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Extended Generalized Quadrangles

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1 Introduction

The aim of my talk is to describe some geometric constructions of extended generalized quadrangles with non-classical point residues and to give several problems related to them. The contents other than the higher dimensional dual arcs are also explained in the nice survey by Del Fra and Pasini [DP].

Recall that an incidence structure (geometry) $(\mathcal{P}, \mathcal{L}; *)$ of rank 2, in which elements of $\mathcal{P}$ and $\mathcal{L}$ are referred to as *points* and *lines* respectively, is called a *generalized quadrangle* (GQ for short) of order (s, t), if the following conditions hold (here the phrase "a point P lies on a line l" or "l goes through P" are used if $P * l$, and we say that two points are *collinear* if there is a line through them):

(GQ1) There is at most one line through two distinct points, and (dually) there is at most one point lying on two distinct lines.

(GQ2) For a point P not lying on a line l, there is a unique point Q on l collinear with P.

(GQ3) There are exactly $s+1$ points (resp. $t+1$ lines) on a line (resp. through a point).

The GQs belong to a class of geometry, called *buildings*, of rank 2 (or *generalized polygons*) and hence they often appear as a "building block" of a larger geometry admitting many automorphisms. As for generalized quadrangles with s, t finite, the standard text is [PT]. See [Th1] and [vM] for generalized polygons.

We recall two more definitions. A geometry of rank 2 is called a *generalized diagon* if each point is incident to all lines. A geometry of rank 2 is called a *circle geometry*, if each line is incident to exactly two points and if every pair of points is incident to a unique line. (This is equivalent to say that the geometry is isomorphic to the geometry of the sets of vertices and edges of a complete graph with incidence given by inclusion.)

To define extended generalized quadrangles, we need the notion of "residues". Let $(\mathcal{P}, \mathcal{L}, \mathcal{U}; *)$ be a geometry of rank 3, in which elements of $\mathcal{P}$, $\mathcal{L}$ and $\mathcal{U}$ are called *points*, *lines* and *planes* respectively. For a point p, consider the geometry $Res(p) = (\mathcal{L}(p), \mathcal{U}(p); *')$ of rank 2 consisting of the ordered pair of the sets $\mathcal{L}(p)$ and $\mathcal{U}(p)$ of lines and planes incident to p, respectively, with the incidence $*'$ inherited from $*$. This geometry is called the *residue* at a point p. Similarly, we define the *residue* $Res(l) = (\mathcal{P}(l), \mathcal{U}(l); *')$ at a line l and the *residue* $Res(u) = (\mathcal{P}(u), \mathcal{L}(u); *')$ at a plane u.

A geometry $(\mathcal{P},\mathcal{L},\mathcal{U};*)$ of rank 3 is called an *extended generalized quadrangles of order* (s,t) ($EGQ(s,t)$ for short) if its residues have the following structures:

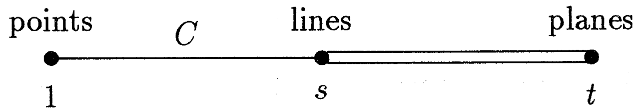
(EGQ1) At each point p, the residue $Res(p) = (\mathcal{L}(p),\mathcal{U}(p);*')$ is a GQ of order (s,t).

(EGQ2) At each line l, the residue $Res(l) = (\mathcal{P}(l),\mathcal{U}(l);*')$ is a generalized digon.

(EGQ3) At each plane u, the residue $Res(u) = (\mathcal{P}(u),\mathcal{L}(u);*')$ is a circle geometry.

These conditions are usually represented by the following diagram.

$(c.C_2)$



It is easy to see that $Res(u)$ consists of $s+2$ points, and hence the residues at planes are isomorphic to each other. The residue at a line consists of 2 points and $t+1$ planes, and so the resideus at lines are isomorphic. However, note that $Res(p)$ may not be isomorphic to $Res(p')$ for distinct points p, p', though they are GQs having the same order. (You will see in later sections that this in fact happens in some EGQs.)

If the point residues are isomorphic to a fixed GQ X, we also say that the EGQ is an *extension* of X.

Several EGQs are known to admit the automorphism groups which acts transitively on the maximal flags and are isomorphic to sporadic simple groups. In fact, EGQs admitting flag-transitive automorphism groups (that is, transitive on the maximal flags) are classified, assuming that the point residues are classical GQs (that is, GQs consisting of the isotropic points and lines of a vector space of a projective Witt index 1 with respect to a non-degenerate symplectic, orthogonal or unitary form, see [PT, 3.1.1]). This result has been established by combining the two joint papers, one by Del Fra, Ghinelli, Meixner and Pasini [DGMP], and the other by Weiss and the author [WY]. There are 13 such EGQs up to isomorphism, and among their automorphism groups the sporadic simple groups McL, Suz and HS (in fact $Aut(HS)$) appear.

