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Limits of Schottky groups and the boundary of Teichmüller space

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§0. Introduction.

This paper has the following two aims: (1) to consider the difference between the Schottky space and the classical Schottky space; (2) to state limits of Schottky groups and the boundary of Teichmuller space. Furthermore, we will state the augmented Schottky space and Jørgensen's numbers.

§1. Schottky space and classical Schottky space.

DEFINITION 1.1. Let $C_1, C_{g+1}; \cdots; C_g, C_{2g}$ be a set of $2g$ $(g \geq 1)$ mutually disjoint Jordan curves on the Riemann sphere which comprise the boundary of a 2g-ply connected region ω. Suppose there are g Möbius transformations $A_1, \cdots, A_g$ which have the property that A_j maps C_j onto C_{g+j} and $A_j(\omega) \cap \omega = \emptyset$, $1 \leq j \leq g$. Then the g necessarily loxodromic transformations A_g generate a *marked Schottky group* $G = \langle A_1, \cdots, A_g \rangle$ of genus g with ω as a fundamental region. In particular, if all the C_j $(j = 1, 2, \cdots, 2g)$ are circles, then we call $A_1, \cdots, A_g$ a *set of classical generators of* G. A *classical Schottky group* is a Schottky group for which there exists some set of classical generators.

We denote by Möb the group of all Möbius transformations. We say two marked subgroups $G = \langle A_1, \ldots, A_g \rangle$ and $\hat{G} = \langle \hat{A}_1 \ldots, \hat{A}_g \rangle$ of Möb to be *equivalent* if there exists a Möbius transformation T such that $\hat{A}_j = T A_j T^{-1}$ for $j = 1, 2. \ldots, g$.

DEFINITION 1.2. The *Schottky space* (resp. *the classical Schottky space*) of genus g, denoted by $\mathbf{S}_g$ (resp. $\mathbf{S}_g^0$), is the set of all equivalence classes of marked Schottky groups (resp. marked classical Schottky groups) of genus $g \geq 1$.

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Fundamental Problem 1. What characterizes the difference between the Schottky space and the classical Schottky space?

In 1974 Marden obtained the following result.

THEOREM 1.3 (Marden [13]). $\mathbf{S}_g \neq \mathbf{S}_g^0$.

In 1975 Zarrow [34] claimed to have obtained an explicit example of a nonclassical Schottky group. However, Sato [24] pointed out as an application of the general theory on the shape of the classical Schottky space [26] that the group given by Zarrow is a clssical Schottky group. Yamamoto [33] gave an explicit example of a nonclassical Schottky group.

Problem 1. Given a compact Riemann surface S of genus g, there exists a Schottky group G such that $\Omega(G)/G = S$. Does there exist a classical Schottky group G_0 such that $\Omega(G_0)/G_0 = S$?

§2. The boundary of Schottky space.

There are many kinds of boundaries of the Schottky space according to the space embedded. The following definition of the Schottky space is due to Marden [13]. Consider the compact manifold $\mathbf{P}_3^g$ $(g \geq 2)$, where $\mathbf{P}_3$ denote the complex projective 3-space with the natural topology. We represent points of this space by g-tuples of 2×2 complex matrices $(A_1, \cdots, A_g)$ (with the natural equivalence relation). Let X be the variety determined by the equation $\prod \det A_j = 0$ and set $V = \mathbf{P}_3^g - X$. Fix a Schottky group G of genus g and a set of free generators $A_1, \cdots, A_g$. This set of generators determines the point $(A_1, \cdots, A_g) \in V$. To any homomorphism $\theta : G \to H$, where H is a subgroup of Möb, we associate the point $(\theta(A_1), \cdots, \theta(A_g)) \in V$. For simplicity we will use the notation (H, θ) for the point. Conversely, a point $(B_1, \cdots, B_g) \in V$ can be expressed as (H, θ), where H is the group generated by $B_1, \cdots, B_g$ and θ is the homomorphism determined by $\theta(A_j) = B_j$. The topology of V corresponds to the "pointwise convergence" topology in the group H. Namely, $(H_n, \theta_n) \to (H, \theta)$ in V if and only if $\theta_n(A_j) \to \theta(A_j)$ for each $j, 1 \leq j \leq g$. Define the Schottky space as follows:

$$\mathbf{S}_g^1 = \{(H, \theta) \mid H : a\ Schottky\ group, \theta : an\ isomorphism\}/\sim .$$

We remark that $\mathbf{S}_g = \mathbf{S}_g^1$. Thus if there is no confusion, we write $\mathbf{S}_g$ for $\mathbf{S}_g^1$. The boundary $\partial \mathbf{S}_g^1$ of the Schottky space $\mathbf{S}_g^1$ is defined as the relative boundary in V. Namely, for each $(H, \theta) \in \partial \mathbf{S}_g^1$ there exists a sequence $(H_n, \theta_n) \in \mathbf{S}_g^1$ converging to (H, θ). The classical Schottky space $\mathbf{S}_g^{01}$ and its boundary are similarly defined.

