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Some Pseudo-Order of Fuzzy Sets on $\mathbb{R}^n$

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Abstract

The aim of this paper is to define an order on a class of fuzzy sets which is extending a pseudo-order for fuzzy numbers, and its characterization and several relations of the previous results are discussed. The idea comes from a set-relation in n-dimensional Euclid space given by Kuroiwa, Tanaka and Ha (1997). We induce the order of a class of fuzzy sets by a closed convex cone and characterize it by using the projection into the dual cone. Especially, a structure of the lattice is described for the class of rectangle-type fuzzy sets.

Keywords: Pseudo-order, fuzzy max order, multidimensional fuzzy sets, rectangle-type fuzzy sets.

1. Introduction and notations

In the theory of optimization it is a quite important problem how to induce a natural definition of order on the class of considering systems. Since it isn't a simple problem about a order on the fuzzy set theory, many auther tried to consider its natural extension.

Ramík and Řimánek [8] has introduced a partial order on the set of fuzzy numbers, called the fuzzy max order. The present authors also tried to optimize the dynamic fuzzy system [4]. Also there are various types of order relations on the class of fuzzy numbers. See [3], [11] and their references. Congxin and Cong [1] have descrived the fuzzy number lattice.

This paper is to extend the fuzzy max order of fuzzy numbers to a class of fuzzy sets defined on $\mathbb{R}^n$. The pseudo order for fuzzy sets is induced by a closed convex cone K in $\mathbb{R}^n$ and characterized by the projection in the dual cone K^+. Also, the structure of a lattice is discussed for the class of rectangle-type fuzzy sets. By our works we can imagine the much wider application to the fuzzy optimization problem. Our idea of the motivation originates from a set-relation in $\mathbb{R}^n$ given by Kuroiwa, Tanaka and Ha [5] and Kuroiwa [6], in which various types of set-relations in $\mathbb{R}^n$ are used in set-valued optimizations.

In the remainder of this section, we will give some notations and review a vector ordering of $\mathbb{R}^n$ by a convex cone. Let $\mathbb{R}$ be the set of all real numbers and $\mathbb{R}^n$ an n-dimensional Euclidean space. We write fuzzy sets on $\mathbb{R}^n$ by their membership functions $\tilde{s}:\mathbb{R}^n\to[0,1]$ (see Novák [7] and Zadeh [10]). The α-cut ($\alpha\in[0,1]$) of the fuzzy set $\tilde{s}$ on $\mathbb{R}^n$ is defined as

$$\tilde{s}_\alpha := \{x\in\mathbb{R}^n \mid \tilde{s}(x)\geq\alpha\}\ (\alpha>0) \quad \text{and} \quad \tilde{s}_0 := \mathrm{cl}\{x\in\mathbb{R}^n \mid \tilde{s}(x)>0\},$$

where cl denotes the closure of the set. A fuzzy set $\tilde{s}$ is called convex if

$$\tilde{s}(\lambda x + (1-\lambda)y) \geq \tilde{s}(x) \wedge \tilde{s}(y) \quad x, y \in \mathbb{R}^n, \ \lambda \in [0,1],$$

where $a \wedge b = \min\{a, b\}$. Note that $\tilde{s}$ is convex iff the α-cut $\tilde{s}_\alpha$ is a convex set for all $\alpha \in [0,1]$. Let $\mathcal{F}(\mathbb{R}^n)$ be the set of all convex fuzzy sets whose membership functions $\tilde{s} : \mathbb{R}^n \to [0,1]$ are upper-semicontinuous and normal ($\sup_{x \in \mathbb{R}^n} \tilde{s}(x) = 1$) and have a compact support. When the one-dimensional case $n = 1$, the fuzzy sets are called fuzzy numbers and $\mathcal{F}(\mathbb{R})$ denotes the set of all fuzzy numbers.

Let $\mathcal{C}(\mathbb{R}^n)$ be the set of all compact convex subsets of $\mathbb{R}^n$, and $\mathcal{C}_r(\mathbb{R}^n)$ be the set of all rectangles in $\mathbb{R}^n$. For $\tilde{s} \in \mathcal{F}(\mathbb{R}^n)$, we have $\tilde{s}_\alpha \in \mathcal{C}(\mathbb{R}^n)$ ($\alpha \in [0,1]$). We write a rectangle in $\mathcal{C}_r(\mathbb{R}^n)$ by

$$[x, y] = [x_1, y_1] \times [x_2, y_2] \times \cdots [x_n, y_n]$$

for $x = (x_1, x_2, \cdots, x_n), y = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$ with $x_i \leq y_i$ ($i = 1, 2, \cdots, n$). For the case of $n = 1$, $\mathcal{C}(\mathbb{R}) = \mathcal{C}_r(\mathbb{R})$ and it denotes the set of all bounded closed intervals. When $\tilde{s} \in \mathcal{F}(\mathbb{R}^n)$ satisfies $\tilde{s}_\alpha \in \mathcal{C}_r(\mathbb{R}^n)$ for all $\alpha \in [0,1]$, $\tilde{s}$ is called a rectangle-type. We denote by $\mathcal{F}_r(\mathbb{R}^n)$ the set of all rectangle-type fuzzy sets on $\mathbb{R}^n$. Obviously $\mathcal{F}_r(\mathbb{R}) = \mathcal{F}(\mathbb{R})$.

