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Physical Equivalence on Quantum Spaces

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Abstract

Physical equivalence based on the experimental uncertainties is shown to be described as the equivalence of eigenstates based on free ultra-filters on $^{*}\mathcal{H}$. The coarse-graining is realized by means of the equivalent treatment with respect to the equivalent eigenstates. Two different types of decoherence between quantum states are simultaneously derived in the coarse-graining process. One represents the decoherence for realizing microcanonical ensemble of statistical mechanics and the other is that required in quantum theory of measurements.

1. Introduction

There is no experiment accompanied by no experimental error. It is important that in the analysis of experiments these uncertainties play an essential role for the determination of the equivalence between different experimental results. In fact we understand that an experimental result is equivalent to the other, if their difference are smaller than the uncertainty associated with the experiment. Recognition of this equivalence (we shall call this equivalence the physical equivalence) as a very fundamental concept is important for our understanding of nature. We, however, have no theory which involves such a fundamental concept, that is, the physical equivalence based on our observations with some uncertainties. Note that the uncertainties discussed here are not those derived from the usual uncertainty relations involved in quantum mechanics, but those are purely caused by our observations. There are some different origins of experimental uncertainties. For instance, we cannot prepare completely monotonic beams and sometimes we cannot cover all the region of parameters such as some regions of scattering angle and etc. Such uncertainties are usually not very difficult to analyze in experiments. The most difficult one for the analysis arises from the indeterminacy of physical states of experimental apparatuses (detectors), because in general their structure are so complicated that some fundamental physical quantities required to determine their Hamiltonian, such as the total number of constituents (N), cannot be determined. That is to say, this problem may be represented by

"How can we describe macroscopic objects involving some indeterminable
fundamental quantities in quantum mechanics?"

Similar situation also appears in the description of thermal equilibrium in statistical mechanics. In statistical mechanics the existence of macroscopic objects such as heat-bath is a fundamental recognition. We actually know that a system composed of completely free particles without any interactions with heat-bath (their elastic scattering with walls may be postulated) do never reach any thermal equilibrium, unless they are incidentally in thermal equilibrium. However, we also know the empirical fact that such systems reach thermal equilibrium. Quantum-mechanically speaking, there must exist some thermal interactions between the system and the heat-bath, which lead the system to thermal equilibrium. Of course, effects of the thermal interactions must be negligibly small in the evaluations of all thermodynamical observables. This means that systems in thermal equilibrium also involve some indeterminable fundamental quantities, thermal interactions in their Hamiltonians, which induce energy uncertainties ΔE. It is interesting that in statistical mechanics this fact is expressed by the equivalent treatment with respect to different states covered by the energy uncertainties ΔE, that is, the principle of equal *a*

priori probability. Thus this equivalence caused by the energy uncertainty is also a fundamental concept for describing statistical mechanics. We may say that this equivalence in statistical mechanics is one of the physical equivalence.

It is also noticeable that another common feature in these processes, quantum observations and thermal equilibrium, is decoherence between quantum states. In other words their density matrices are described not by pure states of quantum mechanics but mixed states. We may expect that there is a certain relation between the physical equivalence and the decoherence of quantum states. In this paper we shall study the physical equivalence in the quantum-mechanical description of macroscopic objects and also the decoherence of quantum states derived from the equivalence.

An interesting model for the description of macroscopic objects was proposed by van Kampen.[1] In his idea the physical equivalence is introduced by representing macroscopic observables in terms of coarse-grained operators. That is to say, within an experimental margin of uncertainty ΔE which is much larger than the average distance between the quantum levels δE there exist an enormous number of eigenstates of the Hamiltonian H describing the macroscopic objects. Then the corresponding eigenstates span a multidimensional subspace of H. It means that an actual measurement does not measure H, but only some coarse-grained version of H, which will be denoted by $\hat{H}$ here. Let us decompose the sequence of the energy eigenvalues E_n into blocks of lengths ΔE. In each block all eigenvalues E_n are replaced by some intermediate value E_ϵ, where $|E_n - E_\epsilon| < \Delta E$ must be fulfilled. The resulting operator is the coarse-grained observable $\hat{H}$. We see that the physical equivalence with respect to the states contained in the same block of the coarse-grained Hamiltonian $\hat{H}$ is realized in this method. The operator $\hat{H}$ commutes with H and each eigenvalue E_ϵ is associated with an eigenspace of $\Delta E/\delta E$ dimensions. This idea is quite interesting and seems to be sufficient to interpret experimental situations. From the theoretical point of view, however, we find some ambiguities in the theory. For instance, there is an ambiguity relating to the uncertainty ΔE in the division of the whole operator H into blocks having the eigenspaces of $\Delta E/\delta E$ dimensions. That is to say, the border line between two blocks can be changed within ΔE and then the elements of the orthogonal subsets for the blocks depend on the choice of the border lines. Therefore, it is not apparent that this ambiguity does not affect any important results such as the probability for the system to be in the Jth of blocks, which has been given by

$$P_J = \sum_{l=1}^{g_J} |b_{J_l}|^2$$

for the state of the system expanded as $|\psi> = \sum_J \sum_l b_{J_l} |\xi_{J_l} >$,[1] where ξ_{J_l} $(l = 1, 2, ..., g_J)$ are the orthogonal eigenstates of the J block (J cell). (In details, see Ref.1.) Since the orthogonal subsets representing the blocks depend on the choice of the border lines, the probability generally has the ambiguity arising from the choice of the border lines. Even though this model has some ambiguous points from theoretical point of view, the idea for the coarse-graining of macroscopic observables accompanied by experimental uncertainties is very interesting for realizing the physical equivalence.