DEFINITION 2.1. A boundary group G of $\partial \mathbf{S}_g$ is called a *cusp* if G contains a parabolic element.

THEOREM 2.2 (Chuckrow [4]).

(i) *A boundary group* (H, θ) *of* $\mathbf{S}_g$ *is a g-generator group which does not contain an elliptic element.*

(ii) *There exists a cusp on* $\partial \mathbf{S}_g$.

(iii) *There exists a point which is not a cusp on* $\partial \mathbf{S}_g$.

THEOREM 2.3 (Maskit [14]). *If a finitely generated discontinuous group (Kleinian group of the second kind) G is free and purely loxodromic, then G is a Schottky group.*

THEOREM 2.4 (Marden [13]).

(i) *Each group on* $\partial \mathbf{S}_g$ *is a discrete group.*

(ii) *Each group on* $\partial \mathbf{S}_g^0$ *is a discontinuous group.*

(iii) *There is a not discontinuous group on* $\partial \mathbf{S}_g$.

DEFINITION 2.5. A group G is called *geometrically finite* if G has a fundamental polyhedron having a finite number of faces.

THEOREM 2.6 (Jørgensen-Marden-Maskit [8]). *If* $G \in \partial \mathbf{S}_g^0$, *then* G *is geometrically finite.*

§3. Augmented Schottky space and global coordinates.

DEFINITION 3.1. *A closed Riemann surface with nodes* S, is a compact complex space each point P of which has a neighborhood isomorphic either to a disk $|z| < 1$ in $\mathbf{C}$ (with P corresponding to $z = 0$) or to the set $|z| < 1, |w| < 1, zw = 0$ in $\mathbf{C}^2$ (with P corresponding to $z = w = 0$). In the later case, P is called a *node.*

Bers [3] introduced the augmented Schottky space in his sense by using the multipliers and fixed points of generators , namely the augmented Schottky space in the sense of Bers means the space which consists of all Schottky groups of genus $g \geq 2$ and all extended Schottky groups representing Riemann surfaces with only nondividing nodes. Furthermore, he constructed $(2q-1)(g-1)$ numbers of linearly independent automorphic forms of weight $(-2q)$ on the fiber space $F\hat{\mathbf{S}}_g(= \hat{\mathbf{S}}_g \times \Omega(G)) - \{\text{thin sets}\}$, where $\Omega(G)$ is the region of discontinuity of G. For the general case containing Riemann surfaces with dividing nodes, Sato [18,19] defined the augmented Schottky space and constructed $(2q-1)(g-1)$ numbers of linearly independent automorphic forms of weight $(-2q)$ on the fiber space $F\hat{\mathbf{S}}_g(= \hat{\mathbf{S}}_g \times \Omega(G)) - \{\text{thin sets}\}$.

Problem 2. Construct $(2q-1)(g-1)$ numbers of linearly independent automorphic forms of weight $(-2q)$ on the fiber space $F\hat{\mathbf{S}}_g = \hat{\mathbf{S}}_g \times \Omega(G)$.

The coordinates introduced for defining the augmented Schottky space by Bers [3] and Sato [18] are not invariant under a Möbius transformation. Then Sato [21] introduced new coordinates which are invariant under a Möbius transformation and which have geometric means. We will state the coordinates in the case of $g = 2$ in the next section. By using these coordinates a uniformization of Riemann surfaces with nodes is obtained (Sato [20]).

THEOREM 3.2 (Sato [20]). *Given a Riemann surface with nodes S, there exists a point in the augmented Schottky space $\hat{\mathbf{S}}_g(\Sigma)$ which represents S, where Σ denotes a maximal dissection of S.*

Problem 3. Given a Riemann surface with nodes S, does there exist a point in the augmented classical Schottky space $\hat{\mathbf{S}}^0_g(\Sigma)$ which represents S.

