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On the activity level increasing rationality condition in multichoice games

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Abstract.

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1 Introduction.

In [1] Hsiao and Raghavan introduced a class of multichoice cooperative games and found for it the Shapley value using an axiomatic approach. Later, Nouweland [2] determined the Shapley value for multichoice cooperative games following its probabilistic interpretation. It is happened that these two methods lead to quite different values. We should also mention the determination of the Shapley value through a potential function proposed by Calvo and Santos [3]. In our paper while avoiding the problem of inconsistency of the Shapley value between Hsiao-Raghavan and Nouweland, we consider a necessary and sufficient condition for the Shapley value by Nouweland to be in the core of a multichoice cooperative game.

It is well known that in the class of usual cooperative games with the characteristic function form the Shapley value is in the core if the characteristic function is either convex ([4]), average convex ([5]), or totally convex ([6]). In the last paper it has been shown that the class of totally convex games includes that of average convex games. We are interested in conditions leading the Shapley value to be in the core on the class of multichoice cooperative games as well.

2 Multichoice cooperative game.

First of all, we describe the multichoice cooperative game (MCG) introduced in [1]. Let $N = \{1, 2, \ldots, n\}$ be the set of players, $M_i = \{0, 1, 2, \ldots, m_i\}$ the set of activity levels of player i for $i \in N$, but we assume that $m_i = m$ for all $i \in N$ as considered in [1]. A coalition in this game is denoted by a vector $s = (s_1, \ldots, s_n)$, where for each $i \in N$, $s_i \in M_i$ shows activity of player i in the coalition s. If a player does not participate in the coalition, his level of activity is zero. Hence, the "empty" coalition is $0 = (0, \ldots, 0)$. We denote the set of all feasible coalitions by M, that is, $M = M_1 \times \cdots \times M_n$. Throughout this paper, a coalition $s \wedge t = (\min\{s_1, t_1\}, \min\{s_2, t_2\}, \ldots, \min\{s_n, t_n\})$ is considered as the intersection of coalitions s and t, and a coalition $s \vee t = (\max\{s_1, t_1\}, \max\{s_2, t_2\}, \ldots, \max\{s_n, t_n\})$ is admitted as the union of s and t. Within given notations a superadditive function $v: M \to R^1$ with $v(0) = 0$ is called a characteristic function of a MCG. We denote such MCG by $G(v, N)$.

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The imputations of a MCG are presented by $(m+1) \times n$ – dimensional matrices. Let $I(v, N)$ be the set of all imputations in $G(v, N)$, that is, $I(v,N) = \Big\{\xi = \{\xi_{ij}\} \,\Big|\, \sum_{j=0}^{s_i} \xi_{ij} \geq v\big((0,\ldots,0,s_i,0,\ldots,0)\big)\, \forall i \in N \text{ and } \forall s_i \in M_i, \text{ and } \sum_{i=1}^{n}\sum_{j=0}^{m} \xi_{ij} = v\big((m,\ldots,m)\big)\Big\}$. We shall say that the set $C(v,N) = \Big\{\xi \in I(v,N) \,\Big|\, \sum_{i:s_i\neq 0}\sum_{j=0}^{s_i} \xi_{ij} \geq v(s) \text{ for all } s \in M\Big\}$ is the core of $G(v, N)$.

3 The Shapley value.

In [2], the following procedure of the Shapley value construction was proposed. Suppose that a given coalition $s \in M$ is formed step–by–step, starting from zero coalition $0 = (0, \ldots, 0)$ and on every stage one of the players increases his level of activity by 1. So after $k(s) = \sum_{i:s_i\neq 0} s_i$ steps the coalition s will be created. Let $w = \{w_1, \ldots, w_{k(m)}\}$ be an order of the coalition $m = \{m, \ldots, m\}$ construction, where w_t is the activity level of some player on a step t of the procedure. We denote the set of steps when player i increases own activity level by T_i. The order w is called admissible if for each player $i \in N$ from $t' < t''$, where t', $t'' \in T_i$, it follows that $w_{t'} < w_{t''}$. The total number of the admissible orders is

$$\Omega(m) = \frac{(\sum_{i\in N} m_i)!}{\prod_{i\in N}(m_i!)} = \frac{(mn)!}{(m!)^n}.$$

Further we will consider only admissible orders. Take an arbitrary coalition $s \in M$ and fix player $l \in N$, $s_l \neq 0$. Suppose that by an order w the given coalition s is created after the first $k(s)$ steps, with the player l completing formation s. The number of such orders is

$$\Omega_l(s) = \frac{\left(\sum_{i:(s|s_l-1)_i\neq 0} s_i\right)! \left(\sum_{i\in N}(m-s_i)\right)!}{\prod_{i:(s|s_l-1)_i\neq 0}(s_i!) \prod_{i\in N}\left((m-s_i)!\right)},$$

where $s|s_l - 1 = (s_1, \ldots, s_{l-1}, s_l - 1, s_{l+1}, \ldots, s_n)$. It is admitted that $\Omega_l(s) = 0$ if $s_l = 0$. Nouweland [2] showed that $\phi = \{\phi_{ij}\}$, where $i = 1, \ldots, n$, $j = 0, \ldots, m$, and

$$\phi_{ij} = \sum_{s:s_i=j} \frac{\Omega_i(s)}{\Omega(m)}[v(s) - v(s|s_i - 1)], \tag{3.1}$$

is the Shapley value of $G(v, N)$. If there is realized a coalition s in the game $G(v, N)$, then the payoff of player i equals $\phi_i(s) = \sum_{j=0}^{s_i} \phi_{ij}$. We call $\phi(s) = \sum_{i:s_i\neq 0} \phi_i(s)$ the payoff of the coalition s according the Shapley value ϕ.

