacz studied the existence of periodic solution of an ordinary differential equation. Our results are the following:

Theorem 1. *Let X be a separable Banach space and let A be an m-accretive subset of $X \times X$ such that $\overline{D(A)}$ is convex and $-A$ generates a compact semigroup. Let $T > 0$ and let f be a Carathéodory mapping from $[0,T] \times \overline{D(A)}$ into X. Assume that there exist $r > 0$ and $\varepsilon > 0$ such that $B_r \cap \overline{D(A)}$ is nonempty,*

$$\int_0^T \sup_{x \in \overline{D(A)} \cap B_{r+\varepsilon}} \|f(t,x)\|\, dt < \infty,$$

and for every $(x,y) \in A$ *with* $r-\varepsilon \le \|x\| \le r+\varepsilon$, *there exists* $z \in Jx$ *such that*

$$\langle y - f(t,x), z\rangle \ge 0 \quad \text{for almost every } t \in (0,T),$$

where $B_r = \{u \in X : \|u\| \le r\}$ *and* J *is the duality mapping from* X *into its topological dual. Then there exists at least one* T*-periodic, integral solution of*

$$u'(t) + Au(t) \ni f(t,u(t)) \quad \text{for } 0 \le t \le T.$$

Theorem 2. *Let* X, A, T *and* f *be as in Theorem 1. Assume that for every* $\rho > 0$ *there exists* $a_\rho \in L^1(0,T)$ *such that* $\|f(t,x)\| \le a_\rho(t)$ *for almost every* $t \in [0,T]$ *and for every* $x \in \overline{D(A)}$ *with* $\|x\| \le \rho$. *Assume also that there exist* $r > 0$, $c > 0$ *and* $b \in L^1(0,T)$ *such that for every* $(x,y) \in A$ *with* $\|x\| \ge r$, *there exists* $z \in Jx$ *such that*

$$\langle y - f(t,x), z\rangle \ge c\|x\|^2 - b(t)\|x\| \quad \text{for almost every } t \in (0,T).$$

Then for every $h \in L^1(0,T;X)$, *there exists at least one* T*-periodic, integral solution* u *of*

$$u'(t) + Au(t) \ni f(t,u(t)) + h(t) \quad \text{for } 0 \le t \le T.$$

2. Preliminaries

Throughout this paper, all vector spaces are real, we denote by $\mathbb{N}$ and $\mathbb{R}$, the set of all positive integers and the set of all real numbers, respectively, and by homology, we understand the Čech homology with rational coefficients; see [8, 9].

Let Y and Z be topological spaces. Let $\mathcal{T}$ be a subset of $Y \times Z$. We identify the set $\mathcal{T}$ with a multivalued mapping $\mathcal{T}$ from Y into Z by $\mathcal{T}y = \{z \in Z : (y,z) \in \mathcal{T}\}$ for every $y \in Y$. We denote by $D(\mathcal{T})$ and $R(\mathcal{T})$, the sets $\{y \in Y : \mathcal{T}y \ne \emptyset\}$ and $\bigcup\{\mathcal{T}y : y \in D(\mathcal{T})\}$, respectively. We say that $\mathcal{T}$ is upper semicontinuous if for every $y_0 \in Y$ and open set V in Z with $\mathcal{T}y_0 \subset V$, there exists a open neighborhood U of y_0 such that $\mathcal{T}y \subset V$ for every $y \in U$.

The following fixed point theorem was obtained in [9, 17]. Since [17] is written in Japanese, we give the proof in Appendix.

Proposition 1 (Górniewicz, Shioji). *Let* Y *be a convex subset of a locally convex, Hausdorff topological vector space* E *and let* K *be a compact subset of* Y. *Let* $\mathcal{T}$ *be an upper semicontinuous mapping from* Y *into* K *such that for every* $y \in Y$, $\mathcal{T}y$ *is a nonempty, acyclic, compact subset of* K. *Then there is an element* y *of* Y *such that* $y \in \mathcal{T}y$.

Let X be a Banach space, let D be a subset of X and let $r > 0$. We denote by $\overline{D}$, the closure of D and we denote by B_r, the closed ball in X with center 0 and radius r. Let X^* be the topological dual of X. The value of $x^* \in X^*$ at $x \in X$ will be denoted by $\langle x, x^*\rangle$. Let J be the multivalued mapping from X into X^* defined by $Jx = \{x^* \in X^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}$ for every $x \in X$. We call J the duality mapping from X into X^*. For every $(x,y) \in X \times X$, we define

$$[x,y]_+ = \lim_{t \downarrow 0} \frac{\|x+ty\| - \|x\|}{t}.$$

We know that $(x,y) \mapsto [x,y]_+$ is an upper semicontinuous function from $X \times X$ into $\mathbb{R}$. We say a subset $A \subset X \times X$ is accretive if $[x_1 - x_2, y_1 - y_2]_+ \ge 0$ for every $(x_1,y_1),(x_2,y_2) \in A$. We know that A is accretive if and only if for every $(x_1,y_1),(x_2,y_2) \in A$, there exists $w^* \in J(x_1 - x_2)$ such that $\langle y_1 - y_2, w^*\rangle \ge 0$. We say an accretive set A is m-accretive if $R(I + \lambda A) = X$ for every $\lambda > 0$. Let $a, b \in \mathbb{R}$ with $a < b$. We denote by $C(a,b;X)$, the space of all continuous functions from $[a,b]$ into X. For $1 \le p < \infty$, we also denote by $L^p(a,b;X)$, the space of all strongly measurable, p-integrable, X-valued functions defined almost everywhere on $[a,b]$.