The author proposed a model for the description of macroscopic objects in terms of non-standard analysis.[2-5] On non-standard spaces, however, we have no such ambiguities pointed out in the van Kampen's model, because all experimental numbers which are represented by real numbers have their own infinitesimal neighborhoods denoted as monads (the definition of monad will be given in the following review of non-standard analysis) and then we can determine the borders between any two real numbers without any ambiguities in non-standard spaces. We shall show that all the eigenstates belonging to one neighborhood (cell) characterized by one real number are equivalent to each other and they are represented by ultra-eigenstates on non-standard Hilbert spaces. The physical equivalence

is introduced in the equivalent treatment with respect to equivalent ultra-eigenstates. We have also showed that the equivalent sum over the equivalent states derives decoherence between the states. We may expect that the idea for the coarse-graining of operators for macroscopic physical-observables will be realized in a mathematically rigorous form on non-standard Hilbert spaces and the decoherence problem will also be shed light on.

In §2 some fundamentals of non-standard analysis[6] and equivalence between quantum states on non-standard Hilbert spaces[2-5,7] are briefly reviewed. The description of macroscopic objects on non-standard Hilbert spaces, that is, the method for coarse-graining of macroscopic observables are proposed in §3. The quantum-mechanical equivalence on non-standard Hilbert space and the derivation of decoherence are discussed in §4. In §5 an example for thermal equilibrium is presented. Remarks are given in §6.

2. Fundamentals of non-standard analysis and equivalence on non-standard Hilbert spaces

Let us briefly review some fundamentals of nonstandard analysis [6] and a few important results of nonstandard extension of quantum mechanics.[2-5,7]

2.1 Fundamentals in nonstandard analysis[6]

(i) Free ultra-filters

The set of real numbers $\mathcal{R}$ can be extended to the set of numbers ($^*\mathcal{R}$) containing infinitesimal and infinity in terms of free ultra-filters($\mathcal{F}$) over $\mathcal{N}$ ($\mathcal{N} = (0, 1, 2, ...)$ denotes the set of natural numbers). The free ultra-filters satisfy the following properties:

(a) $\mathcal{N} \in \mathcal{F}$, $\phi(\text{empty set}) \notin \mathcal{F}$,

(b) $A, B \in \mathcal{F} \implies A \cap B \in \mathcal{F}$,

(c) $A \in \mathcal{F}$, $A \subseteq B \implies B \in \mathcal{F}$,

(d) $\mathcal{F}$ contains no finite set, (the filter having this property is called free),

(e) either $E \in \mathcal{F}$ or $\mathcal{N} - E \in \mathcal{F}$ for $^\forall E \subseteq \mathcal{N}$ (the filter having this property is called ultra-filter over $\mathcal{N}$).

(ii) Equivalence in terms of free ultra-filter and non-standard extension

We can construct the non-standard extension of $\mathcal{R}$ by introducing an equivalence relation on sequences in $\mathcal{R}^\mathcal{N}$ by means of a ultra-filter $\mathcal{F}$. The equivalence relation is given by

$$f \longleftrightarrow_\mathcal{F} g \tag{1}$$

if and only if $(i \in \mathcal{N} | f(i) = g(i)) \in \mathcal{F}$, where f and g are, respectively, represented by ultra-products

$$f = \prod_{i \in \mathcal{N}} f(i), \qquad g = \prod_{i \in \mathcal{N}} g(i). \tag{2}$$

Note that the sequences associated with the equivalence relation may be expressed by using ultra-powers

$$\prod_{i \in \mathcal{N}} f(i) / \longleftrightarrow_\mathcal{F} . \tag{3}$$

We may write the non-standard extension of $\mathcal{R}$ in terms of the quotient space

$$^*\mathcal{R} = \mathcal{R}^\mathcal{N} / \longleftrightarrow_\mathcal{F} . \tag{4}$$

We also have non-standard extensions of $\mathcal{N}$, $\mathcal{C}$(the set of complex number) and so on, which are denoted as $^*\mathcal{N}$, $^*\mathcal{C}$ and so forth. It is shown that $\mathcal{R} \subset {}^*\mathcal{R}$, $^{\forall *}r \in {}^*\mathcal{R}$ obey similar rules as those in $\mathcal{R}$ as for operations $+$, $\times$, $=$, $<$ and so on, and the magnitudes of the non-standard natural numbers, $^{\forall *}N \in {}^*\mathcal{N} - \mathcal{N}$, are infinity.

(iii) Standard part map(operation)

We have a projection of every finite number($^*r$) of $^*\mathcal{R}$ to a unique element(r) of $\mathcal{R}$, which is called as the standard part map(operation) and written by

$$\text{st}(^*r) = r. \tag{5}$$

All infinitesimals are mapped at zero.