EGQs can be extended to towers of circular extensions of GQs, among which flag-transitive geometries for the Fischer groups F_{2i} ($i = 2,3,4$) are of great interest. They are almost classified by T. Meixner [Me] under the assumption of flag-transitivity. Later Pasechnik [Pase] gives a combinatorial characterizations of those geometries related to the Fischer groups under suitable addtional conditions. (See [BP, 4.3.5 and 4.3.6] for the complete results including classical flag-transitive EGQs.) The author initiated the project of classifying flag-transitive extended dual polar spaces [Yo1],[Yo2], which is now almost completed by the contributions of A. A. Ivanov, G. Stroth and U. Meierfrankenfeldt [Iv1],[Iv2],[IS],[IM]. The sporadic simple groups $Co2$, F_{22}, F_{23}, F'_{24} and $\mathbf{M}$ appear as their automorphism groups. (See also [BP, 4.4.3 and 4.4.3], though the information there is a bit old.)

In a geometry of rank higher than 3 with an EGQ as its residue, the GQ are usually classical. Thus for those who are interested in "extension" of EGQ of rank ≥ 4 only, EGQs with non-classical point residues may be negligible. However, even flag-transitive such EGQs exist. There are four known examples of non-classical GQs with flag-transitive automorphisms: With the notation [PT, 3.1.3], they are $T_2^*(O)$ for O the classical hyperoval O_4 in the desarguesian projective plane $PG(2, \mathbf{F}_4)$ (in this case the GQ is of order $(3,5)$) and for O the Lunelli-Sce hyperoval O_{LS} in $PG(2, \mathbf{F}_{16})$ (the GQ is of order $(15,17)$), together with their duals. In [Yo3] and [Yo4], the author classified the flag-transitive extensions of the GQ $T_2^*(O_4)$ or its dual. Three simply connected new geometries were found up to isomorphism: one EGQ Γ with point-residues $T_2^*(O_4)$ and two EGQs Γ' and Γ'' with point-residues the dual of the GQ $T_2^*(O_4)$. Their automorphism groups are $Aut(\Gamma) \cong 2_+^{1+12}.(A_5 \times A_5).2$, $Aut(\Gamma') \cong 2^{12}.L_3(2)$ and $Aut(\Gamma'') \cong 2_+^{1+12}.3 \cdot A_7$. (See also [BP, 4.3.7].)

The constructions of EGQs Γ, Γ' and Γ'' given in [Yo3] were not geometric, but were given as coset geometries. So their geometric constructions naturally arise as problems. Recently, Del Fra, Pasechnik and Pasini [DPP] gave a neat construction of a family of EGQ of order $(q-1, q+1)$ for $q = 2^e$ for every $e \geq 1$. For $q = 4$, the geometry coincides with the EGQ Γ above. I also constructed a family of EGQ of order $(q+1, q-1)$ for $q = 2^e$ for every $e \geq 1$, starting from a family $\mathcal{Y}$ of projective planes in $PG(5, q)$ with certain properties [Yo5]. For $q = 4$, the geometry coincides with the EGQ Γ'' above.

These constructions together with those of $EGQ(q-1, q+1)$ with q odd will be described later. The higher dimensional dual arcs will also be discussed, since it seems to be a promising object to study as a generalization of the notion of a family $\mathcal{Y}$ above.

Before concluding the introduction, let me give a reason why I do not consider here "extended polygons" in general, though there are many interesting finite examples are known by Weiss et al. (see the bibliography in [BP][1]). Because, a result of Pasini [Pa1] gives an upper bound $s+2$ for the diameter of the collinearity graph of an EGQ of order (s,t), while the universal cover of the extension of a generalized m-gon for $m \geq 6$ is shown to be infinite by Ronan [Ro]. Hence the locally finiteness (the finiteness of s and t) implies the global finiteness of every EGQ of order (s,t), and they are easier to treat.

I conclude this section with the following problems:

Problem 1. Is there any family of EGQs which contains Γ' above as a member?

Problem 2. Classify the flag-transitive $EGQ(15,17)$s with point-residues $T_2^*(O_{LS})$. [2]

Problem 3. Classify the flag-transitive EGQs without assuming the point residues are classical. [3]

[1] Unfortunately, all my papers concernig geometry have been left out there, except one.

[2] It is easy to see that flag-transitive extensions of the dual of $T_2^*(O_{LS})$ do not exist. Professor Nakagawa at Kinki University determined the possible amalgams of parabolics, but the present capacity of memories of computers seems to be not sufficient for determining thier universal completions via coset enumeration.