Let $A \subset X \times X$ be an m-accretive set, let $f \in L^1(a, b; X)$ and let $x \in \overline{D(A)}$. We say a function $u : [a, b] \to X$ is a strong solution of the initial value problem:

$$u(a) = x, \quad u'(t) + Au(t) \ni f(t) \quad \text{for } a \leq t \leq b, \tag{2.1}$$

if u is differentiable almost everywhere on $[a, b]$, u is absolutely continuous, $u(a) = x$ and $u'(t) + Au(t) \ni f(t)$ almost everywhere on $[a, b]$. We say a function $u : [a, b] \to X$ is an integral solution of the initial value problem (2.1), if u is continuous on $[a, b]$, $u(a) = x$, $u(t) \in \overline{D(A)}$ for every $a \leq t \leq b$ and

$$\|u(t) - y\| \leq \|u(s) - y\| + \int_s^t [u(\tau) - y, f(\tau) - z]_+ \, d\tau$$

for every $(y, z) \in A$ and s, t with $a \leq s \leq t \leq b$. If u is a strong solution of (2.1), then u is an integral solution of (2.1). We know from [2, 4] that the initial value problem (2.1) has a unique integral solution. If u and v are the integral solutions of (2.1) corresponding to $(x, f), (y, g) \in \overline{D(A)} \times L^1(a, b; X)$ respectively, then

$$\|u(t) - v(t)\| \leq \|u(s) - v(s)\| + \int_s^t [u(\tau) - v(\tau), f(\tau) - g(\tau)]_+ \, d\tau$$

for $a \leq s \leq t \leq b$.

If $A \subset X \times X$ is m-accretive, then

$$S(t)x = \lim_{n \to \infty} \left(I + \frac{t}{n} A\right)^{-n} x$$

exists for every $x \in \overline{D(A)}$ and uniformly for t on every bounded interval in the set of nonnegative real numbers; see [2, 6]. We say the family $\{S(t) : \overline{D(A)} \to \overline{D(A)}, t \geq 0\}$ is the nonlinear semigroup generated by $-A$. We remark that for every $x \in \overline{D(A)}$, $t \mapsto S(t)x$ is the unique integral solution of $u(0) = x$ and $u'(t) + Au(t) \ni 0$ for $t \geq 0$. We say $\{S(t) : \overline{D(A)} \to \overline{D(A)}, t \geq 0\}$ is compact if $S(t)$ is compact for every $t > 0$.

To prove our theorems, we need the following propositions; see [20, Theorem 2] and [7, Lemma 2]:

Proposition 2 (Vrabie). *Let X be a Banach space and let A be an m-accretive subset of $X \times X$ such that $-A$ generates a compact semigroup. Let B be a bounded subset of $\overline{D(A)}$, let $a, b \in \mathbb{R}$ with $a < b$ and let G be a uniformly integrable subset of $L^1(a, b; X)$. Then the set of all integral solutions of (2.1) corresponding to $(x, f) \in B \times G$ is relatively compact in $C(d, b; X)$ for every $d \in (a, b)$, and if, in addition, B is relatively compact in X, the set is relatively compact in $C(a, b; X)$.*

Proposition 3 (De Blasi and Myjak). *Let B be a subset of a separable Banach space X and let f be a Carathéodory mapping from $[0, 1] \times B$ into X such that $\int_0^1 \sup_{x \in B} \|f(t, x)\| \, dt < \infty$. Then for every $\varepsilon > 0$, there exists a locally Lipschitz mapping g from $[0, 1] \times B$ into X such that*

$$\int_0^1 \sup_{x \in B} \|f(t, x) - g(t, x)\| \, dt < \varepsilon.$$

3. Proof of Theorem 1

In this section, we give the proof of Theorem 1. Let α be a continuous function from $[0,\infty)$ into $[0,1]$ such that $\alpha(t)=1$ for $t\in[0,r+\varepsilon/2]$ and $\alpha(t)=0$ for $t\in[r+3\varepsilon/4,\infty)$. Define a Carathéodory mapping $\tilde{f}$ from $[0,T]\times\overline{D(A)}$ into X by $\tilde{f}(t,x)=\alpha(\|x\|)f(t,x)$ for $(t,x)\in[0,T]\times\overline{D(A)}$. Since $\int_0^T \sup_{x\in\overline{D(A)}}\|\tilde{f}(t,x)\|\,dt<\infty$, Proposition 3 yields a sequence of locally Lipschitz functions $\{\tilde{f}_n\}$ from $[0,T]\times\overline{D(A)}$ into X such that $\tilde{f}_n(t,x)=0$ for $(t,x)\in[0,T]\times(\overline{D(A)}\setminus B_{r+\varepsilon})$ and

$$\int_0^T \sup_{x\in\overline{D(A)}}\|\tilde{f}(t,x)-\tilde{f}_n(t,x)\|\,dt<\frac{1}{n}. \tag{3.1}$$

For every $x\in\overline{D(A)}\cap B_r$, we set

$$\begin{aligned}\mathcal{S}x=\{u:[0,T]\to\overline{D(A)},\ &u\text{ is an integral solution of}\\ &u(0)=x,\quad u'(t)+Au(t)\ni\tilde{f}(t,u(t))\quad\text{for }0\le t\le T\}.\end{aligned} \tag{3.2}$$