(iv) Monad of $r \in \mathcal{R}$ (Mon(r))

Each real number $r \in \mathcal{R}$ has its own infinitesimal neighborhood defined by the set of $^*r \in {}^*\mathcal{R}$ satisfying

$$\text{st}(^*r - r) = 0. \tag{6}$$

In other words it may be represented by the set of $^*r \in {}^*\mathcal{R}$ satisfying $\text{st}(^*r) = r$. We see that $\text{Mon}(0)$ contains all infinitesimals.

(v) Topology on non-standard spaces

Topologies on non-standard spaces can be introduced by using the monad. (In details, see Ref.6.) Then they are determined by the choice of infinitesimals. It means that physical spaces defined by different limits (where different physical constants are taken to be infinitesimal or infinity) generally have different topologies, even if they are defined on the same extended Hilbert space $^*\mathcal{H}$. It is also important that in non-standard analysis we can rigorously take such limits on $^*\mathcal{H}$ by means of the standard part map in stead of ambiguous procedures for taking limits on usual Hilbert spaces $\mathcal{H}$. This is an important superiority of non-standard analysis to standard analysis.

Finally we point out one very interesting theorem of non-standard analysis.

Theorem: *There is one-to-one correspondence between "the set of monads" and "the set of filters".*

This theorem indicates that a choice of a monad just corresponds to a choice of an filter. Therefore we may consider that the choice of some physical observables as infinitesimal constants corresponds to the determination of the filter.

2.2 Equivalence on non-standard Hilbert space

General argument for quantum mechanics on non-standard spaces was performed by Farrukh[7]. Here we pick up some important results needed in our discussions.

(i) Ultra-eigenvectors

On $^*\mathcal{H}$ eigenvectors for self-adjoint operator A with eigenvalues $\lambda \in {}^*\mathcal{R}$ are extended to ultra-eigenvectors f defined by ultra-eigenequations

$$\parallel Af - \lambda f \parallel / \parallel f \parallel \approx 0, \tag{7}$$

where $\parallel f \parallel$ stands for the norm of f defined on $^*\mathcal{H}$ and $X \approx 0$ means that X is an infinitesimal.

(ii) Equivalence relation and physical space $S(^*\mathcal{H})$

We cannot distinguish the infinitesimal difference between two vectors f and $f + d$, where d is an infinitesimal vector, i.e. $\parallel d \parallel \approx 0$. The set of physical states $S(^*\mathcal{H})$ must be defined by the quotient set

$$S(^*\mathcal{H}) = U(^*\mathcal{H})/ \longleftrightarrow, \tag{8}$$

where $U(^*\mathcal{H})$ stands for the set of unit vectors on $^*\mathcal{H}$ and the equivalence relation $\longleftrightarrow$ on $U(^*\mathcal{H})$ is defined as [7]

> f and g are equivalent on $^*\mathcal{H}$ ($f \longleftrightarrow g$) if and only if $f, g \in {}^*\mathcal{H}$ and
> $\|f\| = \|g\| = 1$ and there exists a $\phi \in {}^*\mathcal{R}$ such that $\|e^{i\phi}f - g\| \approx 0$.

(iii) Basic equations

The ultra-eigenvectors with $\|f\| = 1$ need not satisfy the equation $Af = \lambda f$ exactly. They are required to satisfy it only approximately such that $Af \stackrel{\approx}{\sim} \lambda f$, where the approximate equality $\stackrel{\approx}{\sim}$ is defined by the relation $\|Af - \lambda f\| \approx 0$, that is, the difference $d \equiv Af - \lambda f$

must be infinitesimal vectors on $^*\mathcal{H}$. Schroedinger equation for the stationary states with the energy E is written by the approximate equality with infinitesimal differences

$$^*H\phi(x)\stackrel{\approx}{\approx}E\phi(x), \tag{9}$$

where $\stackrel{\approx}{\approx}$ means $\|(^*H - E)\phi(x)\| \approx 0$. In the case where $\hbar \approx 0$ is taken such as in the classical limit, the infinitesimal vectors $d \equiv (^*H - E)\phi(x)$ may be of the order of some powers of $\hbar$.[2-5] This extension will allow the existence of new solutions which are excluded in the usual quantum mechanics where it is required that $\hbar$ is not an infinitesimal but a finite number of $\mathcal{R}$.[2-4]

3. Coarse-graining of macroscopic observables on $^*\mathcal{H}$

Let us describe a method for quantum-mechanical description of macroscopic objects by using the ultra-power representation of non-standard analysis.[8] We can get our idea into the following scheme:

(i) Determine Hamiltonian H_N for the fixed number of constituents N. Thus the original Hilbert space $\mathcal{H}_N$ is determined.

(ii) Write down the complete set of the eigenstates in terms of the direct products of N-independent modes $\{|n_i>; n_i = 1, 2, \cdots$ for $(i = 1, 2, \cdots, N)\}$

$$|[n_i]>\equiv \prod_{i=1}^{N} |n_i>, \tag{10}$$

where $[n_i] \equiv (n_1, n_2, \cdots, n_N)$. The eigenstates of H_N are generally written by the superposition of (10).

