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A note on equalities in group algebras and character values

Tsuyoshi Atsumi(厚見寅司)
Department of Mathematics
Faculty of Science, Kagoshima University Kagoshima, 890 Japan
atsumi@sci.kagoshima-u.ac.jp

1 Introduction

In this report I reproduce my 20 minutes talk given at Kyoto RIMS on March 9, 1998.

Let G and H be finite groups of order n. A mapping f from G into H is called a planar function of degree n if, for each element $\tau\in H$ and $u\in G^{*}=G-\{1\}$, there exists exactly one $x\in G$ such that $f(ux)f(x)^{-1}=\tau$. In [2] Hiramine has shown that if both G and H are abelian groups of order $3p$ with $p(\geq 5)$ a prime, then there exists no planar function from G into H. To prove this he has established two results on character values. Their proofs are slightly complicated. In this note we shall give short proofs.

We follow the notation and terminology of [2].

2 Planar Functions and Equations in Group Algebras

Let G and H be finite groups of order n. Throughout this article elements of G will be denoted by small Roman letters and elements of H by small Greek letters.

Let f be a mapping from G into H and $S_{\alpha}=\{x\in G|f(x)=\alpha\}$, $\alpha\in H$. If $S_{\alpha}\neq\emptyset$, we set $\hat{S}_{\alpha}=\sum_{x\in S_{\alpha}}x\in C[G]$ and $\hat{S}_{\alpha}^{-1}=\sum_{x\in S_{\alpha}}x^{-1}\in C[G]$, otherwise $\hat{S}_\alpha = \hat{S}_\alpha^{-1} = 0$, where $C[G]$ is the group algebra of G over the complex number field C. Let $G_0 = G \times H$ be the direct product of groups G, H.

To prove the results we need two propositions.

The following is Proposition 2.1 [2].

Proposition 1 *The following are equivalent.*

(i) The function f is planar.

(ii) In the group algebra $C[G]$ of G,

$$\sum_{\alpha\in H} \hat{S}_{\tau\alpha}\hat{S}_\alpha^{-1} = \sum_{\alpha\in H} \hat{S}_{\alpha\tau}^{-1}\hat{S}_\alpha = \begin{cases} \hat{G}+n-1 & \text{if } \tau = 1, \\ \hat{G}-1 & \text{otherwise.} \end{cases}$$

REMARK 1.If $\tau \neq 1$, then it follows from the equation in (ii) of the proposition above that in the group algebra $C[G_0]$ of G_0,

$$\sum_{\alpha\in H} \hat{S}_{\tau\alpha}\tau\alpha\hat{S}_\alpha^{-1}\alpha^{-1} = (\hat{G}-1)\tau.$$

We prove the following

Proposition 2 *We have in $C[G_0]$,*

$$(\sum_{\alpha\in H} \hat{S}_\alpha\alpha)(\sum_{\beta\in H} \hat{S}_\beta^{-1}\beta^{-1}) = \hat{G}+n-1+\sum_{\tau\in H,\,\tau\neq 1}(\hat{G}-1)\tau.$$

In order to prove proposition 2 we need the Remark 1 above and an easy

Lemma 1

$$(\sum_{\alpha\in H}\alpha)(\sum_{\beta\in H}\beta^{-1}) = \sum_{\tau\in H}\sum_{\beta\in H}(\beta\tau)\beta^{-1}.$$

3 Proofs of Hiramine' Results

We start with the following well-known facts about character theory. These facts play important parts in the proofs of his results.