For every $n\in\mathbb{N}$ and $\sigma\in[0,T]$, we denote by $F_{n,\sigma}$, the function from $[0,T]\times\overline{D(A)}$ into X defined by

$$F_{n,\sigma}(t,x)=\begin{cases}\tilde{f}(t,x) & \text{if }(t,x)\in[0,\sigma]\times\overline{D(A)},\\ \tilde{f}_n(t,x) & \text{if }(t,x)\in(\sigma,T]\times\overline{D(A)}.\end{cases}$$

For every $n\in\mathbb{N}$ and $x\in\overline{D(A)}\cap B_r$, we also set

$$\begin{aligned}\mathcal{S}_nx=\bigcup_{\sigma\in[0,T]}\{u:[0,T]\to\overline{D(A)},\ &u\text{ is an integral solution of}\\ &u(0)=x,\quad u'(t)+Au(t)\ni F_{n,\sigma}(t,u(t))\quad\text{for }0\le t\le T\}.\end{aligned} \tag{3.3}$$

Since $\int_0^T \sup_{x\in\overline{D(A)}}\|\tilde{f}(t,x)\|\,dt<\infty$, $\int_0^T \sup_{x\in\overline{D(A)}}\|\tilde{f}_n(t,x)\|\,dt<\infty$ and $-A$ generates a compact semigroup, we know from [18, Theorem 2] or [21, Theorem 3.8.1] that there exist integral solutions for (3.2) and (3.3) on a small interval $[0,\delta]$, and we also know from [21, Theorem 3.2.2] that such integral solutions are continuable on $[0,T]$. So $\mathcal{S}x$ and $\mathcal{S}_nx$ are nonempty for every $x\in\overline{D(A)}\cap B_r$ and $n\in\mathbb{N}$.

Lemma 1. *For every $x\in\overline{D(A)}\cap B_r$ and $u\in\mathcal{S}x$, $\|u(t)\|\le r$ for every $t\in[0,T]$.*

Proof. Let $x\in\overline{D(A)}\cap B_r$ and let $u\in\mathcal{S}x$. Let $[T_0,T_1]$ be an interval contained in $[0,T)$ such that $u(T_0)=r$ and $r-\varepsilon/4\le\|u(t)\|\le r+\varepsilon/4$ for every $t\in[T_0,T_1]$. Since u is an integral solution of $u'(t)+Au(t)\ni\tilde{f}(t,u(t))$ on the interval $[T_0,T_1]$, for every $\delta\in(0,\varepsilon/4)$, there exist $t_0,\cdots,t_N\in[0,T]$, $x_0,\cdots,x_N\in\overline{D(A)}$, $f_0,\cdots,f_N\in X$ such that

$$T_0=t_0<t_1<\cdots<t_{N-1}<T_1\le t_N,\quad \max(t_i-t_{i-1})\le\delta,$$

$$\sum_{i=1}^N\int_{t_{i-1}}^{t_i}\|f_i-\tilde{f}(t,u(t))\|\,dt\le\delta,$$

$$\frac{x_i-x_{i-1}}{t_i-t_{i-1}}+Ax_i\ni f_i\quad\text{for }i=1,2,\ldots,N \tag{3.4}$$

and

$$\|v(t)-u(t)\|\le\delta\quad\text{for every }t\in[T_0,T_1], \tag{3.5}$$

where

$$v(t) = \begin{cases} x_0 & \text{if } t = t_0, \\ x_i & \text{if } t \in (t_{i-1}, t_i], \quad i = 1, 2, \dots, N; \end{cases}$$

see [13]. From the hypothesis of our theorem, (3.4), (3.5) and $0 < \delta < \varepsilon/4$, for every $i = 1, 2, \dots, N$, there exists $z_i^* \in Jx_i$ such that

$$\left\langle f_i - \frac{x_i - x_{i-1}}{t_i - t_{i-1}} - \tilde{f}(t, x_i), z_i^* \right\rangle \geq 0 \quad \text{for almost every } t \in [0, T].$$

So we have

$$\|x_j\| \leq \|x_0\| + \sum_{i=1}^{j} \int_{t_{i-1}}^{t_i} \|f_i - \tilde{f}(t, x_i)\| \, dt \leq r + 2\delta + \sum_{i=1}^{j} \int_{t_{i-1}}^{t_i} \|\tilde{f}(t, u(t)) - \tilde{f}(t, x_i)\| \, dt$$

for every $j = 1, 2, \dots, N$, which implies

$$\|u(t)\| \leq r + 3\delta + \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \|\tilde{f}(t, u(t)) - \tilde{f}(t, x_i)\| \, dt$$

for every $t \in [T_0, T_1]$. Since $\delta \in (0, \varepsilon/4)$ is arbitrary, we obtain $\|u(t)\| \leq r$ for every $t \in [T_0, T_1]$. This completes the proof. $\square$

Lemma 2. *For every $n \in \mathbb{N}$ and $x \in \overline{D(A)} \cap B_r$, $\mathcal{S}_n x$ is compact, where $\mathcal{S}_n x$ is endowed with the $C(0, T; X)$ topology.*