(iii) Extend $\mathcal{H}_N$ to the non-standard Hilbert space $^*\mathcal{H}$ by using the ultra-products

$$|^*[n_i]>\equiv \prod_{i\in\mathcal{N}} |n_i>, \tag{11}$$

where $^*[n_i] \equiv (n_1, n_2, \cdots\cdots)$ is an infinite series of integers.

Up to now no infinitesimal physical-observable is specified. Let us specify it.

(iv) Following the experimental situation of observed quantities, take observables Q which are covered by experimental uncertainties as infinitesimals. From the ultra-eigenequation $\|(^*H - E)|\psi_E > \| \approx o(Q)$, ultra-eigenstates are obtained and generally written as superposed states of (11). Thus the extended Hilbert space $^*\mathcal{H}_Q$ describing the experimental situation is determined.

Let us specify the monad corresponding to the physical observation accompanied by the uncertainties. As noted in (v) of §2.1. the theorem of non-standard analysis certifies that the determination of the monad just corresponds to the determination of the filter $\mathcal{F}_Q$ for the choice of a topology on $^*\mathcal{H}$. We understand that

ultra-filters $\mathcal{F}_Q$ represent apparatuses for searching on non-standard quantum-mechanical spaces and these apparatuses catch different observables, depending on the choice of infinitesimal physical-constants Q.

(v) In terms of the filter $\mathcal{F}_Q$ specified by the infinitesimal physical quantity Q we can introduce the equivalence relation $\leftrightarrow_Q$ and then the non-standard physical-space $S(^*\mathcal{H})_Q$ is determined by means of the equivalence relation on $^*\mathcal{H}_Q$. (See (ii) of §2.2.) Sets of equivalent ultra-eigenstates are uniquely determined by the equivalence relation. Thus the blocks in van Kampen's model are rigorously defined by these equivalent-sets in this model. In general an equivalent-set contains an infinite number of ultra-eigenstates.

(vi) All evaluations should be carried out on $S(^*\mathcal{H})_Q$ and the physical results on the

observed physical-space $\mathcal{H}_Q^{\text{obs}}$ are obtained by taking the standard part map of the evaluations on $S(^*\mathcal{H})_Q$. Precisely speaking, the expectation value $< A|\mathcal{O}|A >$ for the state $|A >$ is calculated on $S(^*\mathcal{H})_Q$ and then the final result representing the experimental value is obtained by taking the standard part map $\text{st}< A|\mathcal{O}|A >$. The observed physical-space $\mathcal{H}_Q^{\text{obs}}$ is just the physical space which is observed by the experiment accompanied by the uncertainties which are represented by the filter $\mathcal{F}_Q$ on the non-standard Hilbert space $^*\mathcal{H}$. Note that the standard part map corresponds to taking some limits in usual evaluations. We need not take any limits on non-standard spaces.

We again stress that in this scheme an equivalent-set just represents a block of van Kampen's idea and it is uniquely determined from experimental uncertainties through the role of ultra-filters on non-standard Hilbert spaces. Thus the coarse-graining of macroscopic observables is rigorously accomplished.

We still have an important problem in the evaluation on the non-standard physical spaces $S(^*\mathcal{H})_Q$. On $S(^*\mathcal{H})_Q$ all the equivalent states must be treated equivalently.

How is it realized in the evaluations on $S(^*\mathcal{H})_Q$?

4. Equivalent sum on $S(^*\mathcal{H})_Q$ and decoherence

Let us study the realization of the equivalence on $S(^*\mathcal{H})_Q$. The most fundamental operator in the evaluation of the expectation values is density matrix. We, therefore, study this problem as the problem for realizing the equivalent treatment with respect to the equivalent eigenstates specified by the equivalence relation $\leftrightarrow_Q$ in density matrices.

Let us investigate the problem between two different equivalent-sets represented by $\{|\psi_i >\}$ ($i = 1, 2, ...$) and $\{|\phi_j >\}$ ($j = 1, 2, ...$). The ultra-eigenstates included in the same set are, of course, equivalent each other, i.e., $|\psi_i > \leftrightarrow_Q |\psi_k >$ for $^\forall(i,k)$ and $|\phi_j > \leftrightarrow_Q |\phi_l >$ for $^\forall(j,l)$. But the two sets are not equivalent, i.e., $|\psi_i > \not\leftrightarrow_Q |\phi_j >$ for $^\forall(i,j)$, and fulfill the orthogonality as

$$< \psi_i|\phi_j >= 0, \quad \text{for } ^\forall(i,j).$$

Generally a state being equivalent to the set $\{|\psi_i >\}$ is written by the superposition of all the elements of the set as

$$|\Psi >= \sum_i c_i|\psi_i >, \quad \text{with } c_i \in {}^*\mathcal{C}. \tag{12}$$

Similarly we can write a general state equivalent to the set $\{|\phi_j >\}$ as

$$|\Phi >= \sum_j d_j|\phi_j >, \quad \text{with } d_j \in {}^*\mathcal{C}. \tag{13}$$

From the normalization condition we have relations

$$\sum_i |c_i|^2 = 1, \quad \sum_j |d_j|^2 = 1. \tag{14}$$

Let us consider a superposed state of the above two states like

$$|S >= a|\Psi > +b|\Phi >, \quad \text{with } |a|^2 + |b|^2 = 1, \tag{15}$$

where $a, b \in {}^*\mathcal{C}$. The density matrix of the state is obtained by

$$|S >< S| = a^*a|\Psi >< \Psi| + b^*b|\Phi >< \Phi| + a^*b|\Psi >< \Phi| + b^*a|\Phi >< \Psi|. \tag{16}$$