[3] This seems rather tough as it contains a solution of Problem 2. But I am feeling that we now have all methods and tools to solve this problem, if we include the classification of doubly transitive groups. I have some partial unpublished results which restrict the possible doubly transitive permutation groups induced on a plane. I hope I can solve this problem in some near future.

2 Families of $EGQ(q-1,q+1)$s

Several classes of non-classical GQs are known [PT, Chap.3], among which the Tits GQ $T^*(O)$ and the Ahrens-Szekeres GQ $AS(q)$ are of order $(q-1,q+1)$. They can be obtained by a general construction of Payne from a GQ of order (q,q) (with a regular point) [PT, 3.1.4,3.2.6]. [4] In this talk, I do not attempt to consider the extensions of larger class of GQs of order $(s-1,s+1)$ in general, but those of $T^*(O)$ and $AS(q)$ only. Thus

Problem 4. Construct an extension of the GQ $P(\mathcal{S},x)$ of order $(s-1,s+1)$ (with the notation of [PT, 3.1.4]) which is not $T_2^*(O)$ or $AS(q)$.

A presentation of $AS(q)$ as $P(W(q),p)$ will be given later in Subsection 3.1. Here we review a presentation of $T^*(O)$ given by Tits [PT, 3.1.3], where O stands for a *hyperoval* in $PG(2,q)$, $q=2^e$, that is, the set of $q+2$ points of $PG(2,q)$ no three of which lie on a line in common. Embedd $PG(2,q)$ in $PG(3,q)$, and define the *points* of $T_2^*(O)$ to be the affine points of $Af(3,q)$, that is, the points of $PG(3,q)$ outside $PG(2,q)$. The *lines* are defined to be the lines of $PG(3,q)$ outside $PG(2,q)$ but intersect $PG(2,q)$ in one point of O. The incidence inherited from $PG(3,q)$. It is easy to see the resulting geometry forms a GQ of order $(q-1,q+1)$.

2.1 A family of $EGQ(q-1,q+1)$s, q even

There are two known families of $EGQ(q-1,q+1)$, one by Del Fra, Pasechnik and Pasini [DPP] and the other by Pasini [Pa2]. The latter is *flat*, that is, every point is incident to every plane, and related to the notion of "tube". See [DP, 2.4],[Pa2] for the detail. Here I briefly introduce the construction of the former family. For the detail, see [DPP].

Let u_∞ be a plane of $PG(4,q)$, $q=2^e$, and choose a hyperoval O on u_∞ and a line l_∞ on u_∞ which does not intersect O. The "upper" residue $Res^+(l_\infty)$ at l_∞ in $PG(4,q)$ consists of the projective planes and 3-subspaces containing l_∞, which is isomorphic to the projective plane $PG(2,q)$. Thus we may choose a hyperoval O^* in $Res^+(l_\infty)$ containing u_∞. In $PG(4,q)$, O^* is a collection of $q+2$ planes u_∞, u_i $(i=0,\dots,q)$ through l_∞ such that no three of them are contained in a 3-subspace.

Let $\Gamma(O,O^*)$ be the geometry in which the *points*, *lines* and *planes* are the projective points on $\cup_{i=0}^q u_i$ but not on l_∞, the lines of $PG(4,q)$ skew to u_∞ but intersecting at least one memeber of O^*, and the planes of $PG(4,q)$ intersecting u_∞ in exactly a point on O, respectively. The incidence inherited from $PG(4,q)$.

Each line of $\Gamma(O,O^*)$ intersects exactly two planes of O^*, as O^* is a hyperoval in $Res^+(l_\infty)$. Then the residue at a plane is a circle geometry on $q+3$ vertices. It is also easy to see that the residue at a point is $T_2^*(O)$. Thus $\Gamma(O,O^*)$ is an extension of $T_2^*(O)$.

When $q=4$, it can be shown that $\Gamma(O,O^*)$ does not depend on the choice of O, O^* and l_∞. It admits a flag-transitive automorphism group, and therefore coincides with the geometry Γ with $Aut(\Gamma)\cong 2_+^{1+12}.(A_5\times A_5).2$ in the introduction. It is known that $\Gamma(O,O^*)$

[4] See the discussion in [PT, pp.47–48]. Note that in the statement of [PT, 3.2.6], $T_2(O)$ should be $T_2^*(O)$.

is not flag-transitive for $q > 4$ [DPP, 2.3]. The universal cover of $\Gamma(O, O^*)$ has not yet been determined if $q > 4$. However, the collinearity graph of $\Gamma(O, O^*)$ is a complete $(q+1)$-partite graph with parts $u_i \setminus l_\infty$ $(i = 0, \ldots, q)$, and hence if O and O^* are classical we may apply the algebraic construction by Cameron [Ca] to get a $q/2$-cover of $\Gamma(O, O^*)$ by finding a suitable 2-cocycle to $\mathbf{F}_q/\mathbf{F}_2$ [DPP, §3].