The definitions of addition and scalar multiplication on $\mathcal{F}(\mathbb{R})$ are as follows: For $\tilde{m}, \tilde{n} \in \mathcal{F}(\mathbb{R})$ and $\lambda \geq 1$,

$$(1.1) \qquad (\tilde{m} + \tilde{n})(x) := \sup_{x_1, x_2 \in \mathbb{R};\ x_1 + x_2 = x} \{\tilde{m}(x_1) \wedge \tilde{n}(x_2)\},$$

$$(1.2) \qquad (\lambda \tilde{m})(x) := \begin{cases} \tilde{m}(x/\lambda) & \text{if } \lambda > 0 \\ I_{\{0\}}(x) & \text{if } \lambda = 0 \end{cases} \quad (x \in \mathbb{R}),$$

where $I_{\{\cdot\}}(\cdot)$ is an indicator. By using set operations $A + B := \{x + y \mid x \in A, y \in B\}$ and $\lambda A := \{\lambda x \mid x \in A\}$ for any non-empty sets $A, B \subset \mathbb{R}$, the following holds immediately.

$$(1.3) \qquad (\tilde{m} + \tilde{n})_\alpha := \tilde{m}_\alpha + \tilde{n}_\alpha \quad \text{and} \quad (\lambda \tilde{m})_\alpha = \lambda \tilde{m}_\alpha \quad (\alpha \in [0,1]).$$

Let K be a non-empty cone of $\mathbb{R}^n$. Using this K, we can define a pseudo-order relation $\preccurlyeq_K$ on $\mathbb{R}$ by $x \preccurlyeq_K y$ iff $y - x \in K$. Let $\mathbb{R}^n_+$ be the subset of entrywise non-negative elements in $\mathbb{R}^n$. When $K = \mathbb{R}^n_+$, the order $\preccurlyeq_K$ will be denoted by $\preccurlyeq_n$ and $x \preccurlyeq_n y$ means that $x_i \leq y_i$ for all $i = 1, 2, \cdots, n$, where $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$.

In Section 2, we will introduce a pseudo-order relation on $\mathcal{F}(\mathbb{R}^n)$ which is characterized by the scaralization technique. In section 3, the lattice structure is discussed for the class of rectangle-type fuzzy sets.

2. A pseudo-order on $\mathcal{F}(\mathbb{R}^n)$

First we introduce a binary relation on $\mathcal{C}(\mathbb{R}^n)$, by which a pseudo-order on $\mathcal{F}(\mathbb{R}^n)$ is given. Henceforth we assume that the convex cone $K \subset \mathbb{R}^n$ is given.

We define a binary relation $\preccurlyeq_K$ on $\mathcal{C}(\mathbb{R}^n)$: For $A, B \in \mathcal{C}(\mathbb{R}^n)$, $A \preccurlyeq_K B$ means the following (C.a) and (C.b) (c.f. [5], [6]):

(C.a) For any $x \in A$, there exists $y \in B$ such that $x \preccurlyeq_K y$.

(C.b) For any $y \in B$, there exists $x \in A$ such that $x \preccurlyeq_K y$.

Lemma 2.1. *The relation $\preccurlyeq_K$ is a pseudo-order on $\mathcal{C}(\mathbb{R}^n)$.*

Proof. It is trivial that $A \preccurlyeq_K A$ for $A \in \mathcal{C}(\mathbb{R}^n)$. Let $A, B, C \in \mathcal{C}(\mathbb{R}^n)$ such that $A \preccurlyeq_K B$ and $B \preccurlyeq_K C$. We will check $A \preccurlyeq_K C$ by two cases (c.a) and (C.b). Case(C.a): Since $A \preccurlyeq_K B$ and $B \preccurlyeq_K C$, for any $x \in A$ there exists $y \in B$ such that $x \preccurlyeq_K y$ and there exists $z \in C$ such that $y \preccurlyeq_K z$. Since $\preccurlyeq_K$ is a pseudo-order on $\mathbb{R}^n$, we have $x \preccurlyeq_K z$. Therefore it holds that for any $x \in A$ there exists $z \in C$ such that $x \preccurlyeq_K z$. Case(C.b):Since $A \preccurlyeq_K B$ and $B \preccurlyeq_K C$, for any $z \in C$ there exists $y \in B$ such that $y \preccurlyeq_K z$ and there exists $x \in A$ such that $x \preccurlyeq_K y$. Since $\preccurlyeq_K$ is a pseudo-order on $\mathbb{R}^n$, we have $x \preccurlyeq_K z$. Therefore it holds that for any $z \in C$ there exists $x \in A$ such that $x \preccurlyeq_K z$.

