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# REMARKS ON BLOCH FUNCTIONS ON WEAKLY PSEUDOCONVEX DOMAINS

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## 1. INTRODUCTION

In this note, we consider certain characterization of Bloch functions due to H. Arai [1] in the case of weakly pseudoconvex tube domains in $\mathbb{C}^2$.

An analytic function $f: D := \{z \in \mathbb{C}; |z| < 1\} \to \mathbb{C}$ is called a *Bloch function* on the unit disk if

$$\sup\{|f'(z)|(1-|z|^2); z \in D\} < \infty.$$

There are many detailed studies about the class of Bloch functions on the unit disk and this class can be characterized in many different ways (ref. [6]). In the case of several complex variables, Hahn [3], Timoney [6] and Krantz-Ma [5] generalized the definition of Bloch function in terms of invariant metrics (the Bergman metric or the Kobayashi metric). The class of Bloch functions in several complex variables has also been characterized in many different ways. The study of Arai [1] is one of these interesting characterizations. He characterized the class of Bloch functions on bounded strongly pseudoconvex domains in terms of invariant geometry, Bergman-Carlson measures and Kähler diffusion process.

By the way the asymptotic expansion of the Bergman kernel due to C. Fefferman plays an important role in the argument of Arai [1]. Since an appropriate asymptotic formula of the Bergman kernel is not generally obtained in the case of domains of finite type until now, Arai's characterization seems difficult to be generalized in this case. But the author [4] obtained an asymptotic expansion of the Bergman kernel in the case of weakly pseudoconvex tube domains of finite type in $\mathbb{C}^2$. The purpose of this note is to show that his expansion can be applied to the characterization of Bloch functions for these domains.

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## 2. Definitions and main results

Let $\Omega$ be a bounded domain in $\mathbb{C}^n$. If $z \in \Omega$ and $\xi \in T_z(\Omega)$, then we denote by $F_K(z, \xi)$ the infinitesimal Kobayashi metric for $\Omega$.

**Definition 2.1** (Krantz-Ma [5]). *A holomorphic function defined on $\Omega$ is said to be a Bloch function $f \in \mathcal{B}(\Omega)$, if*

$$|f_*(z) \cdot \xi| \leq C F_K(z, \xi), \quad z \in \Omega, \, \xi \in T_z(\Omega),$$

*where $f_*(z)$ is the mapping from $T_z(\Omega)$ to $T_{f(z)}(\mathbb{C})$ induced by $f$.*

The Bergman space $B(\Omega)$ is the subspace of $L^2(\Omega)$ consisting of holomorphic $L^2$-functions on $\Omega$. The orthogonal projection $\mathbb{B} : L^2(\Omega) \to B(\Omega)$ can be written by using an integral kernel:

$$\mathbb{B}f(z) = \int_\Omega K(z, w) f(w) dV(w) \quad \text{for } f \in L^2(\Omega),$$

where $dV$ is the Lebesgue measure on $\Omega$. Here $K$ is called as the Bergman kernel of $\Omega$. The *Bergman metric* of $\Omega$ is the function $F_B : \Omega \times \mathbb{C}^n \to \mathbb{R}_+$ defined by

$$F_B(z; \xi) = \left( \sum_{j,k=1}^{n} g_{j\bar{k}}(z) \xi_j \overline{\xi_k} \right)^{1/2}, \tag{2.1}$$

where $g_{j\bar{k}}(z) = \partial / \partial z_j \partial / \partial \bar{z}_k \log K(z, z)$. Let $(g^{\bar{j}k})$ be the inverse matrix of $(g_{j\bar{k}})$. If $f \in C^1(\Omega)$ and $z \in \Omega$, then we denote by $\|\tilde{\nabla} f(z)\|$ the norm of the gradient of $f$ with respect to the Bergman metric of $\Omega$, that is,

$$\|\tilde{\nabla} f(z)\|^2 := \sum_{j,k=1}^{n} g^{\bar{j}k}(z) \frac{\partial f(z)}{\partial z_j} \overline{\frac{\partial f(z)}{\partial z_k}}.$$

A positive measure $\mu$ on $\Omega$ is called as *Bergman-Carleson measure*, if there exists a positive constant $C$ such that

$$\int_\Omega |f(z)|^2 d\mu(z) \leq C \int_\Omega |f(z)|^2 dV(z)$$

for all $f \in B(\Omega)$.

