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Groups and Generalized Polygons

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1 Introduction

Geometric interpretation is a technique that has proved very useful to study certain groups. Especially well disigned for this are the Tits buildings, which are geometric interpretations of groups of Lie type, Chevalley groups, semi simple algebraic groups, groups with a BN-pair, Kac-Moody groups, etc. Conversely, given a certain geometry, for instance a special kind of Tits building, one could raise the question whether there is always a group behind it. This is searching for a geometric characterization of the groups in question. Examples are the spherical Tits buildings of rank ≥ 3, the affine Tits buildings of rank ≥ 4, and certain twin Tits buildings, giving rise to, respectively, semi simple algebraic groups of relative rank ≥ 3 and groups of mixed type, semi simple algebraic groups and mixed type groups of relative rank ≥ 3 with a valuation on the root groups in the sense of BRUHAT & TITS [1972], certain Kac-Moody groups. Moreover, a lot of sporadic finite simple groups have been geometrically characterized by geometries which extend Tits buildings.

The building bricks in all these cases are the buildings of rank 2, the so-called generalized polygons. The main examples of these are constructed from the parabolic subgroups of a rank 2 Tits system, or BN-pair. For instance, in the finite case, one has so-called *classical* examples related to the linear groups $\mathrm{PSL}(3,q)$ (the projective planes or generalized 3-gons), the symplectic groups $\mathrm{PSp}(4,q)$, the orthogonal groups $\mathrm{PSO}(5,q)$ and $\mathrm{PSO}^-(6,q)$, the unitary groups $\mathrm{PSU}(4,q)$ and $\mathrm{PSU}(5,q)$ (generalized quadrangles), Dickson's group $\mathrm{G}_2(q)$ and the triality groups ${}^3\mathrm{D}_4(q)$ (generalized hexagons), and the Ree groups ${}^2\mathrm{F}_4(q)$ (generalized octagons).

Hence it is important to have characterizations of these main examples in terms of groups. This then leads to characterizations of higher rank geometries via groups. For instance in order to classify the finite flag transitive extended generalized quadrangles (rank 3), it would be very helpful to have at one's disposal a classification of all finite flag transitive generalized quadrangles. Though sometimes the fact that the polygon occurs in a rank 3 geometry imposes some extra condition that can be dealt with. This is the case when the polygon sits in a rank ≥ 3 spherical or rank ≥ 4 affine building. Perhaps also the flag transitive extended quadrangles can be handled, but this has not been done yet (personal communication by YOSHIARA).

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In the present paper, we review some characterizations of generalized polygons using groups. It is no surprise that in many cases the finiteness condition will be essential, but there are also a few more general results.

We start with definitions and main results related to generalized polygons.

2 Axioms and results

A *generalized polygon* (of order (s,t), $s,t \geq 1$) is a point-line incidence geometry whose incidence graph has girth $2n$ and diameter n, for some natural number n, $n \geq 2$, in which case we also speak about a *generalized n-gon* (such that there are exactly $s+1$ points incident with any line, and $t+1$ lines incident with any point). A generalized polygon is *thick* if every line is incident with at least 3 points and every point is incident with at least three lines. Every thick generalized polygon has an order, and every non-thick generalized n-gon either arises in some canonical way from a thick generalized m-gon, with m a divisor of n, or has exactly two elements incident with more than 2 elements, or has order $(1,1)$. The latter two cases are trivial ones and so one is reduced to the study of thick generalized polygons, briefly called generalized polygons. In a more geometric way, one might restate the axioms of a generalized n-gons as follows:

(1) there are no ordinary k-gons for $k < n$ as subgeometry,

(2) every pair of elements is contained in an ordinary n-gon,

(3) there exists an ordinary $(n+1)$-gon.

The third axiom is actually equivalent to thickness.