It is noted that the treatment with respect to the equivalent sates in $|\Psi >$ is not equivalent, because the coefficients $\{c_i\}$ are generally different. The situation is same as for $|\Phi >$. We have no reason for giving a state a higher probability than the others in the same equivalent-set. The completely equivalent treatment for the equivalent states can be realized by summing up with respect to the coefficients in all allowed regions such that

$$\rho_Q(|S >) = N_c N_d \prod_j \int d^2 d_j \, {}^*\delta(\sum_l |d_l|^2 - 1) \prod_i \int d^2 c_i \, {}^*\delta(\sum_k |c_k|^2 - 1)|S >< S|, \quad (17)$$

where N_c and N_d are, respectively, introduced for the normalization of integrals with c_i and those with d_j as

$$N_c^{-1} = \sum_k \prod_i \int d^2 c_i \, {}^*\delta(\sum_l |c_l|^2 - 1) \, |c_k|^2 = \prod_i \int d^2 c_i \, {}^*\delta(\sum_l |c_l|^2 - 1),$$

$$N_d^{-1} = \sum_k \prod_j \int d^2 d_j \, {}^*\delta(\sum_l |d_l|^2 - 1) \, |d_k|^2 = \prod_j \int d^2 d_j \, {}^*\delta(\sum_l |d_l|^2 - 1), \quad (18)$$

and ${}^*\delta(x)$ stands for the constraint $x = 0$ in the integrals. In (17) and (18) we may put $\int d^2 c_i = \frac{1}{2} \int_0^1 d|c_i|^2 \int_0^{2\pi} \frac{d\varphi_i}{2\pi}$.

Let us consider the integral of the first term in $|S >< S|$, that is,

$$N_c \prod_i \int d^2 c_i \, {}^*\delta(\sum_l |c_l|^2 - 1)|\Psi >< \Psi|, \quad (19)$$

where the integrations with d_j are dropped from the normalization for d_j given in (18). We can easily perform the integrations with c_i as follows;

$$N_c \, (\prod_i \frac{1}{2} \int_0^1 d|c_i|^2 \, {}^*\delta(\sum_n |c_n|^2 - 1) \int_0^{2\pi} \frac{d\varphi_i}{2\pi}) \sum_l \sum_k |c_l||c_k|e^{i(\varphi_k - \varphi_l)}|\psi_l >< \psi_k|$$

$$= N_c \prod_i \frac{1}{2} \int_0^1 d|c_i|^2 \, {}^*\delta(\sum_n |c_n|^2 - 1) \sum_k |c_k|^2|\psi_k >< \psi_k|$$

$$= \frac{1}{W_\psi} \sum_k |\psi_k >< \psi_k|, \quad (20)$$

where $W_\psi^{-1} = N_c \prod_i \frac{1}{2} \int d^2|c_i|^2 \, {}^*\delta(\sum_n |c_n|^2 - 1) \, |c_k|^2$, for ${}^\forall k$. We see that $W_\psi \equiv \sum_i 1$ is the total number of the equivalent eigenstates contained in the set $\{|\psi_i >\}$. Interference terms between different eigenstates disappear by the integrations over phases in (20). We can obtain a similar result for the second term of $|S >< S|$ corresponding to $|\Phi >< \Phi|$. From the above results we may conclude that the equivalent treatment over all the equivalent eigenstates in the same equivalent-set derives decoherence between the equivalent states in the diagonal terms and the final forms of the diagonal terms are represented by mixed states, in which all the equivalent eigenstates have an equal probability, for instance, $1/W_\psi$ in $|\Psi >< \Psi|$.

Following the same method for the integrations, we easily see that the off-diagonal terms of $|S >< S|$, $|\Psi >< \Phi|$ and $|\Phi >< \Psi|$, vanishes by the integrations over phases of the coefficients. This fact means that decoherence between the two different states $|\Psi >$ and $|\Phi >$ is accomplished. Thus we obtain the final formula of the density matrix for the superposed state $|S >$, which represents the observation through the filter $\mathcal{F}_Q$, as

$$\rho_Q(|S >) = \frac{|a|^2}{W_\psi} \sum_i |\psi_i >< \psi_i| + \frac{|b|^2}{W_\phi} \sum_j |\phi_j >< \phi_j|. \quad (21)$$

The density matrix for the state $|S>$ is represented by a mixed state $\rho_Q(|S>)$ on the non-standard physical space $\mathcal{S}(^*\mathcal{H})_Q$. We may conclude that the equivalent treatment based on the physical equivalence derives the decoherence in the density matrix on $\mathcal{S}(^*\mathcal{H})_Q$. Since the extension of this example to general cases including many different equivalent-sets is trivial, so we do not proceed here.