*Remark.* In this note we can use the Bergman metric instead of the Kobayashi metric in the definition of Bloch function because it is known in [2] that these metrics are comparable on domains of finite type in $\mathbb{C}^2$ .

The following is a main result of this note.

**Theorem 2.1.** *Let $\Omega$ be a tube domain of finite type in $\mathbb{C}^2$. Let $f$ be a holomorphic function in $\Omega$. Then the following conditions are equivalent:*

*(1) $f$ is a Bloch function.*

*(2) $\sup_{z\in\Omega} |\nabla_N f(z)||r(z)| < \infty$, where $r(z)$ is a defining function of $\Omega$ and $\nabla_N$ is the normal derivative. (This assertion is independent of the choice of defining function.)*

*(3) $\sup_{z\in\Omega} \|\tilde{\nabla} f(z)\| < \infty$.*

*(4) The measure*

$$d\mu_f(z) := \|\tilde{\nabla} f(z)\|^2 dV(z)$$

*is a Bergman-Carleson measure on $\Omega$.*

## 3. Analysis of the Bergman kernel and metric

In this section, we investigate the boundary behavior of the Bergman kernel and metric of certain tube domains in $\mathbb{C}^2$, which plays an important role in the proof of Theorem 3.1.

Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f'' \geq 0$ on $\mathbb{R}$ and $f(x) = x^{2m} g(x)$, where $g(0) > 0$ and $m = 2, 3, \ldots$. The tube domain $\Omega_f \subset \mathbb{C}^2$ is defined by $\Omega_f = \mathbb{R}^2 + i\omega_f$, where $\omega_f = \{(x,y) \in \mathbb{R}^2; y > f(x)\}$. The projection $\pi : \mathbb{C}^2 \to \mathbb{R}^2$ is defined by $\pi(z_1, z_2) = (\Im z_1, \Im z_2)$. Note that $\pi^{-1}(\{(0,0)\})$ is a set of weakly pseudoconvex points and their type is $2m$. Let $(x,y) = (\Im z_1, \Im z_2)$. We introduce two variables $(\tau, \eta)$, which are defined by

$$\begin{cases} \tau = \chi(1 - f(x)/y), \\ \eta = y^{1/m}, \end{cases}$$

where the function $\chi \in C^\infty([0,1))$ satisfies the conditions: $\chi'(u) \geq 1/2$ on $[0,1]$, and $\chi(u) = u$ for $0 \leq u \leq 1/3$ and $\chi(u) = 1 - (1-u)^{\frac{1}{2m}}$ for $1 - 1/3^{2m} \leq u \leq 1$. The Bergman kernel of $\Omega_f$ can be clearly expressed in terms of the above variables in [4]:

$$K(z,z) = \frac{\tilde{\Phi}(\tau,\eta)}{\eta^{2m+1}} + \tilde{\tilde{\Phi}}(\tau,\eta)\log\eta, \tag{3.1}$$

where $\tilde{\Phi} \in C^\infty((0,1] \times [0,\epsilon))$ and $\tilde{\tilde{\Phi}} \in C^\infty([0,1] \times [0,\epsilon))$.

Now let us investigate the boundary behavior of the Bergman metric of $\Omega_f$ near weakly pseudoconvex points by using the above asymptotic formula. Let $\mathcal{S}_k$ be the class of all functions which can be written in the form $f(\tau, \eta, \eta^k \log\eta)$ with $f \in C^\infty((0,1] \times [0,\epsilon) \times (-\epsilon, \epsilon))$. Note that if $k < k'$, then $\mathcal{S}_k \supset \mathcal{S}_{k'}$. The boundary behavior of the functions $g_{j,\bar{k}}(z)$ in (2.1) can also be clearly expressed in terms of $(\tau, \eta)$.