Example. As an illustration of the Shapley value for multichoice game we consider a modification of "Land–lord and farm laborers" example given in Vorobjev [7].

Suppose there are $n-1$ farm laborers (players $i = N \setminus \{1\}$) and a land–lord (player 1). The land–lord engages farm laborers and derives the gathered harvest. The farm–laborers work for the land–lord and cannot derive a benifit for themselves. We characterize their wishing to work by the sets $M_i = \{0, 1, \ldots, m_i\}$, $i \in N \setminus \{1\}$. A time length may be one of the simpliest interpretation of m_i. In our example the farm–laborers can work with an equal enforce, i.e., $m_i = m_j$ for any $i \neq j$, $i, j \in N \setminus \{1\}$. The land–lord does not work and his activity have to be defined by two choices: to engage or not to engage. So that, M_1 is given by $\{0,1\}$. Such relations between players is described by the following characteristic function $v\colon \prod_{i\in N} M_i \to R^1$

$$v(s) = \begin{cases} 0, & s_1 = 0 \text{ or } s_i = 0\ \forall i \in N\setminus\{1\} \\ f(s), & s_1 = 1 \text{ and } \exists\, i \in N\setminus\{1\}\colon\ s_i \neq 0 \end{cases},$$

where real-valued function f satisfies $f(s) \leq f(r)$ for $s \leq r$, $s, r \in M = \prod_{i\in N} M_i$. Suppose that for the land–lord it is important only final result and no matter who fulfills job, i.e., if $\sum_{i\in N} s_i = \sum_{i\in N} r_i$ for $s, r \in M$, then $f(s) = f(r)$. It will convinient to use the function v_t determined by $v_t = v(s)$, where $t = \sum_{i\in N} s_i$. Within given conditions let's find the payoff of the land–lord if the profit is shared according with the Shapley value. By formular (3.1) we have

$$\phi_{11} = \sum_{t=2}^{m(n-1)+1} \sum_{s:\sum_{i\in N} s_i = t, s_1 = 1} \frac{\left(\sum_{i\in N}(s|s_1-1)_i\right)!\left(\sum_{i\in N}(m_i - s_i)\right)!}{\left(\sum_{i\in N} m_i\right)!}$$

$$\frac{\left(\prod_{i\in N} m_i\right)!}{\left(\prod_{i\in N}(s|s_1-1)_i\right)!\left(\prod_{i\in N}(m_i - s_i)\right)!} v(s)$$

After denoting

$$\sum_{s:\sum_{i\in N} s_i = t, s_1 = 1} \frac{\left(\prod_{i\in N} m_i\right)!}{\left(\prod_{i\in N}(s|s_1-1)_i\right)!\left(\prod_{i\in N}(m_i - s_i)\right)!} = Q(t)$$

we can rewrite

$$\begin{aligned} \phi_{11} &= \sum_{t=2}^{m(n-1)+1} \frac{(t-1)!(m(n-1)+1-t)!}{(m(n-1)+1)!} Q(t) v^t \\ &= \sum_{t=2}^{m(n-1)+1} \frac{1}{t}{}_{(m(n-1)+1)}C_t Q(t) v_t. \end{aligned}$$

By the symmetry, each farm–laborer $i \in N\setminus\{1\}$ gets payoff $\phi_i(1, m, \ldots, m) = \frac{f(m)-\phi_{11}}{n-1}$ if he works with the enforce m. Now on the example of level m and $m-1$ we discuss the reasonability to work harder for a farm–laborer. We change the game such that $m_i = m-1$ for $i \in N\setminus\{1\}$. Let in the new game the payoff of the land–lord be ϕ_{11}^{m-1} and the farm–laborer's benifit be $\phi_i^{m-1}(1, m-1, \ldots, m-1)$, $i \in N\setminus\{1\}$. It is easily seen that $\phi_{11} \geq \phi_{11}^{m-1}$. Therefore the land–lord is always interested for his workers to increase productivity. In respect to the farm–laborers, in general, there may be function f that $\frac{f(m)-\phi_{11}-(f(m-1)-\phi_{11}^{m-1})}{n-1} \leq 0$. In this case the farm–laborers has no sense to move from the activity $m-1$ to m. Obviously, as land–lord as workers are stimulated to choose the level m, when the Shapley value is in the core.