Problem 5. Find the uiversal cover of $\Gamma(O, O^*)$, assuming that O and O^* are classical.[5]

2.2 A family of $EGQ(q-1, q+1)$s, q odd

Thas [Th3] constructed a family of extensions of $AS(q)$ for q odd prime power, but this is a one-point extension (the number of points increases by just one). This geometry allows an algebraic construction of $(q+1)/2$-cover via [Ca], which turns out to be isomorphic to a family of extensions of $AS(q)$ constructed by Kashikova and Shult [DP, Prop.1]. I leave the details (see [DP, 2.2]), but just give two questions:

Problem 6. Determine the universal cover of Thas's family of extensions of $AS(q)$.

Problem 7. Is there another way of constructing a family of extensions of $AS(q)$?

3 Families of $EGQ(q+1, q-1)$s

3.1 A family of $EGQ(q+1, q-1)$s, q odd

For odd q, so far only one family of extensions of $AS(q)$ (for the detail, see [DP]) is known. The GQ $AS(q)$ can be constructed from the symplectic classical GQ $W(q)$ of order (q, q) as follows ([PT, 3.1.4,3.1.5,3.2.6]): Choose a point p of $W(q)$, and delete every point collinear with p and every line through p. The *points* of $AS(q)$ are the remaining $(q+1)(q^2+1) - 1 - q(q+1) = q^3$ points of $W(q)$, and the set of *lines* of $AS(q)$ consists of the remaining $(q+1)(q^2+1) - (q+1) = q^3 + q^2$ "old" lines together with $q^3/q = q^2$ "hyperbolic" lines $\{p, x\}^{\perp\perp}$ for points x not collinear with p. Incidence is given by inclusion.

The resulting GQ $(\mathcal{P}, \mathcal{L})$ admits a *parallelism*, that is, a map f from the set $\mathcal{L}$ of lines to the set of indicies $\{0, \ldots, q+1\}$ whose restriction onto the residue at each point is a bijection. In fact, taking the lines l_i $(i = 1, \ldots, q+1)$ of $W(q)$ through p, the map f can be given by assigning 0 to each hyperbolic line and the number i to each old line intersecting l_i (note that each line of $W(q)$ not passing through p intersect a unique line through p).

As q is odd, the circle geometry $(\mathcal{P}', \mathcal{L}')$ of vertices and edges of the complete graph with $q+3$ vertices also admits a parallelism f' onto $\{0, \ldots, q+1\}$ (so called the 1-factorization). Then we can apply the technique called "*glueing*", developped by Buekenhout, Huybrechts and Pasini [BHP], to produce an extension $Gl(q)$ of $AS(q)$. Explicitly, take $\mathcal{P}'$ and $\mathcal{P}$ as the sets of points and planes of $Gl(q)$ and define the pairs (l', l) of lines $l' \in \mathcal{L}'$ and $l \in \mathcal{L}$ with

[5] We need a nice description of the collinearity graph of the $q/2$-cover, for otherwise it seems complicated to apply the standard strategy of decomposing based cycles into triangles etc.

$f'(l') = f(l)$ as the lines of $Gl(q)$. Every point of $Gl(q)$ is defined to be incident to every plane of $Gl(q)$. A point or plane x is incident to a line (l', l) whenever x is incident to l' or l in $(\mathcal{P}', \mathcal{L}')$ or $(\mathcal{P}, \mathcal{L})$.

We can show that $Gl(q)$ is an extension of the dual of $AS(q)$. Details on the construction of $Gl(q)$ can be found in [DP, §4]. Unfortunately, the guled geometry $Gl(q)$ is flat by the definition. Natural questions arize:

Problem 8. Determine the universal cover of $Gl(q)$.

Problem 9. Is there any other construction of the extension of the dual of $AS(q)$ which is not flat?

3.2 Y-family

In [Yo5] an infinite family of $EGQ(q+1, q-1)$s was constructed using the following family of planes of $PG(5, q)$.

Definition [Yo5] A *Y-family* is a set $\mathcal{S}$ of $q+3$ projective planes in $PG(5, q)$ with the following properties:

(i) $X \cap Y$ is a projective point for every distinct members X, Y of $\mathcal{S}$, and $X \cap Y \cap Z = \emptyset$ for three distinct members of $\mathcal{S}$.