Proof. Since $\mathcal{S}_n x$ is relatively compact from Proposition 2, we only need to show that $\mathcal{S}_n x$ is closed. Let $n \in \mathbb{N}$ and let $x \in \overline{D(A)} \cap B_r$. Let $\{u_m\}$ be a sequence in $\mathcal{S}_n x$ which converges to u. We shall show $u \in \mathcal{S}_n x$. For every $m \in \mathbb{N}$, there exists $\sigma_m \in [0, T]$ such that u_m is an integral solution of

$$\text{(3.6)} \qquad u_m(0) = x, \quad u_m'(t) + Au_m(t) \ni F_{n,\sigma_m}(t, u_m(t)) \quad \text{for } 0 \leq t \leq T.$$

We may assume that $\{\sigma_m\}$ converges to $\sigma \in [0, T]$. Then $\{F_{n,\sigma_m}(t, u_m(t))\}$ converges to $F_{n,\sigma}(t, u(t))$ in $L^1(0, T; X)$. Since u_m is an integral solution of (3.6), we have

$$\|u_m(t) - y\| \leq \|u_m(s) - y\| + \int_s^t [u_m(\tau) - y, F_{n,\sigma_m}(\tau, u_m(\tau)) - z]_+ \, d\tau$$

for every $(y, z) \in A$, s, t with $0 \leq s \leq t \leq T$ and $m \in \mathbb{N}$. Tending m to infinity, we have $u \in \mathcal{S}_n x$. Hence, $\mathcal{S}_n x$ is closed. $\square$

The following is crucial to prove our theorem. In the proof, we use the method employed in [11, Proposition 3] and [22].

Lemma 3. *For every $n \in \mathbb{N}$ and $x \in \overline{D(A)} \cap B_r$, $\mathcal{S}_n x$ is contractible.*

Proof. Let $n \in \mathbb{N}$ and let $x \in \overline{D(A)} \cap B_r$. For every $s \in [0, 1]$ and $v \in \mathcal{S}_n x$, we denote by $w_{s,v}$, the integral solution $w_{s,v} : [sT, T] \to \overline{D(A)}$ of

$$w_{s,v}(sT) = v(sT), \quad w_{s,v}'(\tau) + Aw_{s,v}(\tau) \ni \tilde{f}_n(\tau, w_{s,v}(\tau)) \quad \text{for } sT \leq t \leq T.$$

Define a function H from $[0, 1] \times \mathcal{S}_n x$ into $\mathcal{S}_n x$ by

$$H(s, v)(t) = \begin{cases} v(t) & \text{if } 0 \leq t \leq sT, \\ w_{s,v}(t) & \text{if } sT \leq t \leq T \end{cases} \qquad \text{for every } (s, v) \in [0, 1] \times \mathcal{S}_n x.$$

We shall show that H is continuous. Let $(s_0, v_0) \in [0,1] \times \mathcal{S}_n x$. Since $\tilde{f}_n$ is locally Lipschitz, for every $\tau \in [0,T]$, there exist $\delta_\tau > 0$ and $K_\tau > 0$ such that $y \in \overline{D(A)}$, $t \in [s_0 T, T]$ with $|t-\tau| < \delta_\tau$ and $\|y - w_{s_0,v_0}(t)\| < \delta_\tau$ imply $\|\tilde{f}_n(t,y) - \tilde{f}_n(t, w_{s_0,v_0}(t))\| \le K_\tau \|y - w_{s_0,v_0}(t)\|$. From the compactness of $[s_0T, T]$, there exists $\{\tau_1, \cdots, \tau_m\} \subset [s_0 T, T]$ such that $[s_0T, T] \subset \bigcup_{i=1}^m (\tau_i - \delta_{\tau_i}, \tau_i + \delta_{\tau_i})$. Set $\delta = \min\{\delta_{\tau_1}, \cdots, \delta_{\tau_m}\}$ and $K = \max\{K_{\tau_1}, \cdots, K_{\tau_m}\}$. Then we have

$$\|\tilde{f}_n(t,y) - \tilde{f}_n(t, w_{s_0,v_0}(t))\| \le K\|y - w_{s_0,v_0}(t)\| \tag{3.7}$$

for every $(t,y) \in [s_0T, T] \times \overline{D(A)}$ with $\|y - w_{s_0,v_0}(t)\| < \delta$. Fix $\eta \in (0, \delta]$. Choose $\rho > 0$ satisfying $\rho < \delta$ and $\rho < \eta/(4e^{KT})$. We can also choose $\zeta \in (0, \rho]$ such that

$$\int_t^{t+\zeta T} \sup_{y \in \overline{D(A)}} \|\tilde{f}(\tau, y)\|\, d\tau < \rho \quad \text{and} \quad \int_t^{t+\zeta T} \sup_{y \in \overline{D(A)}} \|\tilde{f}_n(\tau, y)\|\, d\tau < \rho \quad \text{for every } t \in [0, (1-\zeta)T].$$

From $v_0 \in \mathcal{S}_n x$, there exists $\sigma \in [0,T]$ such that v_0 is an integral solution of

$$v_0(0) = x, \quad v_0'(\tau) + Av_0(\tau) \ni F_{n,\sigma}(\tau, v_0(\tau)) \quad \text{for } 0 \le \tau \le T.$$

We remark that

$$\int_t^{t+\zeta T} \|F_{n,\sigma}(\tau, v_0(\tau))\|\, d\tau < 2\rho \quad \text{for every } t \in [0, (1-\zeta)T].$$

