数理解析研究所講究録
1069巻 1998年 92-104

DEVELOPABLE HYPERSURFACES AND ALGEBRAIC HOMOGENEOUS SPACES IN A REAL PROJECTIVE SPACE

GO-O ISHIKAWA (石川 剛郎, 北海道大学理学研究科)

Department of Mathematics, Hokkaido University, Sapporo 060, Japan.
e-mail: ishikawa@math.sci.hokudai.ac.jp

n 次元実射影空間 $\mathbf{R}P^n$ 内の滑らかな超曲面 M が**可展** (developable) とは，その Gauss 写像 $\gamma: M \to \mathrm{Gr}(n, \mathbf{R}^{n+1}) \cong \mathrm{Gr}(1, (\mathbf{R}^{n+1})^*) = \mathbf{R}P^{n*}$ が $\mathrm{rank}(\gamma) < \dim(M) = n-1$ をみたすときにいう．ここで，$\mathrm{rank}(\gamma)$ は γ の微分写像の階数の M 上での最大値を意味する．

3次元空間の developable surface の古典的例として，cylinder, cone, 空間曲線の tangent developable が知られているが，この中で，射影空間 $\mathbf{R}P^3$ 内で特異点を持たないものは平面に限る．可展超曲面は特異点を持ちやすいので，非特異 compact 可展超曲面は非常に限られると期待できる．たとえば同じことを複素数上で考えると，複素射影空間 $\mathbf{C}P^n$ 内の複素解析的 (すなわちこの場合代数的) compact 可展超曲面は射影超平面に限ることが知られている (Griffiths-Harris 1979)．実射影空間内においても，同次 Monge-Ampère 方程式の接触幾何的考察から次のことがわかっている:

定理 1 (Morimoto-I [IM]) $M^{n-1} \subset \mathbf{R}P^n$ が compact 可展超曲面ならば，$r = \mathrm{rank}(\gamma)$ は偶数であり，$r \neq 0$ ならば，$n < \frac{1}{2}r(r+3)$ である．とくに，$\mathrm{rank}(\gamma) \leq 1$ であるものは射影超平面に限る．また，$M^2 \subset \mathbf{R}P^3$ または $M^4 \subset \mathbf{R}P^5$ のときは，M は射影超平面に限られる． □

定理 1 で階数の条件は本質的である．実際，次のような compact 可展超曲面の例を等質空間とその変形から構成できる:

定理 2 ([I]) $n = 4, 7, 13, 25$ に対して n 次元実射影空間に 3 次実代数的非特異可展超曲面が存在する．それらはそれぞれ群 $SO(3), SU(3), Sp(3), F_4$ の等質空間の構造をもつ．そ

1991 *Mathematics Subject Classification*: Primary 58C27, 53C15; Secondary 53A05, 53A20.
Keywords: projective duality, Cayley's octonians, Veronese embedding, Jordan algebra, Monge-Ampère foliation, Severi varieties, isoparametric hypersurface.

の射影双対は，$K = \mathbf{R}, \mathbf{C}, \mathbf{H}, \mathbf{O}$(Cayley 8 元代数, Octonians) に関する射影平面 KP^2 の Veronese embedding の linear projection となる．これらの実代数的可展超曲面は，それぞれ 2, 3, 5, 9 個の関数の自由度をもつ (正確には，$KP^2 \subset \mathbf{R}P^n$ の normal bundle の section の空間を無限小変形にもつような) compact C^∞ 可展超曲面族への変形を持つ． □

$\mathbf{R}^n$ の properly embedded 可展超曲面は，$\mathrm{rank}(\gamma) \leq 1$ ならば cylinder に限る (Hartman-Nirenberg 1959). $\mathbf{C}^n$ の場合も同様の結果が知られている (Abe 1972). $\mathbf{R}^4$ の cylinder でない C^∞ 可展超曲面の最初の例は Sacksteder(1960) により与えられた: $M = \{x_4 = x_1 \cos x_3 + x_2 \sin x_3\}$. また，Mori(1994) は deformable submanifolds の研究との関連で，$\mathbf{R}^4$ の cylinder でない可展超曲面族の例を与えている．また，Akivis(1987) は $\mathbf{R}P^4$ の C^∞ complete 可展超曲面で射影超平面でないものの存在を微分式系の理論から証明しているが具体例は与えていない．最近 Fischer-Wu(1995) により $\mathbf{C}P^n, \mathbf{C}^n, \mathbf{R}^n$ の余次元の高い場合も含めた可展部分多様体が研究されている．Wu の論文の中で，$\mathbf{R}^4$ 内の cylinder でない実代数的可展超曲面の例 (Bourgain によるもの，unpublished) を紹介している: $M = \{x_1 x_4^2 + x_2(x_4 - 1) + x_3(x_4 - 2) = 0\}$. しかし，この例では (Sacksteder の例同様) $\overline{M} \subset \mathbf{R}P^4$ は特異点を持っている．