For relationship between limits of Schottky groups and limits of Riemann surfaces, the following are studied (Sato [22]): Give a point τ in the augmented Schottky space $\hat{\mathbf{S}}_g(\Sigma)$ representing a Riemann surface with nodes S, for a sequence $\{\tau_n\}$ of points convergence to the point τ does the sequence $\{S(\tau_n)\}$ of Riemann surfaces represented by τ_n converge to S? The answer to this problem is negative in the general case, that is, for almost all maximal dissection. However the answer is affirmative in a special case, namely in the case that Σ is a standard system of Jordan curves (see [22] for the definition). Sato [22] showed that to what Riemann surfaces does the sequence of Riemann surfaces $\{S(\tau_n)\}$ converge for the general case.

§4. Jørgensen's numbers.

Fundamental Problem 2. Find conditions which gurantee a subgroup of Möb to be a discrete group (Kleinian group).

Fot this problem the following are well-known:
(1) Jørgensen's inequality as a necessary condition.
(2) Poincaré's theorem as a sufficient condition.

Here we will consider only Jørgensen's inequality and Jørgensen's numbers.

THEOREM 4.1 (Jørgensen [7]). If $G = \langle A_1, A_2\rangle$ is a non-elementary discrete group, then

$$|\mathrm{tr}^2(A_1) - 4| + |\mathrm{tr}(A_1A_2A_1^{-1}A_2^{-1}) - 2| \geq 1,$$

where the lower bound 1 is best possible.

DEFINITION 4.2.
(1) The *Jørgensen's number* for a marked two-generator group $\langle A_1, A_2\rangle$ is

$$J(\langle A_1, A_2\rangle) := |\mathrm{tr}^2(A_1) - 4| + |\mathrm{tr}(A_1A_2A_1^{-1}A_2^{-1}) - 2|.$$

(2) The *Jørgensen's number* for a group $G \subset$ Möb is

$$||J(G)|| := \inf\{J(\langle A_1, A_2\rangle) | \langle A_1, A_2\rangle \subset G, A_1^m \neq A_2^n (m, n \in \mathbf{Z})\}.$$

Jørgensen-Lascurain-Pignataro [10] and Sato-Yamada [28] studied two-generator groups with $J(\langle A_1, A_2\rangle) = 1$.

Now we will introduce the global coordinates in the Schottky space in the case of genus $g = 2$, which is announced in the previous section. We denote by $\mathbf{M}_2$ the set of all equivalence classes $[\langle A_1, A_2\rangle]$ of marked groups $\langle A_1, A_2\rangle$ generated by loxodromic transformations A_1 and A_2 whose fixed points are all distinct. Let $[\langle A_1, A_2\rangle] \in \mathbf{M}_2$. For $j = 1, 2$, let λ_j ($|\lambda_j| > 1$), p_j and q_j be the multipliers, the repelling and the attracting fixed points of A_j, respectively. We define t_j ($j = 1, 2$) by setting $t_j = 1/\lambda_j$. Thus $t_j \in D^* = \{z \mid 0 < |z| < 1\}$. We define ρ by $\rho = 1/(p_1, q_1, p_2, q_2)$, where (p_1, q_1, p_2, q_2), is the cross-ratio of p_1, q_1, p_2, q_2:

$$(p_1, q_1, p_2, q_2) := \frac{p_1 - p_2}{p_1 - q_2} \frac{q_1 - q_2}{q_1 - p_2}.$$

Thus $\rho \in \mathbf{C} - \{0, 1\}$. We can define a mapping α of the space $\mathbf{M}_2$ into $(D^*)^2 \times (\mathbf{C} - \{0, 1\})$ by setting $\alpha([\langle A_1, A_2\rangle]) = (t_1, t_2, \rho)$. Then we say $[\langle A_1, A_2\rangle]$ represents (t_1, t_2, ρ) and (t_1, t_2, ρ) corresponds to $[\langle A_1, A_2\rangle]$ or $\langle A_1, A_2\rangle$. We can identify $\mathbf{M}_2$ with $\alpha(\mathbf{M}_2)$. Similarly we can define the mapping α^* of $\mathbf{S}_2$ or $\mathbf{S}_2^0$ into $(D^*)^2 \times (\mathbf{C} - \{0, 1\})$ by restricting α to this space, and identify $\mathbf{S}_2$ (resp. $\mathbf{S}_2^0$) with $\alpha^*(\mathbf{S}_2)$ (resp. $\alpha^*(\mathbf{S}_2^0)$). From now on we denote $\alpha(\mathbf{M}_2), \alpha^*(\mathbf{S}_2)$ and $\alpha^*(\mathbf{S}_2^0)$ by $\mathbf{M}_2, \mathbf{S}_2$ and $\mathbf{S}_2^0$, respectively.