Let $(s,v) \in [0,T] \times \mathcal{S}_n x$ such that $|s - s_0| < \zeta$ and $\|v - v_0\| < \zeta$. For every $t \in [s_0T, sT]$ in the case of $s \ge s_0$ or for every $t \in [sT, s_0T]$ in the case of $s_0 \ge s$, we have

$$\begin{aligned}
&\|H(s,v)(t) - H(s_0,v_0)(t)\| \\
&\quad \le \begin{cases} \|v(t) - v_0(t)\| + \displaystyle\int_{s_0T}^t \|F_{n,\sigma}(\tau, v_0(\tau)) - \tilde{f}_n(\tau, w_{s_0,v_0}(\tau))\|\, d\tau & \text{if } s \ge s_0, \\ \|v(sT) - v_0(sT)\| + \displaystyle\int_{sT}^t \|\tilde{f}_n(\tau, w_{s,v}(\tau)) - F_{n,\sigma}(\tau, v_0(\tau))\|\, d\tau & \text{if } s_0 \ge s, \end{cases} \\
&\quad \le 4\rho.
\end{aligned}$$

Then we have

$$\|H(s,v)(t) - H(s_0,v_0)(t)\| \le 4\rho + \int_{T'}^t \|\tilde{f}_n(\tau, w_{s,v}(\tau)) - \tilde{f}_n(\tau, w_{s_0,v_0}(\tau))\|\, d\tau \tag{3.8}$$

for every $t \in [T', T]$, where $T' = \max\{sT, s_0T\}$. We shall show that $\|H(s,v)(t) - H(s_0,v_0)(t)\| < \eta$ for every $t \in [0,T]$. Suppose not. Then there exists $t_0 \in (T', T]$ such that $\|H(s,v)(t_0) - H(s_0,v_0)(t_0)\| = \eta$ and $\|H(s,v)(t) - H(s_0,v_0)(t)\| < \eta$ for every $t \in [T', t_0)$. By (3.7), (3.8) and Gronwall's inequality, we have $\|H(s,v)(t_0) - H(s_0,v_0)(t_0)\| \le 4\rho e^{KT} < \eta$, which is a contradiction. So, we have $\|H(s,v)(t) - H(s_0,v_0)(t)\| < \eta$ for every $t \in [0,T]$. Hence H is continuous.

On the other hand, for every $v \in \mathcal{S}_n x$, $H(1,v) = v$ and $H(0,v) = w$, where w is the integral solution of

$$w(0) = x, \quad w'(t) + Aw(t) \ni \tilde{f}_n(t, w(t)) \quad \text{for } 0 \le t \le T.$$

Therefore $\mathcal{S}_n x$ is contractible. □

Lemma 4. *For every $x \in \overline{D(A)} \cap B_r$, $\mathcal{S}x$ is compact and acyclic.*

Proof. Let $x \in \overline{D(A)} \cap B_r$. Since $F_{n,T} = \tilde{f}$ for every $n \in \mathbb{N}$, we have $\mathcal{S}x \subset \bigcap_{n=1}^{\infty} \mathcal{S}_n x$. We shall show the opposite inclusion. Let $u \in \bigcap_{n=1}^{\infty} \mathcal{S}_n x$. Then for every $n \in \mathbb{N}$, there exists $\sigma_n \in [0,T]$ such that u is an integral solution of

$$u(0) = x, \qquad u'(t) + Au(t) \ni F_{n,\sigma_n}(t, u(t)) \quad \text{for } 0 \le t \le T.$$

Then, from (3.1), we have

$$\|u(t) - y\| \le \|u(s) - y\| + \int_s^t [u(\tau) - y, \tilde{f}(\tau, u(\tau)) - z]_+ \, d\tau + \frac{1}{n}$$

for every $(y,z) \in A$, s, t with $0 \le s \le t \le T$ and $n \in \mathbb{N}$. Tending n to infinity, we obtain $u \in \mathcal{S}x$. So we have $\mathcal{S}x = \bigcap_{n=1}^{\infty} \mathcal{S}_n x$. From Lemma 2, Lemma 3 and the continuity property of the Čech homology, we have that $\mathcal{S}x$ is compact and acyclic. $\square$

Now, we can give the proof of our theorem.

Proof of Theorem 1. Let $Z_0 = \bigcup\{v(T) : y \in \overline{D(A)} \cap B_r, v \in \mathcal{S}y\}$. From Lemma 1, Z_0 is a nonempty subset of $\overline{D(A)} \cap B_r$. Let Z be the closed, convex hull of Z_0. From Proposition 2, Z_0 is relatively compact and hence Z is compact. Let Y be the set $\{u \in C(0,T;X) : u(t) \in \overline{D(A)} \cap B_r$ for every $t \in [0,T]$ and $u(T) \in Z\}$ and let $\mathcal{T}$ be a multivalued mapping from Y into $C(0,T;X)$ defined by

$$\mathcal{T}u = \mathcal{S}u(T) \quad \text{for every } u \in Y,$$

i.e., $\mathcal{T}$ is the composition of $u \mapsto u(T)$ and $\mathcal{S}$. From the compactness of Z, Proposition 2, Lemma 1 and Lemma 4, $\mathcal{T}(Y)$ is contained in a compact subset of Y and $\mathcal{T}u$ is a nonempty, acyclic, compact subset of Y for every $u \in Y$. We shall show that $\mathcal{T}$ is upper semicontinuous. Suppose not. Then there exist $u \in Y$, a open neighborhood V of $\mathcal{T}u$, $\{u_n\} \subset Y$ and $\{v_n\} \subset Y$ such that $\{u_n\}$ converges to u and $v_n \in \mathcal{T}u_n \setminus V$ for every $n \in \mathbb{N}$. From Proposition 2, we may assume $\{v_n\}$ converges to v, and hence $v \notin V$. Since $v_n \in \mathcal{T}u_n$, we have $v_n(0) = u_n(T)$ and

