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A family of group association schemes with the same intersection numbers

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We gave an example of a family of infinite pairs of non-isomorphic group association schemes with the same intersection numbers.

1 Introduction

Definitions (1) Let G be a finite group with conjugacy classes $\mathcal{C}_0=\{1\},\mathcal{C}_1,\ldots,\mathcal{C}_d$. The ***group association scheme*** $\mathcal{X}(G)$ for group G is a set G with relations $\{R(\mathcal{C}_i)\}_{i=0}^{d}$ defined by $(x,y)\in R(\mathcal{C}_i)$ iff $x^{-1}y\in \mathcal{C}_i$ for each i.

(2) Let G and H be finite groups with the same numbers of conjugacy classes $\{\mathcal{C}_i\}_{i=0}^{d}$ and $\{\mathcal{D}_i\}_{i=0}^{d}$ respectively. The group association schemes $\mathcal{X}(G)$ and $\mathcal{X}(H)$ are called ***isomorphic*** when there is a bijection from G to H which sends each relation $R(\mathcal{C}_i)$ to $R(\mathcal{D}_i)$ for each i.

(3) An ***automorphism*** of a group association scheme $\mathcal{X}(G)$ is an automorphism of the set G which sends each relation $R(\mathcal{C}_i)$ to $R(\mathcal{C}_i)$ for each i. The group of all automorphisms of $\mathcal{X}(G)$, denoted by $Aut(\mathcal{X}(G))$, is called the ***full automorphism group*** of $\mathcal{X}(G)$.

From the definition, isomorphic group association schemes have the same intersection numbers, but the converse is known to be false. The only known example is the group association schemes for extensions of $\mathbb{F}_2^3$ by $SL(3,2)$ found by Yoshiara [4].

We found a family of infinite pairs of non-isomorphic group association schemes with the same intersection numbers as follows:

Theorem 1.1 *Let q be any power of 2 greater than 8, V be the column vector space over $\mathbb{F}_q$ of degree 2. Set the group E_0 be the split extension of V by $SL(2,q)$, and E_1 be a non-split one.*

The group association schemes $\mathcal{X}(E_0)$ and $\mathcal{X}(E_1)$ have the same intersection numbers but $\mathcal{X}(E_0)\not\cong\mathcal{X}(E_1)$.

Bell shows that E_1 exists if $q\geq 8$. (See [1].)

Table 1: The irreducible characters of E_k

$1(k)$	$\mathcal{V}^\sharp(k)$	$\mathcal{T}_i(k)$ $(i=1,\dots,(q-2)/2)$	$\mathcal{S}_j(k)$ $(j=1,\dots,q/2)$	$\mathcal{U}_\beta(k)$ $(\beta\in\mathbf{F}_q)$
1	q^2-1	$q^3(q+1)$	$q^3(q-1)$	$q(q^2-1)$
1	1	1	1	1
q	q	1	-1	0
$q+1$	$q+1$	$\eta^i+\eta^{-i}$	0	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$q+1$	$q+1$	$\eta^{mi}+\eta^{-mi}$	0	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$q+1$	$q+1$	$\eta^{i(q-2)/2}+\eta^{-i(q-2)/2}$	0	1
$q-1$	$q-1$	0	$-(\xi^j+\xi^{-j})$	-1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$q-1$	$q-1$	0	$-(\xi^{nj}+\xi^{-nj})$	-1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$q-1$	$q-1$	0	$-(\xi^{jq/2}+\xi^{-jq/2})$	-1
(q^2-1)	-1	0	0	$K(\psi;\beta,b')(b'\in\mathbf{F}_q)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
(q^2-1)	-1	0	0	$K(\psi;\beta,b)(b\in\mathbf{F}_q)$

2 Outline of the Proof

About intersection numbers It is known that the group association schemes $\mathcal{X}(G)$ and $\mathcal{X}(H)$ have the same intersection numbers if and only if G and H have the same character tables ([2, (7.1), pp. 42–43]). The character table of E_k $(k=0,1)$ are those given in Table 1 ([3]), where the first row is class names, the second row is the size of class, η is a primitive $(q-1)$st-root of unity in $\mathbf{C}$, ξ is a primitive $(q+1)$st-root of unity in $\mathbf{C}$, and for the additive character $\psi:\ \mathbf{F}_q \ni \alpha \mapsto (-1)^{Tr_{\mathbf{F}_q/\mathbf{F}_2}(\alpha)} \in \mathbf{C}^\times$,

$$K(\psi;\beta,b) := \sum_{u\in\mathbf{F}_q^\times} \psi(u^{-\frac{1}{2}}\beta + bu).$$

The groups E_k have $2q+1$ irreducible characters, but their values do not depend on the choice of k. Hence $\mathcal{X}(E_0)$ and $\mathcal{X}(E_1)$ have the same intersection numbers.