Generalized polygons were introduced by TITS [1959] when constructing the groups $G_2(K)$ and the triality groups ${}^3D_4(K')$, for all fields K and suitable fields K'. The main result for finite generalized n-gons is due to FEIT & HIGMAN [1964] and says that (thick) generalized n gons, $n \geq 3$, only exist for $n = 3,4,6,8$. Note that the generalized 3-gons are the projective planes. The generalized 2-gons are the trivial incidence structures for which every point is incident with every line. Hence we may restrict ourselves to the case of (thick) generalized n-gons, $n \geq 3$, from now on. For an extensive survey including most proofs, we refer to VAN MALDEGHEM [19**b]. For the finite case, with emphasis on the generalized quadrangles, see THAS [1995]. For finite generalized quadrangles, see PAYNE & THAS [1984].

Some more terminology now. The incidence graph of a generalized n-gon Γ induces a distance function on pairs of elements of Γ. The maximal distance between two elements is easily seen to be n. In that case, the two elements are called *opposite*. A *path* of length d is a sequence of $d+1$ consecutively incident elements. A *geodesic* is a path of some length d such that the extremities are at distance d from each other. A *collineation* of a generalized polygon is a permutation of the set of points which induces (via the incidence relation) a permutation of the set of lines.

3 Characterizations with groups

3.1 The Moufang condition

The Moufang condition is perhaps the most important group-theoretical condition in the theory of generalized polygons. The classification of all Moufang polygons has been an open problem for more than twenty years, and when it was finally completed (in 1997), it turned out that the main conjecture on generalized quadrangles was wrong. Indeed, one class of Moufang quadrangles had to be added to the list. The classification of Moufang n-gons for $n \neq 4$, has been achieved much earlier (in the fifties for $n = 3$, in the sixties for $n = 6$, in the seventies for $n \neq 3, 4, 6, 8$ and in the eighties for $n = 8$).

Let $\gamma = (x_1, x_2, \ldots, x_{n-1})$ be a geodesic of length $n-2$ in a generalized n-gon Γ. Then the group of collineations of Γ fixing all elements incident with one of the x_i, $1 \leq i \leq n-1$, acts semi-regularly on the set of elements different from x_1 and incident with any prescribed element x_0, $x_0 \neq x_2$, incident with x_1. If this action is transitive (and hence regular), then we say that γ is a *Moufang path.* The corresponding collineations are called *elations.* If all geodesics of length $n-2$ are Moufang paths, then we say that Γ *satisfies the Moufang condition*, or that Γ is a *Moufang polygon.* For $n = 3$, this amounts to the usual definition of Moufang projective plane. The classification of all Moufang polygons is recently being reviewed and written down by TITS & WEISS [19**]. Roughly speaking, the result says that every Moufang polygon is related to either a classical group, an algebraic group, a group of mixed type, or a Ree group in characteristic 2 (all of relative Tits rank 2). Three remarks: (1) Moufang n-gons exist only for $n = 3, 4, 6, 8$; (2) the recently discovered class of Moufang quadrangles was proved to be related to groups of mixed type F_4 of relative rank 2 by MÜHLHERR & VAN MALDEGHEM [19**]; (3) all finite classical examples mentioned above are Moufang polygons, and every finite Moufang polygon is, up to diality, one of the classical examples.

3.2 Weakenings of the Moufang condition

Let Γ be a generalized n-gon. Let $\gamma = (x_1, \ldots, x_{k-1})$ be a geodesic of length $k-2$, $2 \leq k \leq n$. Then, if $k > 2$, the group of all collineations fixing all elements incident with one of the x_i, $1 \leq i \leq k-1$, acts semi-regularly on the set of geodesics of length n starting with x_1 and containing γ and an arbitrary but fixed element x_k incident with x_{k-1}, $x_k \neq x_{k-2}$. If this action is transitive (and hence regular if $k \neq 2$), then we say that γ is a *Moufang path.* If all geodesics of length $k-2$ are Moufang paths, then we say that Γ is a *k-Moufang polygon.* It is easily seen that the Moufang condition is in fact the n-Moufang condition. But there is more. By a result of VAN MALDEGHEM & WEISS [1992], k-Moufang is equivalent to Moufang for $4 \leq k \leq n$. And finite 3-Moufang (respectively 2-Moufang) generalized polygons are Moufang polygons (and vice versa), see also VAN MALDEGHEM & WEISS [1992] (respectively VAN MALDEGHEM [19**a]).