$$\|v_n(t) - y\| \le \|v_n(s) - y\| + \int_s^t [v_n(\tau) - y, \tilde{f}(\tau, v_n(\tau)) - z]_+ \, d\tau$$

for every $(y,z) \in A$, s,t with $0 \le s \le t \le T$ and $n \in \mathbb{N}$. Tending n to infinity, we obtain $v \in \mathcal{T}u$, which contradicts $\mathcal{T}u \subset V$ and $v \notin V$. So, $\mathcal{T}$ is upper semicontinuous. Hence, by Proposition 1, there exists a point $u \in Y$ such that $u \in \mathcal{T}u$. By the definition of $\mathcal{T}$, $u(0) = u(T)$ and u is an integral solution of $u'(t) + Au(t) \ni \tilde{f}(t, u(t))$ for $0 \le t \le T$. From Lemma 1, u is also an integral solution of $u'(t) + Au(t) \ni f(t, u(t))$ for $0 \le t \le T$. This completes the proof. $\square$

4. Proof of Theorem 2

In this section, we give the proof of Theorem 2. Let $h \in L^1(0,T;X)$. Let M, R and ρ be real numbers such that $M = \int_0^T |b(s)|\, ds + \int_0^T \|h(s)\|\, ds$, $R = \max\{r + M + 2, (1 + 1/(cT))(M+1)\}$ and $\rho = R + M + 4$. From the hypothesis of Theorem 2, there exists $a_\rho \in L^1(0,T)$ such that $\|f(t,x)\| \le a_\rho(t)$ for almost every $t \in [0,T]$ and for every $x \in \overline{D(A)}$ with $\|x\| \le \rho$. Let α be a continuous function from $[0,\infty)$ into $[0,1]$ which satisfies

$$\alpha(\tau) = \begin{cases} 1 & \text{if } \tau \le \rho - 1, \\ 0 & \text{if } \tau \ge \rho \end{cases} \qquad \text{for } \tau \ge 0.$$

Define a function $\tilde{f} : [0,T]\times\overline{D(A)} \to X$ by $\tilde{f}(t,x) = \alpha(\|x\|)f(t,x)$ for every $(t,x) \in [0,T]\times\overline{D(A)}$. Since $\int_0^T \sup_{x\in\overline{D(A)}} \|\tilde{f}(t,x)\|\, d\tau \leq \int_0^T a_\rho(\tau)\, d\tau < \infty$, Proposition 3 yields a sequence of locally Lipschitz functions $\{\tilde{f}_n\}$ from $[0,T] \times \overline{D(A)}$ into X such that

$$\int_0^T \sup_{x\in\overline{D(A)}} \|\tilde{f}(t,x) - \tilde{f}_n(t,x)\|\, dt < \frac{1}{n} \quad \text{for every } n \in N. \tag{4.1}$$

For every $n \in \mathbb{N}$ and $x \in \overline{D(A)} \cap B_R$, we set $F_n x = u(T)$, where u is the unique integral solution of

$$u(0) = x, \quad u'(t) + Au(t) \ni \tilde{f}_n(t, u(t)) + h(t) \quad \text{for } 0 \leq t \leq T. \tag{4.2}$$

From [21, Theorem 3.2.2], and [18, Theorem 2] or [21, Theorem 3.8.1], we know that F_n is well defined.

Lemma 5. *Let $n \in \mathbb{N}$, let $x \in \overline{D(A)} \cap B_R$ and let u be the integral solution of* (4.2). *Then $\|u(t)\| \leq R + M + 1$ for every $t \in [0,T]$.*

Proof. Suppose not. Then there exist $T_0, T_1 \in [0,T]$ such that $T_0 < T_1$, $\|u(T_0)\| = R$, $R \leq \|u(t)\| \leq R + M + 2$ for every $t \in [T_0, T_1]$ and $\|u(T_1)\| > R + M + 1$. Since u is the integral solution of $u'(t) + Au(t) \ni \tilde{f}_n(t, u(t)) + h(t)$ on the interval $[T_0, T_1]$, by the same method as in the proof of Lemma 1, we have

$$\begin{aligned} \|u(T_1)\| &\leq \|u(T_0)\| + \int_{T_0}^{T_1} \|h(s)\|\, ds + 1 - c\int_{T_0}^{T_1} \|u(s)\|\, ds + \int_{T_0}^{T_1} |b(s)|\, ds \\ &\leq R + M + 1. \end{aligned}$$

So we obtain a contradiction. This completes the proof. □

Lemma 6. *For every $n \in \mathbb{N}$, F_n is a mapping from $\overline{D(A)} \cap B_R$ into itself.*