4 Total convexity.

Now we turn to the game $G(v, N)$. and for every coalition $s \in M$ define subgame G^s, with the characteristic function v^s being a restriction of v on the set $M^s = \{t \in M \,|\, 0 \leq t_i \leq$

s_i for each $i \in N\}$. We omit the explicit description of the games G^s, $s \in M$, because it is very similar to the definition of the game $G(v, N)$. Denote the Shapley value of G^s by $\phi^s = \{\phi^s_{ij}\}$, $i = 1, \ldots, N$, $j = 0, \ldots, s_i$. Now we find a condition for ϕ to be in $C(v, N)$.

For the sake of comfortability we introduce functions $\delta_i(s) = v(s) - v(s|s_i - 1)$, $s \in M$, $s_i - 1 \geq 0$, $i \in N$. Let t be an arbitrary given coalition in M. From (3.1) we have

$$\begin{aligned}\phi(t) &= \sum_{i:t_i \neq 0} \phi_i(t) \\ &= \sum_{i:t_i \neq 0} \sum_{j=0}^{t_i} \phi_{ij} \\ &= \sum_{i:t_i \neq 0} \sum_{j=0}^{t_i} \sum_{s:s_i = j} \frac{\Omega_i(s)}{\Omega(m)} \delta_i(s) \\ &= \sum_{i:t_i \neq 0} \sum_{s:(s \wedge t)_i \leq t_i} \frac{\Omega_i(s)}{\Omega(m)} \delta_i(s).\end{aligned} \tag{3.2}$$

Note that $t \leq m$ and hence

$$\sum_{s:(s \wedge t)_i \leq t_i} \frac{\Omega_i(s)}{\Omega(m)} \geq \sum_{r:r \leq t} \frac{\Omega_i(r)}{\Omega(t)}.$$

Hence, if

$$\sum_{i:t_i \neq 0} \sum_{s:(s \wedge t)_i \leq t_i} \frac{\Omega_i(s)}{\Omega(m)} (\delta_i(s) - \delta_i(s \wedge t)) \geq 0, \tag{3.3}$$

then expression (3.2) is greater than or equal to

$$\sum_{i:t_i \neq 0} \sum_{r:r \leq t} \frac{\Omega_i(r)}{\Omega(t)} \delta_i(r) = \sum_{i:t_i \neq 0} \sum_{j=0}^{t_i} \sum_{r:r_i = j} \frac{\Omega_i(r)}{\Omega(t)} \delta_i(r) = \sum_{i:t_i \neq 0} \phi^t_i(t) = v(t). \tag{3.4}$$

By inequality (3.3), it is easily seen that

$$\sum_{i:t_i \neq 0} \sum_{s:(s \wedge t)_i \leq t_i} = \sum_{s \in M} \sum_{i:(s \wedge t)_i \leq t_i},$$

with the last summation being zero if $s \wedge t = 0$.

Definition $G(v, N)$ is called a totally convex multichoice game if for all coalition $t \in M$

$$\sum_{s \in M} \sum_{i:(s \wedge t)_i \leq t_i} \frac{\Omega_i(s)}{\Omega(m)} (\delta_i(s) - \delta_i(s \wedge t)) \geq 0. \tag{3.5}$$

Moving backwards, from (3.4) to (3.2) we come to the fact that if the Shapley value of $G(v, N)$ lies in the core, then $G(v, N)$ is totally convex. Thus, we have proved the following theorem.

Theorem. The necessary and sufficient condition for the Shapley value ϕ of MCG $G(v, N)$ to be in the core $C(v, N)$ is total convexity of $G(v, N)$.

Note that the proof of the theorem is valid for the games where players may have different numbers of activity levels: $m_i \neq m_j$ for $i \neq j$, where $i, j \in N$.

Finally, we consider that the definition of a totally convex multichoice game coincides with another definition given by Izawa and Takahashi [6] on the class of usual n-person cooperative games with characteristic form. For the sake of simplicity of explanation, we draw on the definition of total convexity proposed by Izawa–Takahashi.

Definition A cooperative game (v, N) with the set of players $N = \{1, \dots, n\}$ and characteristic function v is totally convex if for any subset T of N,

$$\sum_{S\subset N}\sum_{i\in S\cap T}\frac{(|S|-1)!(n-|S|)!}{n!}\Big[v(S)-v(S\setminus\{i\})-v(S\cap T)+v(S\cap T\setminus\{i\})\Big]\geq 0, \quad (3.6)$$

where the summation $\sum_{S\subset N}$ is taken over all nonempty subsets S of N.

Note that game (v, N) is equivalent to the MCG $G'(v, N)$ such that $M_i = \{0, 1\}$, $i \in N$ and one–to–one correspondence between M and 2^N is constructed as follows: $s_i = 0 \Leftrightarrow i \notin S$ and $s_i = 1 \Leftrightarrow i \in S$ for each $i \in N$. Then $\frac{\Omega_i(s)}{\Omega(m)} = \frac{(|S|-1)!(n-|S|)!}{n!}$, and $\delta_i(s) - \delta_i(s \wedge t) = v(S) - v(S \setminus \{i\}) - v(S \cap T) + v(S \cap T \setminus \{i\})$, where $s \in M$ is related to $S \subset N$. Thus (3.5) coincides with (3.6) on the class of cooperative games.

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