DEFINITION 4.3. We call $G = \langle A_1, A_2\rangle$ a marked group of *real type* (resp. marked Schottky group and marked classical Schottky group of real type) if $(t_1, t_2, \rho) \in \mathbf{R}^3 \cap \mathbf{M}_2$, (resp. $(t_1, t_2, \rho) \in \mathbf{R}^3 \cap \mathbf{S}_2$, and $(t_1, t_2, \rho) \in \mathbf{R}^3 \cap \mathbf{S}_2^0$,) that is, t_1, t_2 and ρ are all real numbers, where (t_1, t_2, ρ) corresponds to $G = \langle A_1, A_2\rangle$.

Then we can classify marked groups of real type into eight types as follows.

DEFINITION 4.4 (Sato [23]).
(1) G is of the first type (Type I) if $t_1 > 0,\ t_2 > 0,\ \rho > 0.$
(2) G is of the second type (Type II) if $t_1 > 0,\ t_2 < 0,\ \rho > 0.$
(3) G is of the third type (Type III) if $t_1 > 0,\ t_2 < 0,\ \rho < 0.$
(4) G is of the fourth type (Type IV) if $t_1 > 0,\ t_2 > 0,\ \rho < 0.$
(5) G is of the fifth type (Type V) if $t_1 < 0,\ t_2 > 0,\ \rho > 0.$
(6) G is of the sixth type (TypeVI) if $t_1 < 0,\ t_2 < 0,\ \rho > 0.$
(7) G is of the seventh type (Type VII) if $t_1 < 0,\ t_2 < 0,\ \rho < 0.$
(8) G is of the eighth type (type VIII) if $t_1 < 0,\ t_2 > 0,\ \rho < 0.$

The components of the coordinates (t_1, t_2, ρ) have the following meaning. If ρ is positive (resp. negative), then the axes of A_1 and A_2 are disjoint (resp. intersect). If $t_j > 0$ (resp. $t_j < 0$) for $j = 1, 2$, then A_j leaves the upper half plane invariant (resp. A_j interchanges the upper and the lower half planes). Concequently, $G = \langle A_1, A_2\rangle$ is a Schottky group of Type I or Type IV, that is, a Fuchsian Schottky group if and only if both t_1 and t_2 are positive. For geometrical meaning of t_j and ρ, see Sato [18], [20], [21].

For each $k = \mathrm{I}, \mathrm{II}, \cdots, \mathrm{VIII}$, we call the set of all equivalence classes of marked groups (resp. marked Schottky groups and marked classical Schottky groups) of

Type k the ***real space*** (resp. the ***real Schottky space*** and the ***real classical Schottky space***) of Type k , and denote it by $R_k\mathbf{M}_2$ (resp. $R_k\mathbf{S}_2$ and $R_k\mathbf{S}_2^0$). Marden [13] and Sato [25] showed that $R_{\mathrm{I}}\mathbf{S}_2 = R_{\mathrm{I}}\mathbf{S}_2^0, R_{\mathrm{IV}}\mathbf{S}_2 = R_{\mathrm{IV}}\mathbf{S}_2^0$.

Problem 4. $R_k\mathbf{S}_2 = R_k\mathbf{S}_2^0$ $(k = \mathrm{I}, \mathrm{II}, \ldots, \mathrm{VIII})$?

THEOREM 4.5. *Let $G = \langle A_1, A_2\rangle$ be a marked Schottky group and let $J(G)$ be its Jørgensen's number. Then*
(i) $J(G) > 16$ *if G is of the first type* (Gilman[6] , Sato[27]),
(ii) $J(G) > 16$ *if G is of the second type* (Sato[31]),
(iii) $J(G) > 4$ *if G is of the third type* (Sato[32]),
(iv) $J(G) > 4$ *if G is of the fourth type* (Gilman[6], Sato[27]),
(v) $J(G) > 4(1+\sqrt{2})^2$ *if G is of the fifth type* (Sato[31]),
(vi) $J(G) > 16$ *if G is of the sixth type* (Sato[32]),
(vii) $J(G) > 4(1+\sqrt{2})^2$ *if G is of the seventh type* (Sato[31]),
(viii) $J(G) > 16$ *if G of the eighth type* (sato[32]).
The lower bounds are all best possible.