Proof. Let $n \in \mathbb{N}$, let $x \in \overline{D(A)} \cap B_R$ and let u be the integral solution of (4.2). First, we consider the case that there exists $T_2 \in [0,T]$ such that $\|u(T_2)\| < R - M - 1$. Suppose $\|u(T)\| > R$. Then there exists $T_3 \in (T_2, T]$ such that $\|u(T_3)\| = R - M - 1$ and $\|u(t)\| \geq R - M - 1$ for every $t \in [T_3, T]$. By the same method as in the proof of Lemma 1, we have

$$\|u(T)\| \leq \|u(T_3)\| + \int_{T_3}^{T} \|h(s)\|\, ds + \frac{1}{n} - c\int_{T_3}^{T} \|u(s)\|\, ds + \int_{T_3}^{T} |b(s)|\, ds \leq R,$$

which is a contradiction. So we have $\|u(T)\| \leq R$. Next, we consider the case that $\|u(t)\| \geq R - M - 1$ for every $t \in [0,T]$. Then, we have

$$\begin{aligned} \|u(T)\| &\leq \|u(0)\| + \int_0^T \|h(t)\|\, dt + \frac{1}{n} - c\int_0^T \|u(t)\|\, dt + \int_0^T |b(t)|\, dt \\ &\leq \|u(0)\| + M + 1 - cT(R - M - 1) \leq R. \end{aligned}$$

Hence F_n is a mapping from $\overline{D(A)} \cap B_R$ into itself. □

Lemma 7. *For every $n \in \mathbb{N}$, F_n is continuous.*

Proof. Let $n \in \mathbb{N}$. Let $x \in \overline{D(A)} \cap B_R$ and let u be the integral solution of (4.2). Since $\tilde{f}_n$ is locally Lipschitz, by the same method to prove (3.7), there exist $K > 0$ and $\eta > 0$ such that

$$\|\tilde{f}_n(t, y) - \tilde{f}_n(t, u(t))\| \leq K\|y - u(t)\| \tag{4.3}$$

for every $(t, y) \in [0, T] \times \overline{D(A)}$ with $\|y - u(t)\| < \eta$. Let $\varepsilon \in (0, \eta)$ and let $\delta > 0$ satisfying $\delta e^{KT} < \varepsilon$. Let $y \in \overline{D(A)} \cap B_R$ be a point satisfying $\|x - y\| < \delta$ and let v be the integral solution of $v(0) = y$ and $v'(t) + Av(t) \ni \tilde{f}_n(t, v(t)) + h(t)$ for $0 \le t \le T$. We shall show $\|u(t) - v(t)\| < \varepsilon$ for every $t \in [0, T]$. Suppose not. Then there exists $t_0 \in (0, T]$ such that $\|u(t_0) - v(t_0)\| = \varepsilon$ and $\|u(t) - v(t)\| < \varepsilon$ for every $t \in [0, t_0)$. By (4.3), Gronwall's inequality implies $\|u(t_0) - v(t_0)\| \le \delta e^{KT} < \varepsilon$, which is a contradiction. Hence F_n is continuous. □

Proof of Theorem 2. From Lemma 6, Lemma 7 and Proposition 2, F_n is a compact mapping from $\overline{D(A)} \cap B_R$ into itself for every $n \in \mathbb{N}$. Hence, for every $n \in \mathbb{N}$, there exists a fixed point of F_n by Schauder's fixed point theorem, i.e., there exists an integral solution u_n of $u_n'(t) + Au_n(t) \ni \tilde{f}_n(t, u_n(t)) + h(t)$ for $0 \le t \le T$ such that $u_n(0) = u_n(T)$ and $\|u_n(0)\| \le R$. From Proposition 2, we may assume $\{u_n\}$ converges to u in $C(0, T; X)$. Let $(x, y) \in A$ and let $s, t \in \mathbb{R}$ with $0 \le s \le t \le T$. From (3.7), we have

$$\|u_n(t) - x\| \le \|u_n(s) - x\| + \int_s^t [u_n(\tau) - x, \tilde{f}(\tau, u_n(\tau)) + h(\tau) - y]_+ \, d\tau + \frac{1}{n}$$

for every $n \in \mathbb{N}$. Tending n to infinity, we obtain that u is an integral solution of $u'(t) + Au(t) \ni \tilde{f}(t, u(t)) + h(t)$ for $0 \le t \le T$ such that $u(0) = u(T)$ and $\|u(0)\| \le R$. From Lemma 5, u is also an integral solution of $u'(t) + Au(t) \ni f(t, u(t)) + h(t)$ for $0 \le t \le T$. Hence we obtain the desired result. □

5. An example

Let Ω be a bounded domain in $\mathbb{R}^n (n \ge 2)$ with smooth boundary Γ. We consider the following nonlinear differential equation:

$$\frac{\partial u}{\partial t} - \Delta \rho(u) = g(t, x, u(t, x)) + h(t, x) \quad \text{on } \mathbb{R} \times \Omega \tag{5.1}$$

with a boundary condition

$$\rho(u) = 0 \quad \text{on } \mathbb{R} \times \Gamma. \tag{5.2}$$

Theorem 3. *Let $\rho \in C(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\})$ such that $\rho(0) = 0$ and there exist $C > 0$ and $a > \frac{n-2}{n}$ with*

$$\rho'(r) \ge C|r|^{a-1} \quad \textit{for every } r \in \mathbb{R} \setminus \{0\}.$$

Let $g : \mathbb{R} \times \Omega \times \mathbb{R} \to \mathbb{R}$ such that g is T-periodic in its first variable, $g(t, x, \cdot)$ is continuous for almost every $(t, x) \in \mathbb{R} \times \Omega$ and $g(\cdot, \cdot, u)$ is measurable for every $u \in \mathbb{R}$. Assume that there exist $a \in L^1(0, T)$ and $b \in L^1((0, T) \times \Omega)$ such that $|g(t, x, u)| \le a(t)|u| + b(t, x)$ for $(t, x, u) \in [0, T] \times \Omega \times \mathbb{R}$ and that

$$\overline{\lim_{|u| \to \infty}} \operatorname*{ess\,sup}_{(t,x) \in \mathbb{R} \times \Omega} \frac{g(t, x, u)}{u} < 0.$$

Then for every $h \in L^1((0, T) \times \Omega)$, (5.1) and (5.2) have at least one T-periodic integral solution $u \in C(\mathbb{R}, L^1(\Omega))$.

