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Character Products and Q-Polynomial Group Association Schemes

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1 Motivation

In order to study the representation theory of association schemes, the study of Q-polynomial association schemes gives a start point. But there is no systematic study of Q-polynomial association schemes yet. We start with the classification of Q-polynomial group association schemes as we can apply character theory of finite groups fully. Since Q-polynomial association schemes are symmetric, it is important to consider the symmetrization of group association schemes. Now the condition that the scheme becomes Q-polynomial can be interpreted by the terminologies of the condition on the decomposition of the character product. We obtained a new result on the decomposition of the square of a character, which leads us to a desired classification of the symmetrization of group association schemes, which have Q-polynomial properties.

A special case is treated in a previous paper [12] of the second author. The preprint with the same title [15] is available upon request.

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2 Character Product

Let G be a finite group and let $\mathrm{Irr}(G)$ be the set of absolutely irreducible ordinary (complex) characters of G. For $\phi \in \mathrm{Irr}(G)$, let

$$\hat{\phi} = \begin{cases} \phi & \text{if } \phi \text{ is real valued,} \\ \phi + \bar{\phi} & \text{otherwise,} \end{cases}$$

where $\bar{\phi}$ is the character of the contragredient of the representation affording ϕ. Hence $\bar{\phi}(g) = \overline{\phi(g)} = \phi(g^{-1})$ for $g \in G$. Let

$$\mathrm{RIrr}(G) = \{\hat{\phi} \mid \phi \in \mathrm{Irr}(G)\}.$$

In the following, for characters ϕ, ψ of G, let

$$< \phi, \psi >= \frac{1}{|G|} \sum_{g \in G} \phi(g)\overline{\psi(g)}.$$

Theorem 2.1 *Let G be a finite group. Suppose for some $\chi, \psi \in \mathrm{Irr}(G)$, $\hat{\chi}^2$ is a linear combination of 1, $\hat{\chi}$ and $\hat{\psi}$, where 1 denotes the principal character of G. Then the degree of χ is at most 3.*

By a well-known classification of finite subgroups of $GL(2, \boldsymbol{C})$, and $GL(3, \boldsymbol{C})$ in [3, 8], we can determine the groups satisfying the conditions in Theorem 2.1.

Corollary 2.2 *Let G be a finite group of order at least 3. Suppose for some $\chi, \psi \in \mathrm{Irr}(G)$, $\hat{\chi}^2$ is a linear combination of 1, $\hat{\chi}$ and $\hat{\psi}$, where 1 denotes the principal character of G. If χ is faithful, then one of the following holds.*

(1) $G \simeq A_5$, *the alternating group of degree* 5 *and* $\chi(1) = \hat{\chi}(1) = 3$.
(2) $G \simeq A_4$, *the alternating group of degree* 4 *and* $\chi(1) = \hat{\chi}(1) = 3$.
(3) $G \simeq S_3$, *the symmetric group of degree* 3 *and* $\chi(1) = \hat{\chi}(1) = 2$.
(4) $G =< x, y, z \mid x^2 = y^3 = z^3 = xyz >\simeq SL(2,3)$ *and* $\chi(1) = \hat{\chi}(1) = 2$.
(5) $G =< x, y, z \mid x^2 = y^3 = z^4 = xyz >\simeq SL(2,3) \cdot 2$ *and* $\chi(1) = \hat{\chi}(1) = 2$.
(6) $G =< x, y, z \mid x^2 = y^3 = z^5 = xyz >\simeq SL(2,5)$ *and* $\chi(1) = \hat{\chi}(1) = 2$.
(7) $G = F_{21}$, *the Frobenius group of order* 21 *and* $2 \cdot \chi(1) = \hat{\chi}(1) = 6$.
(8) $G = \boldsymbol{Z}_n$, *a cyclic group of order* n *and* $2 \cdot \chi(1) = \hat{\chi}(1) = 2$.

Some related results concerning the decomposition of a power of a character are found in [4, 11, 12].

3 Sketch of Proof

In this section we sketch the proof of the special case of the main theorem, i.e., when both χ and ψ are irreducible.

Assume $\chi = \hat{\chi}$, $\psi = \hat{\psi} \in \mathrm{Irr}(G)$, with $n = \chi(1) > 3$. We may assume χ is faithful.

$$\chi^2 = 1 + a \cdot \chi + c \cdot \psi. \tag{1}$$

- $c = 1$.