These are obtained by using the shape of the classical Schottky spaces, the Schottky modular groups acting on the spaces and fundamental regions for the Schottky modular groups considered in Sato [23,26,29].

Problem 5.
(1) Find the infimum of lower bounds of Jørgensen's numbers for classical Schottky groups: $\inf\{\| J(G) \| \mid G \in \mathbf{S}_g^0\}$.
(2) Find the infimum of lower bounds of Jørgensen's numbers for Schottky groups: $\inf\{\| J(G) \| \mid G \in \mathbf{S}_g\}$.

There are seventeen kinds of elementary groups. The Jørgensen's number for each case is obtained (Sato [30]).

§5. Limits of Schottky groups and the boundary of Teichmüller space.

In this section we will state Gallo's results [5]. We consider the boundary of Teichmüller set by embedding in the space of all holomorphic automorphic forms of weight (-4).

Let Γ be a torsion free Fuchsian group acting on the upper half plane $\mathbf{H}$ such that $\mathbf{H}/\Gamma$ is a compact Riemann surface of genus two. We denote by $B_2(\Gamma)$ the set of all holomorphic automorphic forms of weight (-4) with the norm $\| \varphi \| = \sup\{|\varphi(z)|(2\mathrm{Im} z)^2 \mid z \in \mathbf{H}\}$.

Let $\varphi \in B_2(\Gamma)$. To φ we associate a projective structure $(f_\varphi, \chi_\varphi)$ on $\mathbf{H}/\Gamma$, where $f_\varphi : \mathbf{H} \to \hat{\mathbf{C}}$ is a meromorphic, local homeomorphism normalized by the conditions

$f_\varphi(i) = 0, f'_\varphi(i) = 1, f''_\varphi(i) = 0$, and $\chi_\varphi : \Gamma \to PSL(2, \mathbf{C})$ is a homomorphism which satisfies $f_\varphi \circ \gamma(z) = \chi_\varphi(\gamma) \circ f_\varphi(z)$ for all $z \in \mathbf{H}$ and all $\gamma \in \Gamma$. Moreover, $\varphi = S(f_\varphi)$, where $S(f)$ is the Schwarzian derivative given by

$$S(f) = (f''/f')' - \frac{1}{2}(f''/f')^2.$$

There is a bijective correspondence between normalized projective structures and $B_2(\Gamma)$. We set

$$C_2(\Gamma) := \{\varphi \in S_2(\Gamma) \mid f_\varphi : a \ covering \ map\}.$$

The following is a classical result. Let $A_1, B_1, \cdots, A_g, B_g$ be a canonical generators for Γ; tht is, $\prod_{i=1}^g [A_i, B_i] = id$, where $[A_i, B_i] = B_i^{-1} A_i^{-1} B_i A_i$. Let $N = \langle A_1, \cdots, A_g \rangle$ be the normal subgroup of Γ generated by the elements A_i $(i = 1, \ldots, g)$. Then Γ/N is canonically isomorphic to a Schottky group G with the following properties:

(i) $\Omega(G)/G = \mathbf{H}/\Gamma$,

(ii) there exists a normalized meromorphic covering map $f : \mathbf{H} \to \hat{\mathbf{C}}$ such that $f \circ \gamma = \chi(\gamma) \circ f$, for $\gamma \in \Gamma$, where $\chi : \Gamma \to \Gamma/N = G$ is the natural homomorphism, and

(iii) $\chi(B_i)$ $(i = 1, \ldots, g)$, are free generators for G.

THEOREM 5.1 (Gallo [5]). *Let $\varphi \in \partial T(\Gamma)$, where the genus of $\mathbf{H}/\Gamma$ is two. Then there exists a sequence $\varphi_n \in C_2(\Gamma)$, with $\chi_{\varphi_n}(\Gamma)$ a Schottky group and $\varphi_n \to \varphi$.*

A differential $\varphi \in \partial T(\Gamma)$ is called a ***cusp*** if there exists a hyperbolic transformation $\gamma \in \Gamma$ for which $\chi_\varphi(\gamma)$ is a parabolic. A cusp φ is called *maximal* if there exists a maximal collection α_i $(i = 1, \ldots, 3g-3)$ of non-homotopic, disjoint, simple closed curves in $\mathbf{H}/\Gamma$, called a *maximal dissection* of $\mathbf{H}/\Gamma$, each of which is represented by a hyperbolic element γ_i with $\chi_\varphi(\gamma_i)$ parabolic. If we show that every maximal cusp can be approximated by Schottky structures, then the proof of Theorem 5.1 will be complete by the following McMullen's result [16]: Maximal cusps are dense in $\partial T(\Gamma)$.