定理 2 の構成に必要な主な方法は，Jordan 代数上の実射影幾何 (接触幾何) である．一般に，可展部分多様体には Monge-Ampère foliation と呼ばれる，各 leaf 上で接空間が一定であるような foliation があるが，たとえば F_4 の場合，それは fibration

$$\mathbf{O}P^1 \cong \mathrm{Spin}(9)/\mathrm{Spin}(8) \to F_4/\mathrm{Spin}(8) \to F_4/\mathrm{Spin}(9) \cong \mathbf{O}P^2$$

の fiberwise $\mathbf{Z}/2\mathbf{Z}$ quotient として得られる．

定理 2 で構成される例は，すでに，Zak(1985) による代数閉体上 (たとえば $\mathbf{C}$ 上) の射影空間の Severi varieties の分類，Cartan(1939) 等による球面の isoparametric 超曲面の分類などに類似した形で現われていたものである．それらの対象の内在的な関係を現在考察中である．

[I] : G. Ishikawa, Developable hypersurfaces and algebraic homogeneous spaces in a real projective space, Preprint.

[IM] : G. Ishikawa, T. Morimoto, Solution surfaces of Monge-Ampère equations, Hokkaido Univ. Preprint Series, 376 (1997).

以下の論文は現在投稿中である.

0. Introduction

In this paper we present new examples of dvelopables hypersurfaces, which are algebraic and homogeneous, in real projective spaces. All constructions are explained in an explicit manner.

A C^∞ hypersurface M in the n-dimensional real projective space $\mathbf{R}P^n$ is called **developable** if its Gauss map

$$\gamma : M \to \mathrm{Gr}(n, \mathbf{R}^{n+1}) \cong \mathrm{Gr}(1, (\mathbf{R}^{n+1})^*) = \mathbf{R}P^{n*}$$

defined by $\gamma(x) = \hat{T}_x M \subset \mathbf{R}^{n+1}$ $(x \in M)$ has $\mathrm{rank}(\gamma) < \dim(M) = n-1$. Here, we mean by $\hat{T}_x M$ the linear subspace defined by $T_x M \subset \mathbf{R}P^n$ considered as a projective subspace, by $\mathbf{R}P^{n*}$ the dual projective space, and by $\mathrm{rank}(\gamma)$ the maximum of the rank of differential maps $\gamma_* : T_x M \to T_x \mathbf{R}P^{n*}$ $(x \in M)$ of γ. See [FW][W] for developable submanifolds of arbitrary codimension. Here we treat mainly on hypersurfaces.

It is well-known, as classical examples of developable surfaces in the three dimensional space, cylinders, cones and tangent developables of space curves [Cay][I]: Among them, only the planes have no singularities in the projective space. Observing the singularities of developable hypersurfaces, we expect, also in the general case, that non-singular compact developable hypersurfaces are heavily restrictive. In fact, it is known that a non-singular complex algebraic developable hypersurface in $\mathbf{C}P^n$ is necessarily a projective hyperplane ([GH][W][L1]). Also in a real projective space, we see the following analogy, via the geometrical investigation of homogeneous Monge-Ampère equations based on projective duality:

Theorem 1 ([IM]). *For a compact developable C^∞ hypersurface M in $\mathbf{R}P^n$, the maximal rank $r = \mathrm{rank}(\gamma)$ of the Gauss map $\gamma : M \to \mathbf{R}P^{n*}$ is an even integer and satisfies $n < (1/2)r(r+3)$, provided $r \neq 0$. In particular, if $r \leq 1$, then M is necessarily a projective hyperplane of $\mathbf{R}P^n$. Any compact developable C^∞ hypersurfaces in $\mathbf{R}P^3$ or $\mathbf{R}P^5$ are projective hyperplanes.*

It is essential the rank condition appeared in Theorem 1; in fact we will show in the present paper the following result.

Theorem 2. *For $n = 4, 7, 13, 25$, there exists a real algebraic cubic non-singular developable hypersurface in $\mathbf{R}P^n$. These developable hypersurfaces have the structure of homogeneous spaces of groups $SO(3), SU(3), Sp(3), F_4$, respectively. Their projective duals are linear projections of Veronese embeddings of projective planes $\mathbf{K}P^2$, for $\mathbf{K} = \mathbf{R}, \mathbf{C}, \mathbf{H}, \mathbf{O}$ (the Cayley's octonians). Each of these real algebraic developable hypersurfaces admits deformations to C^∞ developable hypersurfaces with $2, 3, 5, 9$ functional parameters, or more rigorously, with the space of sections of normal bundles to $\mathbf{K}P^2 \subset \mathbf{R}P^n$ as the infinitesimal space of C^∞ developable deformations.*