In the above argument two different types of decoherence are maintained, that is, one is the decoherence between equivalent states contained in the same equivalent-set and the other the decoherence between different equivalent-sets. It should be noted that the former, which realizes mixed states fulfilling the principle of equal *a priori* probability like that in (20), represents the decoherence for deriving microcanonical ensemble of statistical mechanics,[9-11] whereas the later does the decoherence required of states which describe experimental apparatuses(detectors) in quantum theory of measurements.[12-14] In the density matrix ρ_Q the physical equivalence is expressed by the equivalent sum with respect to equivalent states contained in the same equivalent-set.

5. An example of the coarse-graining for thermal equilibrium

An example for the macroscopic motion of N oscillator system was carried out in Ref.5. Here we shall study the scheme for the coarse-graining, presented in §§3 and 4, in the quantum-mechanical description of thermal equilibrium for N oscillator system, which has been discussed in the usual quantum-mechanical framework.[9-11]

5.1 Hamiltonian and eigenstates on original Hilbert space[9]

Following procedures (i) and (ii) of §3, let us determine an original Hilbert and eigenstates. In this model Hamiltonian is given by (for details of the following discussions, see Ref. 9)

$$H([\theta]) = H_0 + H_I([\theta]), \tag{22}$$

where H_0 is the oscillator Hamiltonian

$$H_0 = \epsilon \sum_{j=1}^{N} a_j^\dagger(\theta) a_j(\theta) \tag{23}$$

and $H_I([\theta])$ stands for the relative-phase interaction

$$H_I([\theta]) = \epsilon_g g(\hat{N}) \frac{1}{N} \sum_j (\frac{1}{N} \sum_k \alpha_j^\dagger \alpha_k e^{i\theta_{jk}} + |0>_j \ _j<0|). \tag{24}$$

In H_0 the annihilation and creation operators with phases are defined by $a_j(\theta) = a_j e^{-i\theta_j}$ and $a_j^\dagger(\theta) e^{i\theta_j}$, respectively. In $H_I([\theta])$ ϵ_g is an energy scale, $\alpha_j^\dagger = a_j^\dagger(\hat{N}_j + 1)^{-1/2}$, $\alpha_k = (\hat{N}_k + 1)^{-1/2} a_k$, $\hat{N} = \sum_{j=1}^{N} \hat{N}_j$ (the total excitation-number operator) with $\hat{N}_j = a_j^\dagger a_j$ and the relative phases $\theta_{jk} = \theta_j - \theta_k$.

The eigenstates of H_0 are written in terms of the direct product of the number states with phases as

$$|M; [n_j], [\theta_j] >= \prod_{j=1}^{N} |n_j, \theta_j > \delta_{\sum_{i=1}^{N} n_i, M}, \tag{25}$$

where $|n_j, \theta_j >= |n_j > e^{in_j\theta_j}$ fulfills the equations $a_j^\dagger(\theta)|n_j, \theta_j >= \sqrt{n_j + 1}|n_j + 1, \theta_j >$, $a_j(\theta)|n_j >= \sqrt{n_j}|n_j - 1, \theta_j >$ and $a_j^\dagger(\theta)a_j(\theta)|n_j, \theta_j >= n_j|n_j, \theta_j >$. Here $|n_j >$ is the usual number state fulfilling relations $\hat{N}_j|n_j >= n_j|n_j >$ and $a_j|0 >_j = 0$.

We can also see that the eigenstates of $H_I([\theta])$ are given by the superposition of the eigenstates of H_0 as

$$|M, N, [\theta] > = \sum_{p[n_j]} |M; [n_j], [\theta_j] > /\sqrt{W(M, N)}, \tag{26}$$

where the sum should be taken over all the different combinations of $[n_j] = (n_1, n_2, ..., n_N)$ and the number of the combinations $W(M, N)$ is given by

$$W(M, N) = \frac{(M + N - 1)!}{M!(N - 1)!}. \tag{27}$$

It is obvious that the states $|M, N, [\theta] >$ span only a subspace of the whole physical space of H_0 and the eigenvalues of $H_I([\theta])$ are $\epsilon_g g(M)$.

From the above discussion we easily see that the eigenstates of the total Hamiltonian $H([\theta])$ are the same as those of $H_I([\theta])$ and we have the eigenvalue of $H([\theta])$ as follows;

$$H|M, N, [\theta] > = (\epsilon M + \epsilon_g g(M))|M, N, [\theta] > . \tag{28}$$

It is apparent that the Hilbert spaces having the different phases are different, that is,

$$\mathcal{H}([\theta]) \neq \mathcal{H}([\theta']) \tag{29}$$

for $[\theta] \neq [\theta']$. It is also clear that they are subspaces of the Hilbert space $\mathcal{H}_0$ for the oscillator Hamiltonian H_0, because their eigenstates are described by the superposition of the eigenstates of H_0. It is, however, noted that they are equivalent as physical space. That is to say, all expectation values for operators $\mathcal{O}$ written by the creation and annihilation operators are same on all Hilbert spaces with different phases as

$$< M', N, [\theta^1]|\mathcal{O}(a_j^\dagger(\theta^1), a_k(\theta^1))|M, N, [\theta^1] > = < M', N, [\theta^2]|\mathcal{O}(a_j^\dagger(\theta^2), a_k(\theta^2))|M, N, [\theta^2] > . \tag{30}$$

Note also that all the eigenstates $|M, N, [n_j] >$ of H_0 can be written down in terms of the superposition of the eigenstates of all the Hamiltonians $H([\theta])$ corresponding to the phases $0 <^\forall \theta_j \leq 2\pi$ such that

$$|M, N, [n_j] > = \prod_{j=1}^{N} \int_0^{2\pi} \frac{d\theta_j}{2\pi} e^{-in_j\theta_j} |M, N, [\theta] > . \tag{31}$$

We see that the Hilbert space for the oscillator, $\mathcal{H}_0$, is completely covered by all the Hilbert spaces $\mathcal{H}([\theta])$ for $0 <^\forall \theta_j \leq 2\pi$.