There are two types of maximal dissections in genus two:

(i) a maximal dissection consists of three nondividing curves,

(ii) a maximal dissection consists of one dividing curve and two nondividing curves.

Here we will sketch the proof of Theorem 5.1 for the case (i) due to Gallo [5]. Let $\psi \in \partial T(\Gamma)$ be a maximal cusp. A sequence $\{\varphi_n\}$ is constructed as follows. Let b_1, b_2, b_3 be curves of the dissection. Then we can choose $B_1, B_2 \in \Gamma$ hyperbolic elements representing b_1, b_2, respectively, and $A_1, A_2 \in \Gamma$ such that $\{A_1, B_1, A_2, B_2\}$ is a canonical set of generators for Γ, with $W = C_2 B_1^{-1}$ representing b_3, where $C_2 = A_2^{-1} B_2 A_2$. For each $n \in \mathbf{Z}$ $(n \geq 0)$, we set

$A_{(1,n)} = W^n B_1^n A_1$
$B_{(1,n)} = W^{-n} B_1 W^n$
$A_{(2,n)} = A_2 C_2^n W^n$
$B_{(2,n)} = B_2$
Then

$$\Gamma = \langle A_{(1,n)}, B_{(1,n)}, A_{(2,n)}, B_{(2,n)} \mid \prod_{i=1}^{2} [A_{(i,n)}, B_{(i,n)}] = id \rangle.$$

We set $N_n = \langle W^{-n} B_1^n A_1, A_2 C_2^n W^n \rangle$. Then $\chi_n(\Gamma) \cong \Gamma / N_n$ is a Schottky group $G_n = \langle \chi_n(W^n B_1 W^n), \chi_n(B_2) \rangle$. Furthermore, there exists a normalized meromorphic covering mapping $h : \mathbf{H} \to \hat{\mathbf{C}}$ such that (h_n, χ_n) is a projective structure on $\mathbf{H}/\Gamma$. Let $S(h_n) = \varphi_n \in C_2(\Gamma)$.

Since $C_2(\Gamma)$ is a compact set (Kra-Maskit [12]), we can choose a sequence $\{\varphi_n\} \subset C_2(\Gamma)$ such that $\lim_{n\to\infty} \varphi_n = \varphi$. We can prove the following (1)-(7) by using results in Jørgensen [7] and Jørgensen-Klein [9]:

(1) χ_φ has no elliptic elements.

(2) $\chi_\varphi(B_1) \neq id, \chi_\varphi(B_2) \neq id, \chi_\varphi(W) \neq id.$

(3) $\chi_\varphi(W)$ does not commute with any conjugate of $\chi_\varphi(B_1)$ or of $\chi_\varphi(B_2)$.

(4) $\chi_\varphi(B_2)$ does not commute with any conjugate of $\chi_\varphi(B_1)$.

(5) Let $X_n, Y_n \in PSL(2, \mathbf{C})$ be loxodromic transformations and suppose $X_n \to X, Y_n \to Y$, where X, Y are neither elliptic nor trivial. Furthermore, suppose X, Y share no fixed points and $X_n^n Y_n^n$ converges. Then X, Y are parabolic.

(6) $\chi_\varphi(W), \chi_\varphi(B_1)$, and $\chi_\varphi(B_2)$ are parabolic.

(7) $\chi_\varphi(W), \chi_\varphi(B_1)$, and $\chi_\varphi(B_2)$ belong to distict, non-conjugate, maximal parabolic subgroups of $\chi_\varphi(\Gamma)$.

Finally we can show the following (8) by using Abikoff [1], Kra [11] and Maskit [15]:

(8) $\varphi = \psi_1$.

Theorem 5.1 follows from the above (1)-(8).

Problem 6. Does Theorem 5.1 hold in the case that the genus of $\mathbf{H}/\Gamma$ is greater than two?

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