Now we notice that if n is even, then there are two different families of geodesics of length $n-2$: those starting with a point and those starting with a line. If all members of one family are Moufang paths, then we say that the polygon is a *half Moufang* polygon. It is still an open question whether in general every half Moufang polygon is a Moufang polygon, but for the finite case, this has been solved. THAS, PAYNE & VAN MALDEGHEM [1991] show this for generalized quadrangles, and, using the classification of finite simple groups, BUEKENHOUT & VAN MALDEGHEM [1994] show this for hexagons and octagons.

A *central elation* in a generalized n-gon Γ, $n = 2m$ even, is a collineation fixing all elements at distance at most m from some point. It is shown in VAN MALDEGHEM [19**b] that, if m is odd, and if Γ is a half Moufang $2m$-gon (say, all geodesics of length $n-2$ starting with a point are Moufang paths) with the property that all corresponding elations are central elations, then $n = 6$ and consequently, it follows from RONAN [1980] that Γ is a Moufang hexagon.

3.3 Transitivity conditions

Transitivity conditions have only been considered in the finite case. In the general case, it seems very difficult to deal with it, except if one has an additional structure such as a compact connected topology or a real valued discrete valuation on the polygon. So let us restrict to the finite case. There are four transitivity conditions I would like to review. I mention them from strong to weak.

Consider a finite generalized n-gon, and suppose that there is a group of collineations acting transitively on all ordered ordinary $(n+1)$-gons (an ordered k-gon is just an ordinary k-gon with a distinguished pair of incident elements). Then THAS & VAN MALDEGHEM [1995] (for $n = 4$) and VAN MALDEGHEM [1996] (for $n = 6, 8$) show that this is only possible for some classical examples, and all groups are determined. In particular, no (finite) generalized octagon admits such a collineation group.

Next, consider a finite generalized n-gon Γ, and suppose that there is a group of collineations acting transitively on all ordered ordinary n-gons. This situation amounts to a finite BN-pair of rank 2. Using the classification of finite simple groups, BUEKENHOUT & VAN MALDEGHEM [1994] show that Γ is a finite Moufang polygon, and every finite Moufang polygon admits such a collineation group (as was pointed out by TITS [1974], who was the first to raise the question).

In fact, BUEKENHOUT & VAN MALDEGHEM [1994] prove something stronger. They classify all finite generalized n-gons with a collineation group acting distance transitively on the associated distance regular point graph of Γ. This point graph is the graph with vertices the points of Γ and edges the pairs of collinear points. Besides the Moufang n-gons, there is one other example, namely, for $n = 4$, the unique generalized quadrangle of order $(3, 5)$.

The last condition is a very famous one, but only very partial solutions are known. The condition is that there is a flag transitive group, i.e., a group acting transitively on the pairs of incident elements. The conjecture is that, besides the distance transitive ones of the previous paragraph, there is only one further unique flag transitive polygon and it is a quadrangle of order $(15, 17)$, see KANTOR [1991]. Also, BUEKENHOUT & VAN

MALDEGHEM [1993] show that no finite sporadic simple group can act flag transitively (nor point transitively!) on any generalized polygon.

4 Conclusions

Two types of problems emerge from the previous section. Type 1: there are group theoretical conditions that can and should be handled without the classification of the finite simple groups. Type 2: the same thing, but with the classification of finite simple groups. For type 2, the most important problem is the classification of the flag transitive polygons. The author believes that this is a feasible task for hexagons and octagons. First approximations may be the assumption of a primitive group, or transitivity on short geodesics (such as length 2,3).

It is also important to try to shift solved type 2 problems to a type 1 problem. The classification of all generalized n-gons admitting a group acting transitively on all ordered ordinary n-gons is an important instance of this. Perhaps the case of order (s,t) with $s = t$ can be handled completely, as a first test case.

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