(ii) For each $X \in \mathcal{S}$, the $q+2$ points $X \cap Y$ $(Y \in \mathcal{S},\ Y \neq X)$ form a hyperoval in X.

(iii) The members of $\mathcal{S}$ span $PG(5, q)$.

The condition (i) implies that the points $X \cap Y$ for $Y \in \mathcal{S} - \{X\}$ are mutually distinct and the condition (ii) makes sense. Since X contains a hyperoval, $q = 2^e$ for some e.

Two constructions of Y-families are known, one by the author using the "Veronesean map" and the other by Thas using "the Klein correspondence".

Construction via Veronesean. Recall that the *Veronesean* map ξ is a map from the projective points of $PG(2, q)$ to the points of $PG(5, q)$ defined by

$$[x_0, x_1, x_2] \mapsto [x_0^2, x_1^2, x_2^2, x_0x_1, x_0x_2, x_1x_2].$$

The following facts are well known [HT, 25.1]: The image by ξ of a projective line l of $PG(2, q)$ is a conic (the set of zeros of a non-degenerate quadratic form) on a plane (called *the conic plane* and denoted $\Pi(l)$) in $PG(5, q)$ [HT, the middle of p.150]. For each line l of $PG(2, q)$, the *nucleus* of the conic $\xi(l)$ on the plane $\Pi(l)$ (the point of intersections of all tangent lines to $\xi(l)$) lies on a plane in common, called the *nucleus plane* of the Veronesean $\xi(PG(2, q))$ [HT, the last claim of 25.1.17]. For two distinct lines l, m of $PG(2, q)$, the conic planes $\Pi(l)$ and $\Pi(m)$ intersect in exactly one point [HT, 25.1.11]. Moreover, if three distinct lines of $PG(2, q)$ are not concurrent, there is no point of $PG(5, q)$ contained in the corresponding three conic planes in common. Since a conic together with its nucleus forms a classical hyperoval, the image $\xi(O^*)$ of an arbitrary dual hyperoval O^* of $PG(2,q)$ (the set of $q+2$ lines no three of which are concurrent) satisfies the conditions (i), (ii),(iii) in the definition of a Y-family, in which the hyperoval of intersections on each plane is classical. Now add the nucleus plane N as the $(q+3)$-rd plane. Then we may verify that $\mathcal{Y}(O^*) := \xi(O^*) \cup \{N\}$ forms a Y-family, in which the hyperoval of intersections on the nucleus plane is isomorphic to the hyperoval O dual to O^*. (See [Yo5, Sec.3].)

Construction via Klein correspondence. Immediately after my talk at Assisi conference (September 13, 1996), Jeff Thas suggested me another construction of a Y-family using the Klein correspondence [Th2][6]. Recall that a Klein correspondence is a map κ from the set of lines of $PG(3,q)$ to the set of points of $PG(5,q)$ defined by

$$\begin{aligned}&\langle [a_0,a_1,a_2,a_3],[b_0,b_1,b_2,b_3]\rangle \mapsto \\ &[a_0b_1 - a_1b_0, a_0b_2 - a_2b_0, a_0b_3 - a_3b_0, a_1b_2 - a_1b_2, a_3b_1 - a_1b_3, a_2b_3 - a_3b_2]\end{aligned}$$

The images by κ of all lines through a given point p of $PG(3,q)$ (resp. lying on a given plane Π of $PG(3,q)$) span a totally singular plane, denoted $\kappa(p)$ (resp. $\kappa(\Pi)$), of $PG(5,q)$ with respect to the quadratic form $x_0x_5 - x_1x_4 + x_2x_3$.

Choose a $(q+1)$-*arc* $\mathcal{A}$ in $PG(3,q)$, that is, the set of $q+1$ points no four of which lie on a plane. It is known [Th1, Theorem 7] that up to projective semilinear transformations that $\mathcal{A}$ has the following canonical form for some m, $1 \le m \le e$, prime to e (recall $q = 2^e$).

$$\mathcal{A}_m = \{P_\infty, \dots, P_t | t \in \mathbf{F}_q\},\ P_\infty = [0,0,0,1],\ P_t = [1,t,t^{2^m},t^{2^m+1}].$$

We have $q+1$ planes $X_i := \kappa(P_i)$ ($i = \infty$ or $i \in \mathbf{F}_q$) in $PG(5,q)$. There are two parallel classes $\mathcal{M} = \{m_i\}$ and $\mathcal{N} = \{n_i\}$ of lines of $PG(3,q)$ such that m_i and n_i passing through the point P_i and $m_i \cap n_j \neq \emptyset$ iff $i = j$ $(i,j \in \{\infty\} \cup \mathbf{F}_q)$. The class $\mathcal{M}$ does not lie on a plane, but the image $\kappa(\mathcal{M})$ span a (non-singular) plane X_{q+2} of $PG(5,q)$, similarly $X_{q+3} := \langle \kappa(\mathcal{N})\rangle$. Then we may verify that $\mathcal{Y}(\mathcal{A}) := \{X_i, X_{q+2}, X_{q+3} | i \in \{\infty\} \cup \mathbf{F}_q\}$ forms a Y-family, in which the hyperoval of intersections on X_i is classical for $i \in \{\infty\} \cup \mathbf{F}_q$, whereas that for X_{q+2} or X_{q+3} is the hyperoval $D(2^m)$ of Segre [Th1, p.299].