**Proposition 3.1.**

$$\begin{pmatrix} g_{1\bar{1}} & g_{1\bar{2}} \\ g_{2\bar{1}} & g_{2\bar{2}} \end{pmatrix} = \begin{pmatrix} \dfrac{H_{1\bar{1}}}{\eta} & \dfrac{H_{1\bar{2}}}{\eta^{m+1/2}} \\ \dfrac{H_{2\bar{1}}}{\eta^{m+1/2}} & \dfrac{H_{2\bar{2}}}{\eta^{2m}} \end{pmatrix}, \tag{3.2}$$

*where* $H_{1\bar{1}} \in \mathcal{S}_{2m+1}$, $H_{1\bar{2}} = H_{2\bar{1}} \in \mathcal{S}_{2m}$, $H_{2\bar{2}} \in \mathcal{S}_{2m-1}$.

From the above proposition, we can obtain the complete estimate of the Bergman metric of $\Omega_f$ on an approach region $\mathcal{U}_\alpha := \{(z_1, z_2) \in \Omega_f; \Im z_2 > \alpha f(\Im z_1)\}$ $(\alpha > 1)$.

**Corollary 3.1** ([2]). *If* $z \in \mathcal{U}_\alpha$ *and* $|\xi| = 1$, *then there exist positive constants* $C_\alpha, C'_\alpha$ *depending on* $\alpha$ *such that*

$$C_\alpha \left( \frac{|\xi_1|}{\eta^{1/2}} + \frac{|\xi_2|}{\eta^m} \right) \leq F_B(z;\xi) \leq C'_\alpha \left( \frac{|\xi_1|}{\eta^{1/2}} + \frac{|\xi_2|}{\eta^m} \right)$$

*Proof of Proposition 3.1.* From the asymptotic formula (3.1),

$$F(z) = \phi(\tau, \eta) - (2m+1)\log \eta,$$

where $\phi(\tau,\eta) = \log(\tilde{\Phi}(\tau,\eta) + \tilde{\tilde{\Phi}}(\tau,\eta)\eta^{2m+1}\log\eta)$.

Since $F$ is a function depending only on the variables $(x, y)$, $F_{z_1\bar{z}_1} = 1/4F_{xx}$, $F_{z_1\bar{z}_2} = 1/4F_{xy}$, $F_{z_2\bar{z}_1} = 1/4F_{yx}$ and $F_{z_2\bar{z}_2} = 1/4F_{yy}$.

$$F_x = \phi_\tau \frac{\partial \tau}{\partial x}, \quad F_y = F_\eta \frac{\partial \eta}{\partial y} = \frac{1/m}{\eta^{m-1}} \left\{ \phi_\tau \frac{\partial \tau}{\partial \eta} + \phi_\eta - \frac{2m+1}{\eta} \right\}.$$

$$F_{xx} = \phi_\tau \frac{\partial^2 \tau}{\partial x^2} + \phi_{\tau\tau} \left( \frac{\partial \tau}{\partial x} \right)^2,$$

$$F_{xy} = F_{yx} = F_{x\eta} \frac{\partial \eta}{\partial y} = \frac{1/m}{\eta^{m-1}} \left\{ \phi_\tau \frac{\partial^2 \tau}{\partial \eta \partial x} + \phi_{\tau\eta} \frac{\partial \tau}{\partial x} \right\},$$

$$F_{yy} = F_\eta \frac{\partial^2 \eta}{\partial y^2} + F_{\eta\eta} \left( \frac{\partial \eta}{\partial y} \right)^2 = \frac{1/m(1/m-1)}{\eta^{2m-1}} \left\{ \phi_\tau \frac{\partial \tau}{\partial \eta} + \phi_\eta - \frac{2m+1}{\eta} \right\}$$
$$+ \frac{1/m^2}{\eta^{2m-2}} \left\{ \phi_\tau \frac{\partial^2 \tau}{\partial \eta^2} + \phi_{\tau\eta} \frac{\partial \tau}{\partial \eta} + \phi_{\eta\eta} + \frac{2m+1}{\eta^2} \right\}.$$

If we admit the two lemmas below, then we obtain the proposition from the above equations. □

**Lemma 3.1.**

$$\frac{\partial \tau}{\partial x} = \frac{c_1(\tau,\eta)}{\eta^{1/2}}, \quad \frac{\partial \tau}{\partial \eta} = \frac{c_2(\tau,\eta)}{\eta},$$

$$\frac{\partial^2 \tau}{\partial x^2} = \frac{c_3(\tau,\eta)}{\eta}, \quad \frac{\partial^2 \tau}{\partial \eta \partial x} = \frac{c_4(\tau,\eta)}{\eta^{1+1/2}}, \quad \frac{\partial^2 \tau}{\partial \eta^2} = \frac{c_5(\tau,\eta)}{\eta^2},$$

*where* $c_j \in C^\infty((0,1] \times [0,\epsilon))$ $(j = 1, \dots, 5)$.