From the above (a) and (b), we obtain $A \preccurlyeq_K C$. Thus the lemma holds. Q.E.D.

When $K = \mathbb{R}^n_+$, the relation $\preccurlyeq_K$ on $\mathcal{C}(\mathbb{R}^n)$ will be written simply by $\preccurlyeq_n$ and for $[x, y], [x', y'] \in \mathcal{C}_r(\mathbb{R}^n)$, $[x, y] \preccurlyeq_n [x', y']$ means $x \preccurlyeq_n x'$ and $y \preccurlyeq_n y'$.

Next, we introduce a binary relation $\preccurlyeq_K$ on $\mathcal{F}(\mathbb{R}^n)$: Let $\tilde{s}, \tilde{r} \in \mathcal{F}(\mathbb{R}^n)$. The relation $\tilde{s} \preccurlyeq_K \tilde{r}$ means the following (F.a) and (F.b):

(F.a) For any $x \in \mathbb{R}^n$, there exists $y \in \mathbb{R}^n$ such that $x \preccurlyeq_K y$ and $\tilde{s}(x) \le \tilde{r}(y)$.

(F.b) For any $y \in \mathbb{R}^n$, there exists $x \in \mathbb{R}^n$ such that $x \preccurlyeq_K y$ and $\tilde{s}(x) \ge \tilde{r}(y)$.

Lemma 2.2. *The relation $\preccurlyeq_K$ is a pseudo-order on $\mathcal{F}(\mathbb{R}^n)$.*

Proof. It is trivial that $\tilde{s} \preccurlyeq_K \tilde{s}$ for $\tilde{s} \in \mathcal{F}(\mathbb{R}^n)$. Let $\tilde{s}, \tilde{r}, \tilde{p} \in \mathcal{F}(\mathbb{R}^n)$ such that $\tilde{s} \preccurlyeq_K \tilde{r}$ and $\tilde{r} \preccurlyeq_K \tilde{p}$. We will check $\tilde{s} \preccurlyeq_K \tilde{p}$ by two cases (F.a) and (F.b). Case(F.a):Since $\tilde{s} \preccurlyeq_K \tilde{r}$ and $\tilde{r} \preccurlyeq_K \tilde{p}$, for any $x \in \mathbb{R}^n$ there exists $y \in \mathbb{R}^n$ such that $x \preccurlyeq_K y$ and $\tilde{s}(x) \le \tilde{r}(y)$, and there exists $z \in \mathbb{R}^n$ such that $y \preccurlyeq_K z$ and $\tilde{r}(y) \le \tilde{p}(z)$. Since $\preccurlyeq_K$ is a pseudo-order on $\mathbb{R}^n$, we have $x \preccurlyeq_K z$ and $\tilde{s}(x) \le \tilde{p}(z)$. Therefore it holds that for any $x \in \mathbb{R}^n$ there exists $z \in \mathbb{R}^n$ such that $x \preccurlyeq_K z$ and $\tilde{s}(x) \le \tilde{p}(z)$. Case(F.b) Since $\tilde{s} \preccurlyeq_K \tilde{r}$ and $\tilde{r} \preccurlyeq_K \tilde{p}$, for any $z \in \mathbb{R}^n$ there exists $y \in \mathbb{R}^n$ such that $y \preccurlyeq_K z$ and $\tilde{s}(x) \ge \tilde{r}(y)$, and there exists $x \in \mathbb{R}^n$ such that $x \preccurlyeq_K y$ and $\tilde{s}(x) \ge \tilde{r}(y)$. Since $\preccurlyeq_K$ is a pseudo-order on $\mathbb{R}^n$, we have $x \preccurlyeq_K z$. Therefore it holds that for any $z \in \mathbb{R}^n$ there exists $x \in \mathbb{R}^n$ such that $x \preccurlyeq_K z$ and $\tilde{s}(x) \ge \tilde{p}(z)$.

From the above (a) and (b), we obtain $\tilde{s} \preccurlyeq_K \tilde{p}$. Thus the lemma holds. Q.E.D.