5.2 Non-standard extension of $\mathcal{H}([\theta])$

Let us go to procedure (iii). We can write the direct product of number states in terms of the ultra-product as

$$|M, ^*[n_j] > = \prod_{j \in \mathcal{N}} |n_j, \theta_j > . \tag{32}$$

The eigenstates of the Hamiltonian $^*H([\theta])$, where the constituent number N is taken as an infinity $^*N$ of $^*\mathcal{N} - \mathcal{N}$, are described by the sum of the extended number-eigenstates as

$$|M, [\theta] > = \sum_{p^*[n_j]} |M, ^*[n_j] > /\sqrt{W(M)}, \tag{33}$$

where $W(M)$ denotes the number of the different combinations of $p^*[n_j]$ and belongs to an infinity of $^*\mathcal{N} - \mathcal{N}$. They satisfy the equations

$$\|(^*H([\theta]) - (\epsilon M + \epsilon_g g(M)))|M, [\theta] > \| \approx 0. \tag{34}$$

5.3 Non-standard physical space $S(^*\mathcal{H}([\theta]))_{\epsilon_g}$

Following procedures (iv) and (v), let us specify physical infinitesimals and determine non-standard physical space. In this model the energy uncertainty ΔE is take to be much larger than the eigenvalues of $H_I([\theta])$, $\epsilon_g M$, that is, $\Delta E >> \epsilon_g M$ for $^\forall M \in \mathcal{N}$. This means that ϵ_g is taken to be an infinitesimal of $^*\mathcal{R}$. We can show the following relations;

$$\|(^*H([\theta]) - \epsilon M)|M, [\theta'] > \| = o(\epsilon_g g(M)) \approx 0 \tag{35}$$

for $0 <^\forall \theta'_j \leq 2\pi$, where $o(\epsilon_g g(M))$ stands for the order of $\epsilon_g g(M)$. In the present case where ϵ_g is taken as the infinitesimal, the above equation is understood as the ultra-eigenequation and then all the states $|M, [\theta'] >$ for $0 <^\forall \theta'_j \leq 2\pi$ become the ultra-eigenstates on the non-standard Hilbert space $^*\mathcal{H}([\theta])_{\epsilon_g}$ with the fixed phases $[\theta]$. It is trivial that all the states have the same eigenvalue ϵM. Thus we may consider that all the Hilbert spaces $\mathcal{H}([\theta'])$ with different phases $([\theta'] \neq [\theta])$ are subspaces of the non-standard Hilbert space with a fixed phase $^*\mathcal{H}([\theta])_{\epsilon_g}$ specified by the infinitesimal ϵ_g. It is obvious that the Hilbert space of the oscillator, $\mathcal{H}_0$, is a subspace of $^*\mathcal{H}([\theta])_{\epsilon_g}$, because all the eigenstates of H_0 are written by the superposition of those of $H([\theta])$, as shown in (31). Following the definition given in (ii) of §2.2, we can find the phase

$$\Theta = \sum_j n_j(\theta_j - \theta'_j) \tag{36}$$

between any pair of eigenstates for satisfying the equivalence relation

$$\|e^{i\Theta}|M, [\theta] > -|M, [\theta'] > \| \approx 0. \tag{37}$$

From the definition all the eigenstates with different phases are equivalent. Now we can define the non-standard physical space $S(^*\mathcal{H}([\theta]))_{\epsilon_g}$ by using the above equivalence relation $\leftrightarrow_{\epsilon_g}$. It is apparent that all the states $|M, [\theta'] >$ (for $0 <^\forall \theta'_j \leq 2\pi$) with a fixed M are included in the same equivalent-set, and they have the same energy eigenvalue ϵM. This is nothing but the realization of coarse-graining of Hamiltonian.

The important difference of $S(^*\mathcal{H}([\theta]))_{\epsilon_g}$ from $\mathcal{H}_0$ is represented by the fact that *all the eigenstates with the same eigenvalue ϵM but different phases must be treated equivalently on $S(^*\mathcal{H}([\theta]))_{\epsilon_g}$, while there is no such a constraint on $\mathcal{H}_0$.*