**Lemma 3.2.**

$$\phi_\tau, \phi_{\tau\tau} \in \mathcal{S}_{2m+1}, \quad \phi_\eta, \phi_{\tau\eta} \in \mathcal{S}_{2m}, \quad \phi_{\eta\eta} \in \mathcal{S}_{2m-1}. \tag{3.3}$$

*Proofs of the above two lemmas.* Much computation is necessary for the proofs but it is easy. □

## 4. The proof of Theorem 2.1

By simple transformation, it is sufficient to prove the theorem in the case of the domain $\Omega_f$, which appears in Section 3. Moreover the argument of [5],[1] implies that it is enough to check the equivalence in the theorem on the set $\mathcal{N} := \{(z_1, z_2) \in \Omega_f; \Im z_1 = 0\}$.

(Proof of "(1) ⇔ (2)".) The argument of Theorem 2.1 in [5] can be easily generalized to this case. We remark that if $z \in \mathcal{N}$, then

$$\left|\frac{\partial}{\partial z_2} f(z)\right| \le \frac{C_1}{r} \le \frac{C_2}{\eta^m}$$

implies

$$\left|\frac{\partial}{\partial z_1} f(z)\right| \le \frac{C_3}{r^{1/(2m)}} \le \frac{C_4}{\eta^{1/2}},$$

where $C_j$ is a positive constant depending only on $\Omega_f$ by a similar argument in [6].

(Proof of "(2) ⇒ (3).") Let $(g^{\bar{j}k})$ be the inverse matrix of $(g_{j\bar{k}})$. From Proposition 3.1, the following is obtained by easy computation.

$$\begin{pmatrix} g^{\bar{1}1} & g^{\bar{1}2} \\ g^{\bar{2}1} & g^{\bar{2}2} \end{pmatrix} = \begin{pmatrix} H^{\bar{1}1}\eta & H^{\bar{1}2}\eta^{m+1/2} \\ H^{\bar{2}1}\eta^{m+1/2} & H^{\bar{2}2}\eta^{2m} \end{pmatrix}, \tag{4.1}$$

where $H^{\bar{j}k} \in \mathcal{S}_{2m-1}$ $(j, k = 1, 2)$. From the above we get

$$\|\tilde{\nabla} f(z)\| \le \eta^{1/2} \left|\frac{\partial}{\partial z_1} f(z)\right| + \eta^m \left|\frac{\partial}{\partial z_2} f(z)\right|.$$

Therefore the boundedness of $\|\tilde{\nabla} f(z)\|$ can be shown by the remark in the proof of "(1) ⇒ (2)".

(Proof of "(3) ⇒ (4).") This part is obvious.

(Proof of "(4) ⇒ (2).") For $a = (0, a_2) \in \Omega_f$, let $P(a)$ be a polydisk defined by

$$P(a) = \{(w_1, w_2) \in \Omega_f; |w_1| < \gamma_1(\Im a_2)^{1/(2m)} \text{ and } |w_2 - a_2| < \gamma_2(\Im a_2)\},$$

where $\gamma_j$ is a positive constant depending only on $\Omega_f$. Now the following lemma is analogous to Lemma 3 in [1].

**Lemma 4.1.** *If $w \in P(a)$, then the Bergman kernel $K$ of $\Omega_f$ has the following estimate:*

$$c_1(\Im a_2)^{-2-1/m} \leq K(a,w) \leq c_2(\Im a_2)^{-2-1/m},$$

*where $c_1, c_2$ are positive constants depending only on $\gamma_1, \gamma_2$ and $\Omega_f$.*

*Proof.* We only note that this lemma can be deduced from an integral representation of the Bergman kernel as in [4]. □

By using Lemma 4.1, we can show "(4) ⇒ (2)" by a similar fashion in the argument of [1], p377-379.

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