The following lemma implies the correspondence between the pseudo-order on $\mathcal{F}(\mathbb{R}^n)$ for fuzzy sets and the pseudo-order on $\mathcal{C}(\mathbb{R}^n)$ for the α-cuts.

Lemma 2.3. *Let $\tilde{s}, \tilde{r} \in \mathcal{F}(\mathbb{R}^n)$. $\tilde{s} \preccurlyeq_K \tilde{r}$ on $\mathcal{F}(\mathbb{R}^n)$ if and only if $\tilde{s}_\alpha \preccurlyeq_K \tilde{r}_\alpha$ on $\mathcal{C}(\mathbb{R}^n)$ for all $\alpha \in (0, 1]$.*

Proof. Let $\tilde{s}, \tilde{r} \in \mathcal{F}(\mathbb{R}^n)$ and $\alpha \in (0,1]$. Suppose $\tilde{s} \preccurlyeq_K \tilde{r}$ on $\mathcal{F}(\mathbb{R}^n)$. Then, Two cases (a) and (b) are considered. Case(a): Let $x \in \tilde{s}_\alpha$. Since $\tilde{s} \preccurlyeq_K \tilde{r}$, there exists $y \in \mathbb{R}^n$ such that $x \preccurlyeq_K y$ and $\alpha \leq \tilde{s}(x) \leq \tilde{r}(y)$. Namely $y \in \tilde{r}_\alpha$. Case(b): Let $y \in \tilde{r}_\alpha$. Since $\tilde{s} \preccurlyeq_K \tilde{r}$, there exists $x \in \mathbb{R}^n$ such that $x \preccurlyeq_K y$ and $\tilde{s}(x) \geq \tilde{r}(y) \geq \alpha$. Namely $x \in \tilde{s}_\alpha$.

Therefore we get $\tilde{s}_\alpha \preccurlyeq_K \tilde{r}_\alpha$ on $\mathcal{C}(\mathbb{R}^n)$ for all $\alpha \in (0,1]$ from the above (a) and (b).

On the other hand, suppose $\tilde{s}_\alpha \preccurlyeq_K \tilde{r}_\alpha$ on $\mathcal{C}(\mathbb{R}^n)$ for all $\alpha \in (0,1]$. Then, Two cases (a') and (b') are considered. Case(a'): Let $x \in \mathbb{R}^n$. Put $\alpha = \tilde{s}(x)$. If $\alpha = 0$, then $x \preccurlyeq_K x$ and $\tilde{s}(x) = 0 \leq \tilde{r}(x)$. While, if $\alpha > 0$, then $x \in \tilde{s}_\alpha$. Since $\tilde{s}_\alpha \preccurlyeq_K \tilde{r}_\alpha$, there exists $y \in \tilde{r}_\alpha$ such that $x \preccurlyeq_K y$. And we have $\tilde{s}(x) = \alpha \leq \tilde{r}(y)$. Case(b'): Let $y \in \mathbb{R}^n$. Put $\alpha = \tilde{r}(y)$. If $\alpha = 0$, then $x \preccurlyeq_K x$ and $\tilde{s}(x) \geq 0 = \tilde{r}(y)$. While, if $\alpha > 0$, then $y \in \tilde{r}_\alpha$. Since $\tilde{s}_\alpha \preccurlyeq_K \tilde{r}_\alpha$, there exists $x \in \tilde{s}_\alpha$ such that $x \preccurlyeq_K y$. And we have $\tilde{s}(x) \geq \alpha = \tilde{r}(y)$.

Therefore we get $\tilde{s} \preccurlyeq_K \tilde{r}$ on $\mathcal{F}(\mathbb{R}^n)$ from the above Case (a') and (b'). Thus we obtain this lemma. Q.E.D.

Define the dual cone of a cone K by

$$K^+ := \{a \in \mathbb{R}^n \mid a \cdot x \geq 0 \text{ for all } x \in K\},$$

where $x \cdot y$ denotes the inner product on $\mathbb{R}^n$ for $x, y \in \mathbb{R}^n$. For a subset $A \subset \mathbb{R}^n$ and $a \in \mathbb{R}^n$, we define

$$a \cdot A := \{a \cdot x \mid x \in A\} \ (\subset \mathbb{R}). \tag{2.1}$$

The equation (2.1) means the projection of A on the extended line of the vector a if $a \cdot a = 1$. It is trivial that $a \cdot A \in \mathcal{C}(\mathbb{R})$ if $A \in \mathcal{C}(\mathbb{R}^n)$ and $a \in \mathbb{R}^n$.