  Proof. Since $\chi(1) > 1$, there is an element $g \in G$ such that $\chi(g) = 0$. Then we have that $1 + c \cdot \psi(g) = 0$. This implies $c = 1$ as $\psi(g)$ is an algebraic integer. ■

- $n = \chi(1)$ is odd and $\mathrm{Alt}^2\chi = ((n-1)/2)\chi$. In particular,

$$\chi(x^2) = \chi(x)^2 - (n-1)\chi(x). \tag{2}$$

  Proof. Since

$$\chi^2 = \mathrm{Sym}^2\chi + \mathrm{Alt}^2\chi = 1 + a \cdot \chi + \psi,$$

  we have the assertion by inspection. ■

- For every involution t of G, $\chi(t) = -1$.

  Proof. Since $\chi(t^2) = \chi(1) = n$, by (2) we have that

$$n = \chi(t)^2 - (n-1)\chi(t).$$

  Since χ is faithful, $\chi(t) \neq n$ and we have $\chi(t) = -1$. ■

- There is no element s of order 4.

  Proof. Let s be an element of order 4. Since $t = s^2$ is an involution,

$$\chi^{(2)}(s) = \chi(s^2) = \chi(t) = -1.$$

  Now (2) yields

$$\chi(s)^2 - (n-1)\chi(s) + 1 = 0. \tag{3}$$

  Since the 4-th roots of unity are $\pm\sqrt{-1}, \pm 1$ and χ is real valued, $\chi(s)$ is a rational integer. There is no integer solution for (3) as $n > 3$. ■

- Final contradiction.

  Proof. If $\psi(1) = 1$, then we have $n = \chi(1)$ divides 2 by (1) as $c = 1$. Let $g \in G$ with $\psi(g) = 0$. Since there is no element of order 4, g^2 and g^4 are odd order. Hence $\chi(g^2)$ and $\chi(g^4)$ are algebraic conjugate. By (1) and (2), we have the following.

$$\begin{aligned}
0 &= \chi(g)^2 - a \cdot \chi(g) - 1, \\
\chi(g^2) &= \chi(g)^2 - (n-1)\chi(g) \in \boldsymbol{Q}(\chi(g)) \subset \boldsymbol{Q}(\sqrt{a^2+4}), \\
\chi(g^4) &= \chi(g^2)^2 - (n-1)\chi(g^2) \in \boldsymbol{Q}(\chi(g)) \subset \boldsymbol{Q}(\sqrt{a^2+4}).
\end{aligned}$$

  From these equations, we obtain a nontrivial relation, which finally yields a contradiction. ■

4 Group Association Schemes and Representation

Let G be a finite group and let $C_0, C_1, \ldots, C_d$ be the conjugacy classes. Let

$$R_i = \{(g_1, g_2) \in G \times G \mid g_2 g_1{}^{-1} \in C_i\}.$$

Then $\mathcal{X}(G) = (G, \{R_i \mid i = 0, 1, \ldots, d\})$ becomes a commutative association scheme and $\mathcal{X}(G)$ is called a group association scheme. For a conjugacy class C, let $\hat{C} = \{g, g^{-1} \mid g \in C\}$. If we take $\hat{C}_i$ instead of C_i above, we obtain a symmetric association scheme $\widetilde{\mathcal{X}(G)}$, which is called the symmetrization of the group scheme.

It is well-known that the Bose-Mesner algebra of the group association scheme of a finite group G is isomorphic to $Z(\boldsymbol{C}[G])$, the center of the group algebra. Let $E_0, E_1, \ldots, E_d$ be the primitive idempotents of the group association scheme and let

$$E_i \circ E_j = \frac{1}{|G|} \sum_{h=0}^{d} q_{i,j}^h E_h.$$

Then these primitive idempotents correspond to irreducible modules and $q_{i,j}^h$ can be computed from the irreducible characters $\chi_0, \chi_1, \ldots, \chi_d$ of G by the following formula.

$$q_{i,j}^h = \frac{\chi_i(1)\chi_j(1)}{\chi_h(1)} < \chi_h, \chi_i \chi_j > .$$

The representation diagram of a commutative association scheme $\mathcal{X}(G) = (G, \{R_i \mid i = 0, 1, \ldots, d\})$ with respect to a primitive idempotent $E_h = \bar{E}_h$, denoted by $\mathcal{D}^* = \mathcal{D}^*(\mathcal{X}, E_h)$, is an undirected graph with $\{0, 1, \ldots, d\}$ as the vertex set such that the adjacency for distinct vertices i and j is defined as follows.

$$i \sim j \Leftrightarrow q_{h,i}^j \neq 0.$$

A commutative association scheme is said to be Q-polynomial if the representation diagram with respect to a primitive idempotent is a path.