The two Y-families $\mathcal{Y}(O^*)$ and $\mathcal{Y}(\mathcal{A}_m)$ above coincide only when all hyperovals of intersections are classical, or equivalently, $\mathcal{Y}(O^*)$ when O^* is classical and $m = 1$ (or $m = e-1$, but $\mathcal{A}_1$ is isomorphic to $\mathcal{A}_{e-1}$). Note that the possible classes of hyperovals are restricted for $\mathcal{Y}(\mathcal{A}_m)$, while an arbitrary hyperoval O (the dual of O^*) can be chosen for $\mathcal{Y}(O^*)$.

Problem 10. Is there another construction of Y-families? Is it possible to classify them (under suitable assumptions, if necessarily)?

[6] He drew a picture on a sheet showing the schedule of talks to explain his idea. Based on it Antonio Pasini made a beautiful painting which now Cecil Huybrechts holds.

3.3 A family of $EGQ(q+1, q-1)$s, q even

Let $\mathcal{Y}$ be a Y-family. We embedd $PG(5,q)$ as a hyperplane of $PG(6,q)$, and define the following geometry $A(\mathcal{Y})$ (*affine expansion* of $\mathcal{Y}$):

the *points* := the 3-spaces $\langle f, X\rangle$, for $\langle f\rangle \in PG(6,q) \setminus PG(5,q)$, $X \in \mathcal{Y}$,
the *lines* := the lines $\langle f, X \cap Y\rangle$, for $\langle f\rangle \in PG(6,q) \setminus PG(5,q)$, $X \neq Y \in \mathcal{Y}$,
the *planes* := the affine points $\langle f\rangle$, $\langle f\rangle \in PG(6,q) \setminus PG(5,q)$,
the incidence inherited from $PG(6,q)$.

It is easy to see that the residue at a plane is the circle geometry with $q+3$ vertices and that the residue at a point $\langle f, X\rangle$ is the dual of the Tits GQ $T_2^*(O(X))$, where $O(X)$ is the hyperoval of intersections $X \cap Y$ ($Y \in \mathcal{Y}$, $X \neq Y$) on the plane X. Thus the geometry $A(\mathcal{Y})$ is an EGQ of order $(q+1, q-1)$, but in general there are several isomorphism types of point residues. For $\mathcal{Y} = \mathcal{Y}(O^*)$ with a given dual hyperoval O^* in $PG(2,q)$, the residue at $\langle f, X\rangle$ is the dual of $T_2^*(O(X))$ for the classical hyperoval $O(X)$, if X is not the nucleus plane of the Veronesean. If X is the nucleus plane, the residue at $\langle f, X\rangle$ is the dual of $T_2^*(O)$ for O the dual of O^*. For $\mathcal{Y} = \mathcal{Y}(\mathcal{A}_m)$, the residue at a point $\langle f, X\rangle$ with X the image of a point of the arc $\mathcal{A}$ under the Klein correspondence is the dual of the Tits GQ for the classical hyperoval. If X corresponds to one of the two parallel classes $\mathcal{M}$ and $\mathcal{N}$, the residue is the dual of the Tits GQ for the Segre hyperoval.

The above remark suggests that the EGQ $A(\mathcal{Y})$ is not flag-transitive unless all hyperovals $O(X)$ are classical. In fact, we can show that $A(\mathcal{Y})$ admits a flag-transitive automorphism group if and only if $q = 2$ or $q = 4$ (in these cases, the hyperovals in $PG(2,q)$ are classical). The former is simply connected, and is the double over of an EGQ known as the Fisher-Cameron extension. The latter is not simply connected, but its double cover is the simply connected EGQ Γ'' with $Aut(\Gamma'') \cong 2^{1+12}3A_7$ in the introduction.