Lemma 2.4. *Let $A, B \in \mathcal{C}(\mathbb{R}^n)$. $A \preccurlyeq_K B$ on $\mathcal{C}(\mathbb{R}^n)$ if and only if $a \cdot A \preccurlyeq_1 a \cdot B$ on $\mathcal{C}(\mathbb{R})$ for all $a \in K^+$, where $\preccurlyeq_1$ is the natural order on $\mathcal{C}(\mathbb{R})$.*

Proof. Suppose $A \preccurlyeq_K B$ on $\mathcal{C}(\mathbb{R}^n)$. Consider the two cases (a) and (b). Case(a): For any $a \cdot x \in a \cdot A$, there exists $y \in B$ such that $x \preccurlyeq_K y$. Then $y - x \in K$. If $a \in K^+$, then $a \cdot (y - x) \geq 0$ and i.e. $a \cdot x \leq a \cdot y$. Case(b): For any $a \cdot y \in a \cdot B$, there exists $x \in A$ such that $x \preccurlyeq_K y$. Then $y - x \in K$. If $a \in K^+$, then $a \cdot (y - x) \geq 0$ and i.e. $a \cdot x \leq a \cdot y$. From the above cases (a) and (b), we have that $a \cdot A \preccurlyeq_1 a \cdot B$.

On the other hand, to prove the inverse statement, we assume that $A \preccurlyeq_K B$ on $\mathcal{C}(\mathbb{R}^n)$ does not hold. Then we have the following two cases (i) and (ii). Case(i): There exists $x \in A$ such that $y - x \notin K$ for all $y \in B$. Then $B \cap (x + K) = \emptyset$. Since B and $x + K$ are closed convex, by the separation theorem there exists $a \in \mathbb{R}$ $(a \neq 0)$ such that $a \cdot y < a \cdot x + a \cdot z$ for all $y \in B$ and all $z \in K$. Hence we suppose that there exists $z \in K$ such that $a \cdot z \geq 0$. Then $\lambda z \in K$ for all $\lambda \geq 0$ since K is a cone, and so we have $a \cdot x + a \cdot \lambda z = a \cdot x + \lambda a \cdot z \to -\infty$ as $\lambda \to \infty$. This contradicts $a \cdot y < a \cdot x + a \cdot z$. Therefore we obtain $a \cdot z \geq 0$ for all $z \in K$. Especially taking $z = 0 \in K$, we get $a \cdot y < a \cdot x$ for all $y \in B$. This contradicts $a \cdot A \preccurlyeq_1 a \cdot B$. Case(ii): There exists $y \in B$ such that $y - x \notin K$ for all $x \in A$. Then we derive the contradiction in a similar way to the case (i).

Therefore the inverse statement holds from the results of the above (i) and (ii). The proof of this lemma is completed. Q.E.D.

For $a \in \mathbb{R}^n$ and $\tilde{s} \in \mathcal{F}(\mathbb{R}^n)$, we define a fuzzy number $a \cdot \tilde{s} \in \mathcal{F}(\mathbb{R})$ by

$$a \cdot \tilde{s}(x) := \sup_{\alpha \in [0,1]} \min\{\alpha, 1_{a \cdot \tilde{s}_\alpha}(x)\}, \quad x \in \mathbb{R}. \tag{2.2}$$

where $1_D(\cdot)$ is the classical indicator function of a closed interval $D \in \mathcal{C}(\mathbb{R})$.

We define a partial relation $\preccurlyeq_M$ on $\mathcal{F}(\mathbb{R})$ as follows ([8]): For $\tilde{s}, \tilde{r} \in \mathcal{F}(\mathbb{R}^n)$, $\tilde{s} \preccurlyeq_M \tilde{r}$ means that $\tilde{s}_\alpha \preccurlyeq_1 \tilde{r}_\alpha$ for all$\alpha \in [0,1]$.

The following theorem gives the correspondence between the pseudo-order $\preccurlyeq_K$ on $\mathcal{F}(\mathbb{R}^n)$ and the fuzzy max order $\preccurlyeq_M$ on $\mathcal{F}(\mathbb{R})$.

Theorem 2.1. *For $\tilde{s}, \tilde{r} \in \mathcal{F}(\mathbb{R}^n)$, $\tilde{s} \preccurlyeq_K \tilde{r}$ iff $a \cdot \tilde{s} \preccurlyeq_M a \cdot \tilde{r}$ for all $a \in K^+$.*

Proof. From Lemmas 2.3 and 2.4, $\tilde{s} \preccurlyeq_K \tilde{r}$ iff $a \cdot \tilde{s}_\alpha \preccurlyeq_1 a \cdot \tilde{r}_\alpha$ for all for all $a \in K^+$ and $\alpha \in (0,1]$. Is equivalent to $a \cdot \tilde{s} \preccurlyeq_M a \cdot \tilde{r}$ for all $a \in K^+$. Q.E.D.