The representation diagram of a group G with respect to a real valued character χ, denoted by $\mathcal{D}^* = \mathcal{D}^*(G, \chi)$, is an undirected graph with $\mathrm{Irr}(G)$ as the vertex set such that the adjacency for distinct vertices χ_i and χ_j is defined as follows.

$$\chi_i \sim \chi_j \Leftrightarrow < \chi\chi_i, \chi_j > \neq 0.$$

The real representation diagram $\widetilde{\mathcal{D}}^* = \widetilde{\mathcal{D}}^*(G, \chi)$ is defined similarly by taking $\mathrm{RIrr}(G)$ as the vertex set.

As remarked above the irreducible characters of a group G correspond to the primitive idempotents of the corresponding group association scheme. It is easy to see from the formula for $q_{h,i}^j$ above that the representation diagram of a group with respect to a real valued irreducible character is isomorphic to the corresponding representation diagram of its group association scheme. And each real representation diagram is isomorphic to the corresponding representation diagram of the symmetrization of its group association scheme.

Suppose $\mathcal{D}^* = \mathcal{D}^*(G, \chi)$ is a path, i.e., the corresponding group association scheme is Q-polynomial. Then the group association scheme is necessarily symmetric. So we consider the condition when the symmetrization of a group association scheme becomes Q-polynomial. Then $\hat{\chi}^2$ is a linear combination of a principal character 1, $\hat{\chi}$ and at most one more member of $\mathrm{RIrr}(G)$. Hence it satisfies the assumption of Theorem 2.1 and Corollary 2.2. Now by inspection, we have the following.

Theorem 4.1 *Let G be a finite group with $d+1$ conjugacy classes, and let $\widetilde{\mathcal{X}(G)}$ be the symmetrization of its group association scheme. If $\widetilde{\mathcal{X}(G)}$ is Q-polynomial, then one of the following holds.*
(1) $G \simeq \boldsymbol{Z}_n$, *the cyclic group of order n.*
(2) $G \simeq S_3$, *the symmetric group of degree 3.*
(3) $G \simeq A_4$, *the alternating group of degree 4.*
(4) $G = < x, y, z \mid x^2 = y^3 = z^3 = xyz > \simeq SL(2,3)$.
(5) $G \simeq F_{21}$, *the Frobenius group of order 21.*

5 Generalization

Using the similar methods of proof in Section 3, we can prove a generalization of Theorem 2.1 as follows.

Theorem 5.1 *Let χ, $\psi_1, \ldots, \psi_r \in \mathrm{Irr}(G)$. Suppose*

$$\chi^2 = 1 + a\chi + \psi_1 + \cdots + \psi_r$$

and $\{\psi_1, \ldots, \psi_r\}$ is a single orbit under the action of $Gal(\overline{\boldsymbol{Q}}/\boldsymbol{Q}(\chi))$, where $\overline{\boldsymbol{Q}}$ is the algebraic closure of $\boldsymbol{Q}$. If $n > 3$, then $G/\ker\chi \simeq GF(2^l) : GF(2^l)^*$, where $2^l - 1$ is a Mersenne prime and $GF(2^l)$ is the additive group of a field with 2^l elements and $GF(2^l)^*$ is the multiplicative group of the field. Here $X : Y$ means a semidirect product.

6 Some Remarks

1. Let $\{\chi(g) \mid g \in G,\ g \neq 1\} = \{\alpha_1, \alpha_2, \ldots, \alpha_s\}$. (G, χ) is said to be *sharp*, if

$$|G| = (\chi(1) - \alpha_1)(\chi(1) - \alpha_2) \cdots (\chi(1) - \alpha_s).$$

   Study relation with the condition 'sharp'. See [1, 6].

2. When a representation diagram of an association scheme becomes a tree, the embedding of the points of the scheme on a sphere enjoys a very nice geometrical condition called 'balanced condition'. Since a path is a tree, it defines a class of association schemes containing Q-polynomial association schemes. All finite irreducible subgroups in $GL(2, \boldsymbol{C})$ satisfy this property. Can we determine all such group association schemes?

3. All known balanced 2-groups have $\widetilde{D}_n$ or a star graphs as representation diagrams.

4. Study the dual.

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