About $\mathcal{X}(E_0)\not\cong\mathcal{X}(E_1)$ Let A be the stabilizer of the identity $1=(0,I_2)$ of E_0 in the full automorphism group $Aut(\mathcal{X}(E_0))$. The stabilizer $A=Aut(\mathcal{X}(E_0))_1$ acts on each conjugacy class of E_0 as A preserves each relation with the identity.

For conjugacy classes $\mathcal{C},\mathcal{D}$ of E_0 and $g\in E_0$, denote $\mathcal{C}(g,\mathcal{D})$ the set of elements $\mathcal{C}$

which are adjacent to g in the $R(\mathcal{D})$-graph:

$$\mathcal{C}(g,\mathcal{D}) := \{h \in \mathcal{C} | (g,h) \in R(\mathcal{D})\} = \{h \in \mathcal{C} | g^{-1}h \in \mathcal{D}\}.$$

We consider the equivalence relation on the conjugacy class $\mathcal{V}^\sharp$ defined as follows:

For $g, h \in \mathcal{V}^\sharp$, g and h are equivalent when $\mathcal{U}_0(g, \mathcal{U}_0) = \mathcal{U}_0(h, \mathcal{U}_0)$.

We see that there are $q+1$ equivalence classes parametrized by the 1-dimensional subspaces of V. Each equivalence class consists of $q-1$ elements of the shape $(\mathbf{v}, I)$, where $\mathbf{v}$ ranges over the nonzero vectors of a 1-dimensional subspace of V.

Let $\Delta_0 = \{({}^T[\alpha,\ \alpha], I) | \alpha \in \mathbf{F}_q^\times\}$ be the equivalence class corresponding to the 1-subspace spanned by ${}^T[1,\ 1]$, and let Δ_i $(i = 1, \ldots, q)$ be the other classes. Define Δ to be the set of the equivalence classes Δ_i $(i = 0, 1, \ldots, q)$. Since A preserves each conjugacy classes, A preserves the above equivalence relation on $\mathcal{V}^\sharp$ and hence acts on Δ. This action is triply transitive since $Inn((\mathbf{0}, M))(\mathbf{v}, I) = (M\mathbf{v}, I)$ for $M \in SL(2,q)$, $\mathbf{v} \in V$.

Let N be the kernel of the action of A on Δ:

$$N := \{\sigma \in A | \sigma(\Delta_j) = \Delta_j \text{ for any } j = 0, \ldots, q\}.$$

The following propositions hold.

Proposition 2.1 *We have $N = \{Inn(\mathbf{v}) | \mathbf{v} \in V\} \times \langle\iota\rangle$, where ι is the automorphism inverting each element of E_0.* ■

Proposition 2.2 *If $q \geq 8$, then A/N has the normal subgroup $Inn(E_0)N/N$ which is isomorphic to $SL(2,q)$, and is isomorphic to the normal subgroup of $Aut(SL(2,q))$.* ■

Proof of Theorem Assume $\mathcal{X}(E_0) \cong \mathcal{X}(E_1)$. Then $Aut(\mathcal{X}(E_0)) \cong Aut(\mathcal{X}(E_1))$ and hence

$$A = Aut(\mathcal{X}(E_0))_1 \cong Aut(\mathcal{X}(E_1))_{1'} \geq Inn(E_1) \cong E_1,$$

where $1'$ is the identity of E_1, since the action of $Aut(\mathcal{X}(E_1))$ on E_1 is transitive. By Proposition 2.2, $A/Inn(E_0)N$ is a cyclic group. From Proposition 2.1, we have the commutator group $A' \leq Inn(E_0)N = Inn(E_0)\langle\iota\rangle$. Thus the second commutator group A'' is isomorphic to a subgroup of $Inn(E_0) \cong E_0$, however, A'' has a subgroup which is isomorphic to $E_1'' = E_1$ from the above argument. Comparing the orders, we have $E_0 \cong E_1$ and this is a contradiction. ■

Remark In this note, the proof of Proposition 2.2 is shortened, however, it uses the classification of doubly transitive groups.

After this conference, this Theorem had proved without the classification of doubly transitive groups. It uses only the classification of Zassenhaus groups. (The structure of the full automorphism is as the form $Aut(\mathcal{X}(E_0)) \cong (E_0 \times E_0){:}2$.)

References

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