For $\{\tilde{s}_k\}_{k=1}^\infty \subset \mathcal{F}(\mathbb{R}^n)$ and $\tilde{s} \in \mathcal{F}(\mathbb{R}^n)$, $\lim_{k \to \infty} \tilde{s}_k = \tilde{s}$ means that $\sup_{\alpha \in [0,1]} \rho(\tilde{s}_{k,\alpha}, \tilde{s}_\alpha) \to 0$ $(k \to \infty)$, where $\tilde{s}_{k,\alpha}$ is the α-cut of $\tilde{s}_k$ and ρ is the Hausdorff metric on $\mathcal{C}(\mathbb{R}^n)$.

Lemma 2.5. *Let $\{\tilde{s}_k\}_{k=1}^\infty \subset \mathcal{F}(\mathbb{R})$ and $\tilde{s} \in \mathcal{F}(\mathbb{R})$ such that $\tilde{s}_k \preccurlyeq_M \tilde{s}_{k+1}$ $(k \geq 1)$ and $\lim_{k \to \infty} \tilde{s}_k = \tilde{s}$. Then $\tilde{s}_1 \preccurlyeq_M \tilde{s}$.*

Proof. Trivial. Q.E.D.

Theorem 2.2. *Let $\{\tilde{s}_k\}_{k=1}^\infty \subset \mathcal{F}(\mathbb{R}^n)$ and $\tilde{s} \in \mathcal{F}(\mathbb{R}^n)$ such that $\tilde{s}_k \preccurlyeq_K \tilde{s}_{k+1}$ $(k \geq 1)$ and $\lim_{k \to \infty} \tilde{s}_k = \tilde{s}$. Then $\tilde{s}_1 \preccurlyeq_K \tilde{s}$.*

Proof. From Theorem 2.1, for all $a \in K^+$ we have $a \cdot \tilde{s}_k \preccurlyeq_M a \cdot \tilde{s}_{k+1}$ $(k \geq 1)$ and $\lim_{k \to \infty} a \cdot \tilde{s}_k = a \cdot \tilde{s}$. By Lemma 2.3, $a \cdot \tilde{s}_1 \preccurlyeq_M a \cdot \tilde{s}$ all $a \in K^+$. From Theorem 2.1, $\tilde{s}_1 \preccurlyeq_K \tilde{s}$. Q.E.D.

Remark. Let the map $g : [0,1] \to \mathcal{F}(\mathbb{R}^n)$ be continuous. A point x_0 is said to be efficient if $x_0 \in [0,1]$ and $g(x_0) \preccurlyeq_K g(x)$ for some $x \in [0,1]$ implies $g(x) = g(x_0)$. Then, by applying the same idea as in Lemma 3.2 of Furukawa [2], we observe that there exists at least one efficient point in $[0,1]$. In fact, considering, if necessary, a partial order $\preccurlyeq_K$ on the class of the quotient sets with respect to the equivalence relation $\sim_K$ defined by $\tilde{s} \sim_K \tilde{r}$ iff $\tilde{s} \preccurlyeq_K \tilde{r}$ and $\tilde{r} \preccurlyeq_K \tilde{s}$, we can assume that $\preccurlyeq_K$ is a partial order on $\mathcal{F}(\mathbb{R}^n)$. By theorem 2.2 and the continuity of g, $\{g(x) \mid x \in [0,1]\}$ can be proved to be an inductively ordered set. So, by Zorn's lemma $\{g(x) \mid x \in [0,1]\}$ has an efficient element.

3. Further results

In this section, we investigate a pseudo-order $\preccurlyeq_K$ on $\mathcal{F}_r(\mathbb{R}^n)$ for a polyhedral cone K with $K^+ \subset \mathbb{R}^n$. To this end, we need the following lemma.

Lemma 3.1. Let $a, b \in \mathbb{R}^n_+$ and $A \in \mathcal{C}_r(\mathbb{R}^n)$. Then for any scalars $\lambda_1, \lambda_2 \geq 0$, it holds

$$(\lambda_1 a + \lambda_2 b) \cdot A = \lambda_1(a \cdot A) + \lambda_2(b \cdot A), \tag{3.1}$$

where the arithmetic in (3.1) is defined in (2.1).