Although the universal covers of the families of EGQs I introduced before have not yet been determined, the universal covers of $A(\mathcal{Y})$ for known Y-families $\mathcal{Y}$ are determined by the author [Yo6]. The strategy adopted there is standard, but the point is to succeed to reduce the calculation of the fundamental group of $A(\mathcal{Y})$ to that of an arithmetic in finite fields. Take a point $\langle f, X\rangle$ as the base point for the fundamental group, where X is the nucleus plane if $\mathcal{Y} = \mathcal{Y}(O^*)$ and the plane corresponding to $\mathcal{M}$ or $\mathcal{N}$ if $\mathcal{Y} = \mathcal{Y}(\mathcal{A}_m)$. Let O be the hyperoval of intersections on X. Up to a semilinear transformation, we may assume that $O = \{[1, t, f(t)] | t \in \mathbf{F}_q\} \cup \{[0,1,0]\}$ for some permutation polynomial $f(t)$. A result in [Yo6] claims that if the values $tf(t)$ generate the additive group $\mathbf{F}_q$, then the EGQ $A(\mathcal{Y})$ is simply connected. Applying this, we can establish the simple connectedness of $A(\mathcal{Y}(\mathcal{A}_m))$ and $A(\mathcal{Y}(O^*))$ for every possible m and known classes of hyperovals O^*, except when $q = 4$ (and possibly for O Payne's hyperovals). For $q = 4$, the hyperoval is classical, so we may take $f(t) = t^2$, but $tf(t) = t^3$ only generate $\mathbf{F}_2$, not $\mathbf{F}_4$.

What is the meaning of exception $q = 4$? In 1996, I observed that the double cover of $A(\mathcal{Y}(O_4))$, O_4 the classical hyperoval in $PG(2,4)$, can be realized in $PG(7,4)$ so that the "base" X of each point is isotropic with respect to a unitary form. Pasini later gave a much more general observation:

Definition. A Y-family $\mathcal{Y}$ is called *of polar type* if there is a polar space Π of rank 3 (so $\Pi \cong Q_5^+(q)$, $S_5(q)$ or $H_5(q)$) in which each member of $\mathcal{Y}$ is totally isotropic (singular).

With this terminology, Pasini showed that for a Y-family $\mathcal{Y}$ of polar type $S_5(q)$ (resp. $H_5(q)$) the EGQ $A(\mathcal{Y})$ admits a q (resp. $\sqrt{q}$)-fold cover. I end up with the following question.

Problem 11. Find a Y-family of polar type except that in $H_5(4)$, or classify Y-families of polar type, if possible.

4 Higher dimensional dual Arcs

The notion of Y-families in the last section can be generalized as follows:

Definition. A family $\mathcal{S}$ of d-(projective) dimensional subspaces of $PG(m,q)$ is called a *d-dimensional dual arc* if the following four conditions hold:

(a) Every point of $PG(m,q)$ is contained in at most two members of $\mathcal{S}$.

(b) $X \cap Y$ is a projective point for every distinct members X, Y of $\mathcal{S}$.

(c) For each $X \in \mathcal{S}$, the projective points $X \cap Y$ $(Y \in \mathcal{S}, Y \neq X)$ span X.

(d) The members of $\mathcal{S}$ span $PG(m,q)$.

It is immediate to see that a Y-family satisfies the conditions (a)–(d), and that a Y-family is a 2-dimensional dual arc in $PG(5,q)$ with $q+3$ members.

The conditions (b) and (c) imply that a fixed d-space X of $\mathcal{S}$ contains at least $d+1$ projective points which are intersections of two distinct members of $\mathcal{S}$. In particular,

$$|\mathcal{S}| \geq d+2.$$

On the other hand, it follows from the conditions (a) and (b) that for each projective point p on a fixed d-subspace X of $\mathcal{S}$ there is at most one member of $\mathcal{S}$ distinct from X such that $p = X \cap Y$. Thus the following upper bound for $|\mathcal{S}|$ is obtained:

$$|\mathcal{S}| \leq 1 + \tfrac{q^{d+1}-1}{q-1} = q^d + q^{d-1} + \cdots + q + 2.$$

The equality in the above upper bound holds exactly when $\mathcal{S}$ satisfies the conditions (b)–(d) and the following condition (a'):

(a') Every point of $PG(m,q)$ is contained in exactly 0 or 2 members of $\mathcal{S}$.

When this condition together with (b)–(c) holds, the dual arc $\mathcal{S}$ is called a d-dimensional *dual hyperoval* in $PG(m,q)$. It is a d-dimensional dual arc in $PG(m,q)$ with the maximum number $q^d + q^{d-1} + \cdots + q + 2$ of subspaces.