Fact 1 *Let G be an abelian group and χ an arbitrary (linear) character of G. Then χ is a homomorphism from G into $C^* = C - \{0\}$. So we can extend this homomorphism χ to an algebra homomorphism $\overline{\chi}$ from $C[G]$ into C.*

Fact 2 *Let H_1, H_2 be finite groups and G_1 the direct product of H_1, H_2. Then all irreducible characters of G_1 are obtained as follows. Let $\chi_0, ..., \chi_{s-1}$ be the irreducible characters of H_1, $\rho_0, ..., \rho_{t-1}$ the irreducible characters of H_2. Then G_1 has exactly st irreducible characters $\Psi_{ij}(0 \le i \le s-1,\ 0 \le j \le t-1)$, satisfying $\Psi_{ij}(h_1 h_2) = \chi_i(h_1)\rho_j(h_2)$, where $h_1 \in H_1,\ h_2 \in H_2$.*

PROOF. See[1, p.54].
REMARK 2. In Fact 2 if both χ_i and ρ_j are linear characters, then Ψ_{ij} is a homomorphism from G_1 to C^*. As in Fact 1, we have an algebra homomorphism $\overline{\Psi_{ij}}$ from $C[G_1]$ into C which is an extension of Ψ_{ij}.

In the remainder of this section we assume that f is a planar function and that G is an abelian group of order n. Let $\chi_0(= 1_G), \ldots, \chi_{n-1}$ be the irreducible (linear) characters of G, where 1_G denote the principal character of G. We set

$$d_i^{(\alpha)} = \begin{cases} \sum_{x \in S_\alpha} \chi_i(x) & \text{if } S_\alpha \neq \emptyset, \\ 0 & \text{if } S_\alpha = \emptyset \end{cases}$$

for each $0 \le i \le n-1$ and for each $\alpha \in H$. Now we state Hiramine' results[2] and give our proof to Result 2.

Result 1 *The following hold*

(i) $d_0^{(\alpha)} = |S_\alpha|$ *and*

$$\sum_{\alpha \in H} d_0^{(\tau\alpha)} d_0^{(\alpha)} = \sum_{\alpha \in H} d_0^{(\alpha\tau)} d_0^{(\alpha)} = \begin{cases} 2n-1 & \text{if } \tau = 1, \\ n-1 & \text{otherwise.} \end{cases}$$

(ii) For $i \neq 0$,

$$\sum_{\alpha \in H} d_i^{(\tau\alpha)} \overline{d_i^{(\alpha)}} = \sum_{\alpha \in H} \overline{d_i^{(\alpha\tau)}} d_i^{(\alpha)} = \begin{cases} n-1 & \text{if } \tau = 1, \\ -1 & \text{otherwise.} \end{cases}$$

(Here $\bar{d}$ denotes the complex conjugate of $d \in C$.)

PROOF. We omit our proof of this result.

Result 2 *With the same notation and assumption as in Result 1, suppose that H is abelian and let $\rho_0(= 1_H), \ldots, \rho_{n-1}$ be the irreducible characters of H. Set $z_{ij} = \sum_{\alpha \in H} d_i^{(\alpha)} \rho_j(\alpha)$. Then,*

(i) $z_{0,0} = n$, and $z_{i,0} = 0 (i \neq 0)$

(ii) For $j \neq 0$, $z_{ij}\overline{z_{ij}} = n$.

PROOF. Since χ_i and ρ_j are linear, from Remark 2 we see that $\overline{\Psi}_{ij}$ is an algebra homomorphism from $C[G_0]$ into C. First we shall prove (ii). We apply $\overline{\Psi}_{ij}$ $(j \neq 0)$ to the equation in Proposition 2. Then we get (ii). We have proved (ii). Next we shall prove (i). Similarly by using $\overline{\Psi}_{i0}$ we can prove (i). This completes the proof of Result 2.
REMARK 3. Nakagawa[3] has proved Result 2 by using Gaussian sums.

References

[1] L. Dornhoff: *Group representation theory, Part A*, Marcel Dekker, New York, 1971.

[2] Y. Hiramine: *Planar functions and related group algebras*, J. of Algebra **152**(1992), 135–145.

[3] N. Nakagawa: *Left cyclic planar functions of degree p^n*, Utilitas Math. in press.