Proof. Let A be the set defined by $\{(u, -\Delta\rho(u)) \in L^1(\Omega) \times L^1(\Omega) : \rho(u) \in W_0^{1,1}(\Omega)\}$ and let f be the function from $\mathbb{R} \times L^1(\Omega)$ into $L^1(\Omega)$ defined by $f(t, u)(x) = g(t, x, u(x))$ for every $(t, u, x) \in \mathbb{R} \times L^1(\Omega) \times \Omega$. We know that $-A$ generates a compact semigroup; see [21, Lemma 2.7.2]. From the assumption, there exist $\delta, M > 0$ such that $g(t,x,u)/u \leq -\delta$ for $(t,x,u) \in \mathbb{R} \times \Omega \times \mathbb{R}$ with $|u| \geq M$. Then for every $u \in L^1(\Omega)$ and $z \in Ju$, we have

$$\int_\Omega g(t,x,u(x))z(x)\,dx \leq -\delta\|u\|^2_{L^1(\Omega)} + \left((\delta + a(t))M|\Omega| + \int_\Omega b(t,x)\,dx\right)\|u\|_{L^1(\Omega)}.$$

So, from Theorem 2, for every $h \in L^1((0,T) \times \Omega)$, there exists a T-periodic integral solution for (5.1) and (5.2). □

Appendix

In this appendix, we give the proof of Proposition 1. The following is obtained in [16].

Proposition 4. *Let Y be a subset of a Hausdorff topological vector space and let K be a Hausdorff topological space. Let $\mathcal{T}$ be an upper semicontinuous multivalued mapping from $\mathrm{co}Y$ into K such that for every $y \in \mathrm{co}Y$, $\mathcal{T}y$ is a nonempty, acyclic, compact subset of K, and let $\mathcal{G}$ be a multivalued mapping from Y into K such that for every $y \in Y$, $\mathcal{G}y$ is a closed subset of K, and*

$$\mathcal{T}(\mathrm{co}\{y_1, \cdots, y_n\}) \subset \bigcup_{i=1}^{n} \mathcal{G}y_i \quad \text{for every finite subset } \{y_1, \cdots, y_n\} \text{ of } Y. \tag{5.3}$$

Then $\{\mathcal{G}y : y \in Y\}$ has the finite intersection property.

From Proposition 4, we have the following, which is obtained in [17]. In the following, we can get a coincidence point of $\mathcal{A}$ and $\mathcal{T}$, though there is no relationship between them.

Proposition 5. *Let Y be convex subset of a Hausdorff topological vector space and let K be a compact, Hausdorff topological space. Let $\mathcal{T}$ and $\mathcal{A}$ be multivalued mappings from Y into K such that $\mathcal{T}$ is upper semicontinuous, for every $y \in Y$, $\mathcal{T}y$ is a nonempty, acyclic, compact subset of K and $\mathcal{A}y$ is an open subset of K, and for every $z \in K$, $\mathcal{A}^{-1}z$ is a nonempty, convex subset of Y. Then there is an element y of Y such that $\mathcal{A}y \cap \mathcal{T}y \neq \emptyset$.*

Proof. Assume that the conclusion does not hold. Define a multivalued mapping $\mathcal{G}$ from Y into K by $\mathcal{G}y = K \setminus \mathcal{A}y$ for every $y \in Y$. We shall show (5.3). Suppose not. Then there exist a finite subset $\{y_1, \cdots, y_n\}$ of Y, $y \in \mathrm{co}\{y_1, \cdots, y_n\}$ and $z \in \mathcal{T}y$ such that $z \notin \bigcup_{i=1}^n \mathcal{G}y_i$. So we have $z \in \mathcal{A}y_i$ i.e., $y_i \in \mathcal{A}^{-1}z$ for every $i = 1, \ldots, n$. Since $\mathcal{A}^{-1}z$ is convex, we have $y \in \mathcal{A}^{-1}z$. So we obtain $z \in \mathcal{T}y \cap \mathcal{A}y$, which is a contradiction. Hence by Proposition 4 and the compactness of K, there exists $w \in K$ such that $w \in \bigcap_{y \in Y} \mathcal{G}y$. So we get $w \notin \mathcal{A}y$ for all $y \in Y$, which implies $\mathcal{A}^{-1}w = \emptyset$, and we get a contradiction. This completes the proof. □

Proof of Proposition 1. Let U be an arbitrary, symmetric, convex, open neighborhood of 0 in E. Define a multivalued mapping $\mathcal{A}$ from Y into K by $\mathcal{A}y = y + U$ for every $y \in Y$. By Proposition 5, there is a point $y_U \in Y$ such that $(y_U + U) \cap \mathcal{T}y_U \neq \emptyset$. From the standard compactness argument, we obtain the conclusion. □

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