Examples (1) Consider the case $d = 1$. Given two distinct members X_1 and X_2 of $\mathcal{S}$, every member X intersects X_1 and X_2 at distinct points, and hence the 1-space X is spanned by $X \cap X_1$ and $X \cap X_2$. Thus $PG(m, q)$, the span of memebers of $\mathcal{S}$, is in fact spanned by X_1 and X_2, and we get $d = 2$. Furthermore, the condition (a) implies there is no three distinct concurrent lines of $\mathcal{S}$. Thus a 1-dimensional dual hyperoval in $PG(m, q)$ exists only when $m = 2$, and the notion of 1-dimesional dual hyperovals coincides with the usual dual hyperoval in $PG(2, q)$. This gives the reason why we use the terminology "hyperoval".

(2) I was informed from Antonio Pasini (February, 1998) that for every d, Thas constructed d-dimensional dual arcs in $PG(2d, 2)$.

(3) Inside the 5-dimensional projective space $PG(5, 4)$ over $\mathbf{F}_4$ with a non-degenerate hermetian form h, a 2-dimensional dual hyperoval $\mathcal{U}$ exists in which every member is a maximal totally isotropic subspace with respect to h. Furthermore, the subgroup $G_{\mathcal{U}}$ of $SU_6(4)$ stabilizing the set $\mathcal{U}$ of 22 planes is isomorphic to the non-split triple cover of the Mathieu group M_{22}. The permutaion group induced by $G_{\mathcal{U}}$ on $\mathcal{U}$ is equivalent to the usual triply transitive action of M_{22} on the 22 points. A compact but explicit description of $\mathcal{U}$ is given in [At, p.39, unitary]. Furthermore, $\mathcal{U}$ contains 7 members which form the exceptional Y-family $\mathcal{Y}(O_4)$ in the last section, on which A_7 is induced. [7]

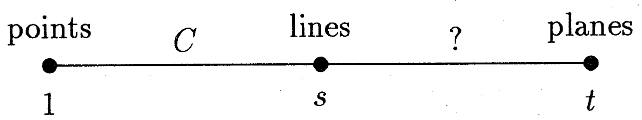
Problem 12. Is there any nice geometric description or construction of the dual hyperoval $\mathcal{U}$ above (if possible, which leads us to good understanding of a 2-local group $2^{1+12}3M_{22}$ in the sporadic simple group J_4, the largest Janko group)?[8]

Starting from any dual arc $\mathcal{S}$ in $PG(m, q)$, we also construct its affine expansion $A(\mathcal{S})$: under a fixed embedding of $PG(m, q)$ into $PG(m + 1, q)$,

the *planes* := the points $\langle f \rangle$ in $PG(m + 1, q)$ outside $PG(m, q)$,
the *lines* := the lines $\langle f, X \cap Y \rangle$ for $\langle f \rangle \in PG(m+1, q) \setminus PG(m, q)$ and $X \neq Y \in \mathcal{S}$,
the *points* := the $(d+1)$-spaces $\langle f, X \rangle$ for $\langle f \rangle \in PG(m+1, q) \setminus PG(m, q)$, $X \in \mathcal{S}$,
the incidence inherited from $PG(m + 1, q)$.

It is easy to see that the residual structures of the geometry $A(\mathcal{S})$ is desribed by the following diagram, where the symbol attatched to the edge joining the middle and the right-most nodes (corresponding to the residue at a point) may vary according to the choice of $\mathcal{S}$.

(c.?)



If $\mathcal{S}$ is a Y-family, the point residue is a GQ of order $(q+1, q-1)$. If $\mathcal{S}$ is a d-dimensional dual hyperoval in $PG(m, q)$, it is the dual of the point-line system in the $(m+1)$-dimensional affine space (usually denoted Af^*).

[7]This exceptional example was known to me since the beginning of 1996, but was mentioned briefly in the summary of my talk for Combinatorics '96 in Assisi.

[8]According to Antonio Pasini, this problem was once discussed by him with Gernot Stroth, but they just found it difficult.

For $\mathcal{S}$ the dual hyperoval in $PG(m,q)$, the geometry $A(\mathcal{S})$ is flag-transitive if the stabilizer $P\Gamma L_{m+1}(q)_{\mathcal{S}}$ in $Aut(PG(m,q))$ of $\mathcal{S}$ acts doubly transitively on $\mathcal{S}$. The recent joint work by Huybrecht and Pasini [HP] almost determined the such dual hyperovals $\mathcal{S}$. For $\mathcal{S}$ a Y-family, [Yo3],[Yo4] determined the Y-families admitting doubly transitive actions.

I conclude my talk with the following questions.

Problem 13. Classify d-dimensional dual arcs in $PG(m,q)$ with doubly transitive actions of the stabilizer in $P\Gamma L_{m+1}(q)$.

Problem 14. Investigate d-dimensional dual arcs of "polar type" (that is, its d-subspaces are totally isotropic or singular with respect to some form on $PG(m,q)$).

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