5.4 Equivalent sum and decoherence of quantum states on $\mathcal{H}_0$

From (20) of §4 the density matrix for the eigenstate $|M >$ with a fixed energy $E = \epsilon M$ is described by the equivalent sum over $0 <^\forall \theta_j \leq 2\pi$ on $S(^*\mathcal{H}([\theta]))_{\epsilon_g}$ such that

$$\begin{aligned}
\rho_{\epsilon_g}(|M >) &= \prod_j \int_0^{2\pi} \frac{d\theta_j}{2\pi} |M, [\theta_j] >< M, [\theta_j]|/W(M) \\
&= (\prod_l \int_0^{2\pi} \frac{d\theta_l}{2\pi} \sum_{p[n_j]} \sum_{p[n'_j]} \prod_j e^{i(n_j - n'_j)\theta_j}) |M, [n_j] >< M, [n'_j]|/W(M), \tag{38}
\end{aligned}$$

where the sum $\sum_k$ in (20) is replaced by the integrations with respect to the phases $[\theta]$. We can easily perform the integrations over the phases and obtain the final expression of the density matrix on $S(^*\mathcal{H}([\theta]))_{\epsilon_g}$

$$|M><M| = \frac{1}{W(M)} \sum_{p^*[n_j]} |M,^*[n_j]><M,^*[n_j]|. \tag{39}$$

Note that the density matrix is represented by completely mixed states with respect to the states $|M,^*[n_j]>$ which are the eigenstates of the oscillator Hamiltonian $^*H_0 = \epsilon \sum_j a_j^\dagger a_j$. We may say that

"*decoherence with respect to the eigenstates of $^*H_0$ is realized.*"

We also see that all the eigenstates of $^*H_0$ with the same eigenvalue ϵM have an equal weight $1/W(M)$, that is, the principle of equal a priori probability is realized. This fact means that this density matrix represents microcanonical ensemble of statistical mechanics on $^*\mathcal{H}_0$. We may conclude that

"*the observed physical space $\mathcal{H}_{\epsilon_g}^{obs}$ which is mapped by means of the standard part operation is the thermal subspace of $\mathcal{H}_0$*".

Note also that the relative-phase interaction is understood as thermal interactions whose existence is a fundamental postulation in thermal equilibrium but it always does not explicitly appear in any discussions of statistical mechanics.

5.5 Remarks on calculations

In the evaluations by using the density matrix the number of constituents N should be taken as an non-standard integer $^*N \in {}^*\mathcal{N} - \mathcal{N}$. In the standard part map of the results, therefore, $^*N$ should be replaced by the number of constituents N, which is, of course, of the order of Avogadro's number and actually an indeterminable macroscopic observable accompanied by a large uncertainty.

6. Concluding rmarks

We introduced the physical equivalence adopted in the interpretation of experimental results into quantum theory, and we showed that the introduction of the equivalence is very important to describe our observations realized by coarse-graining of macroscopic observables on quantum-mechanical spaces. Our main results are summarized as follows:

(1) The physical equivalence based on experimental uncertainties can be well represented by the equivalence based on filters $\mathcal{F}$ for the determination of topology on non-standard Hilbert spaces $^*\mathcal{H}$.

(2) The filter $\mathcal{F}_Q$ for the determination of topology on $^*\mathcal{H}$ is determined by taking physical observables Q, which are so small in comparison with the experimental uncertainties that their observations are impossible in the very experiment, as *infinitesimals* on $^*\mathcal{H}$.

(3) Sets of equivalent eigenstates are uniquely determined in terms of the equivalence based on the filter $\mathcal{F}_Q$ and the non-standard physical space $S(^*\mathcal{H})_Q$ is also determined by means of the same equivalence.

(4) Density matrices on $S(^*\mathcal{H})_Q$ are obtained by the realization of the equivalence over all the equivalent eigenstates, and they are described by mixed states, that is, decoherence of quantum states are realized. All evaluations must be carried out on $S(^*\mathcal{H})_Q$ by using the density matrices.

(5) Two different types of decoherence are maintained. One represents the decoherence for describing microcanonical ensemble of statistical mechanics and the other is that required in quantum theory of measurements.

Note that the equivalent states included in the same equivalent-set are recognized as the equivalent one only when their differences are covered by experimental uncertainties, and the differences are fundamentally measurable in more precise measurements.[10]

We may conclude that quantum mechanics involving the physical equivalence based on our observations with some uncertainties can be described by quantum mechanics on non-standard Hilbert spaces. Through these arguments we may understand the fact;

a choice of an apparatus in an observation process is represented by a choice of
an filter for the determination of topology on non-standard Hilbert spaces.

We peep at only some properties of nature through our filters which are represented by the free ultra-filters on non-standard Hilbert spaces.

Finally I would like to remark on the very strong prejudice against non-standard analysis such that non-standard extension of quantum mechanics does not produce any new results. If non-standard extensions are carried out without taking any physical observables as infinitesimals, the prejudice is right because no filters which have definite physical meanings corresponding to physical observations is involved in such extensions. That is to say, in such extensions non-standard Hilbert spaces $^*\mathcal{H}$ go back to original Hilbert spaces $\mathcal{H}$ by taking the standard part map. In the case where some physical infinitesimal observables Q are introduced, as the model presented in this paper, however, we can derive many new results which have never been obtained in quantum mechanics on real space, such as, decoherence, classical solutions[2,3] and etc. The world observed through physical filters, $\mathcal{H}_Q^{obs}$, is quite different from the original world, $\mathcal{H}$. We cannot reach a right quantum theory involving quantum-mechanical observations, unless we recognize the importance of the fundamental physical-equivalence associated with our observations accompanied by some uncertainties.

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