Proof. Let $\lambda_1 a \cdot x + \lambda_2 b \cdot y \in \lambda_1(a \cdot A) + \lambda_2(b \cdot B)$ with $x, y \in A$. It suffices to show that $\lambda_1 a \cdot x + \lambda_2 b \cdot b \in (\lambda_1 a + \lambda_2 b) \cdot A$. Define $z = (z_1, z_2, \cdots, z_n)$ by

$$\begin{aligned} z_i &= (\lambda_1 a_i x_i + \lambda_2 b_i y_i)/(\lambda_1 a_i + \lambda_2 b_i) \quad \text{if}(\lambda_1 a_i + \lambda_2 b_i) > 0 \\ &= x_i \qquad \text{if}(\lambda_1 a_i + \lambda_2 b_i) = 0 \quad (i = 1, 2, \cdots, n) \end{aligned}$$

Then, clearly $(\lambda_1 a + \lambda_2 b) \cdot z = \lambda_1 a \cdot x + \lambda_2 b \cdot y$. Since $A \in \mathcal{C}_r(\mathbb{R}^n), z \in A$, so that $\lambda_1 a \cdot x + \lambda_2 b \cdot y = (\lambda_1 a + \lambda_2 b) \cdot A$. Q.E.D.

Henceforth, we assume that K is a polyhedral convex cone with $K^+ \subset \mathbb{R}^n$, i.e., there exist vectors $b^i \in \mathbb{R}^n_+ (i = 1, 2, \cdots, m)$ such that

$$K = \{x \in \mathbb{R}^n \mid b^i \cdot x \leq 0 \text{ for all } i = 1, 2, \cdots, m\}.$$

Then, it is well-known (c.f. [9]) that K^+ is expressed as

$$K^+ = \{x \in \mathbb{R}^n \mid x = \sum_{i=1}^{m} \lambda_i b_i,\ \lambda_i \geq 0\ (i = 1, 2, \cdots, m)\}.$$

The above dual cone K^+ is denoted simply by

$$K^+ = \text{conv}\{b^1, b^2, \cdots, b^m\}.$$

The pseudo-order $\preccurlyeq_K$ on $\mathcal{C}_r(\mathbb{R}^n)$ is characterized in the following.

Corollary 3.1. Let $K^+ = \text{conv}\{b^1, b^2, \cdots, b^m\}$ with $b^i \in \mathbb{R}^n_+$. Then, for $A, B \in \mathcal{C}_r(\mathbb{R}^n)$, $A \preccurlyeq_k B$ if and only if $b^i \cdot A \preccurlyeq_1 b^i \cdot B$ for all $i = 1, 2, \cdots, m$, where $\preccurlyeq_1$ is a pseudo-order on $\mathcal{C}_r(\mathbb{R})$.

Proof. We assume that $b^i \cdot A \preccurlyeq_1 b^i \cdot B$ for all $i = 1, 2, \cdots, m$. For any $a \in K^+$, there exists $\lambda_i \geq 0$ with $a = \sum_{i=1}^m \lambda_i b^i$. From Lemma 3.1 we have:

$$a \cdot A = \sum_{i=1}^{m} \lambda_i (b^i \cdot A) \preccurlyeq_1\ = \sum_{i=1}^{m} \lambda_i (b^i \cdot B) = a \cdot B.$$

Thus, by Lemma 2.4, $A \preccurlyeq_K B$ follows. By applying Lemma 2.4 again, the 'only if' part of Corollary holds. Q.E.D.

Lemma 3.2. Let $a, b \in \mathbb{R}^n_+$ and $\tilde{s} \in \mathcal{F}_r(\mathbb{R}^n)$. Then, for any $\lambda_1, \lambda_2 \geq 0$,

$$(\lambda_1 a + \lambda_2 b) \cdot \tilde{s} = \lambda_1(a \cdot \tilde{s}) + \lambda_2(b \cdot \tilde{s}), \tag{3.2}$$

where the arithmetic in (3.2) are given in (1.1), (1.2) and (2.2).

Proof. For any $\alpha \in [0,1]$, it follows from the definition and Lemma 3.1 that

$$\begin{aligned}[(\lambda_1 a + \lambda_2 b)\cdot \tilde{s}]_\alpha &= (\lambda_1 a + \lambda_2 b)\cdot \tilde{s}_\alpha = \lambda_1(a\cdot \tilde{s}_\alpha) + \lambda_2(b\cdot \tilde{s}_\alpha) \\ &= \lambda_1(a\cdot\tilde{s})_\alpha + \lambda_2(b\cdot\tilde{s})_\alpha = [\lambda_1(a\cdot\tilde{s}) + \lambda_2(b\cdot\tilde{s})]_\alpha.\end{aligned}$$

The last equality follows from (1.3). The above shows that (3.3) holds. Q.E.D.

The main results in this section are given in the following.

Theorem 3.1. Let $K^+ = \text{conv}\{b^1, b^2, \cdots, b^m\}$ with $b^i \in \mathbb{R}^n$. Then, for $\tilde{s}, \tilde{r} \in \mathcal{F}_r(\mathbb{R}^n)$,

$$\tilde{s} \preccurlyeq_K \tilde{r} \quad \text{if and only iff} \quad b^i\cdot\tilde{s} \preccurlyeq_M b^i\cdot\tilde{r} \text{ for } i = 1, 2, \cdots, m.$$

Proof. It suffices to prove the 'if' part of Theorem 3.1. For any $a \in K^+$, there exists $\lambda_i \geq 0$ with $a = \sum_{i=1}^m \lambda_i b^i$. Applying Lemma 3.2, we have

$$a\cdot\tilde{s} = \sum_{i=1}^m \lambda_i(b^i\cdot\tilde{s}) \preccurlyeq_M \sum_{i=1}^m \lambda_i(b^i\cdot\tilde{r}) = a\cdot\tilde{r},$$

From Theorem 2.1, $\tilde{s} \preccurlyeq_k \tilde{r}$ follows. Q.E.D.

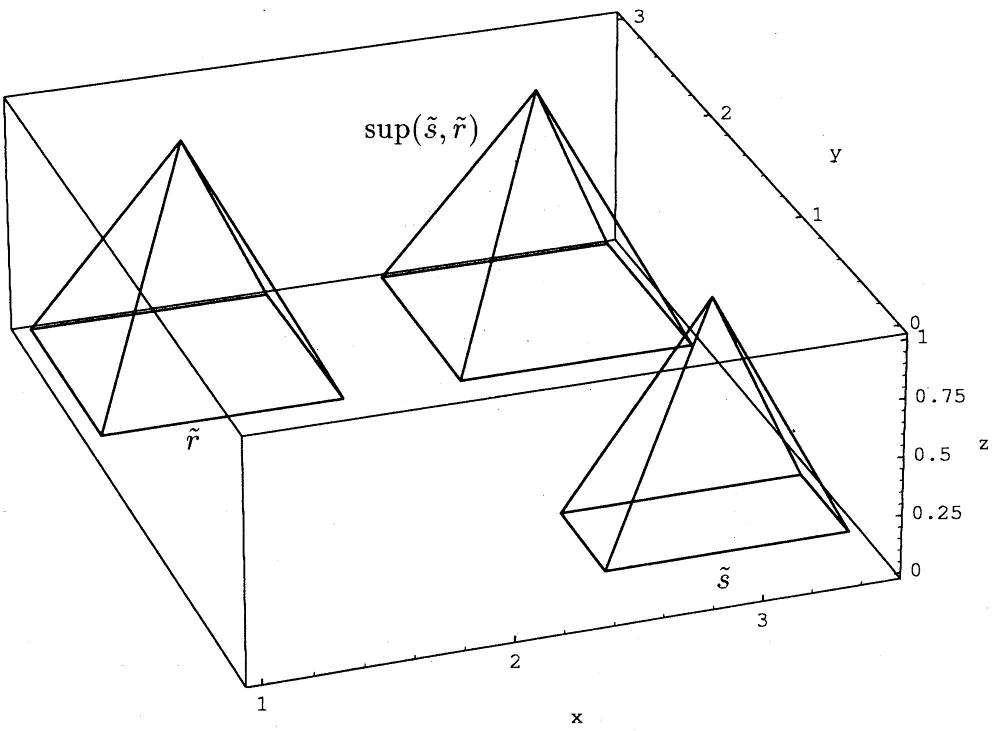

Figure 1: $\tilde{v} = \sup(\tilde{s}, \tilde{r})$

When $K = \mathbb{R}^n$, the pseudo-order $\preccurlyeq_K$ on $\mathcal{F}_r(\mathbb{R}^n)$ will be simply written by $\preccurlyeq_n$. Obviously, $\preccurlyeq_1$ and $\preccurlyeq_M$ are the same.

Congxin and Cong [1] described the structure of the fuzzy number lattice $(\mathcal{F}_r(\mathbb{R}), \preccurlyeq_1)$.

When $K = \mathbb{R}^n$, $K^+ = \mathbb{R}^n$ and $K^+ = \text{conv}\{e^1, e^2, \cdots, e^m\}$. So that, by Theorem 3.1, we see that for $\tilde{s}, \tilde{r} \in \mathcal{F}_r(\mathbb{R}^n)$, $\tilde{s} \preccurlyeq_n \tilde{r}$ means $e^i\tilde{s} \preccurlyeq_1 e^i\tilde{r}$ for all $i = 1, 2, \cdots, n$. Therefore, by applying the same method as [1], we can describe the structure of the fuzzy set lattice $(\mathcal{F}_r(\mathbb{R}^n), \preccurlyeq_n)$. Figure 1 illustrates $\sup(\tilde{s}, \tilde{r})$ for $\tilde{s}, \tilde{r} \in \mathcal{F}_r(\mathbb{R}^2)$.

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