Notice that it is classically known that a properly embedded developable hypersurface in $\mathbf{R}^n$ of $\text{rank}(\gamma) \leq 1$ is necessarily a cylinder (Hartman-Nirenberg's theorem [HN][Ste][Sto]). Similar result is known for $\mathbf{C}^n$ by Abe [Ab]. For this direction, see the survey [B]. The first example of non-cylindrical C^∞ developable hypersurfaces in $\mathbf{R}^4$ is given by Sacksteder [Sac]:

$$M = \{(x_1, x_2, x_3, x_4) \in \mathbf{R}^4 \mid x_4 = x_1 \cos x_3 + x_2 \sin x_3\}.$$

Mori [M] gives an example of families of non-cylindrical developable hypersurfaces in $\mathbf{R}^4$, in connection with the study of deformable submanifolds. On the other hand, Akivis [Ak] proves the existence of C^∞ complete developable hypersurfaces in $\mathbf{R}P^4$ which is not a projective hyperplane, using the theory of differential systems. (See also [AG] Ch. 4, for the method of construction). However it is not given any *concrete* examples. Recently, Fischer and Wu ([FW][W]) study developable submanifolds in $\mathbf{C}P^n, \mathbf{C}^n$ and $\mathbf{R}^n$ of higher codimension. In [W], it is introduced an (unpublished) example of non-cylindrical real algebraic developable hypersurfaces in $\mathbf{R}^4$ by Bourgain:

$$M = \{(x_1, x_2, x_3, x_4) \in \mathbf{R}^4 \mid x_1 x_4^2 + x_2(x_4 - 1) + x_3(x_4 - 2) = 0\}.$$

Then, M is non-singular in $\mathbf{R}^4$ and even in $\mathbf{C}^4$ after complexification, while the Zariski closure $\overline{M} \subset \mathbf{R}P^4$ of M has singularities in $\mathbf{R}P^4$. (The singular loci is an $\mathbf{R}P^2$ in the projective hyperplane at infinity).

In general, developable submanifolds has the **Monge-Ampère foliation** so that the tangent spaces to the submanifold are constant along each leaf. For instance, in the case of F_4 in Theorem 2, the Gauss mapping is a submersion and the Monge-Ampère foliation is given by the fiberwise $\mathbf{Z}/2\mathbf{Z}$ quotient of the fibration

$$\mathbf{O}P^1 \cong \mathrm{Spin}(9)/\mathrm{Spin}(8) \to F_4/\mathrm{Spin}(8) \to F_4/\mathrm{Spin}(9) \cong \mathbf{O}P^2,$$

arising from the filtration $F_4 \supset \mathrm{Spin}(9) \supset \mathrm{Spin}(8)$. Remark that there exists natural identification $\mathbf{O}P^1 \cong S^8$, and the antipodal map induces the involution on $\mathbf{O}P^1$.

In the next section, we recall the notion of projective duality and the seconf fundamental form of submanifold in a projective space. In §3, we prove Theorem 2: The main tool for the construction of Theorem 2 is the real projective-contact geometry [M1][M2] over Jordan algebras.

It is interesting to ask the connection between the construction of Theorem 2 and the classification of Severi varieties in the projective spaces over algebraically closed field of characteristic zero, for instance, over $\mathbf{C}$, by Zak [Z] (cf. [FL][LV, p.15]) and the classification of isoparametric hypersurfaces in the spheres by Cartan [Car] (cf. [CR]), where similar objects appear. See also [L1][L2][K] in complex projective geometry on the second fundamental forms and degenerate secant varieties, related to homogeneous spaces and Clifford algebras.

1. Projective duality and second fundamental forms

Let $M \subset \mathbf{R}P^n$ be a submanifold of dimension m, $(m < n)$. Consider the projective conormal bundle of M:

$$\widetilde{M} = \{(p, q) \in \mathbf{R}P^n \times \mathbf{R}P^{n*} \mid p \in M,\ T_pM \subset q^\vee\},$$

where $q^\vee$ is the hyperplane of $\mathbf{R}P^n$ determined by $q \in \mathbf{R}P^{n*}$, and we identify T_pM as the corresponding m-dimensional plane through p in $\mathbf{R}P^n$. Then we see $\widetilde{M}$ is a C^∞ submanifold in $\mathbf{R}P^n \times \mathbf{R}P^{n*}$ of dimension $n-1$. Let $\rho : \widetilde{M} \to \mathbf{R}P^n$ (resp. $\rho' : \widetilde{M} \to \mathbf{R}P^{n*}$) denotes the projection to the first (second) component. Then $\rho(\widetilde{M}) = M$ and $\rho'(\widetilde{M}) = M^\vee$ is the projective dual of M.

We call M is **developable** if the Gauss map $\gamma : M \to \mathrm{Gr}(m+1, \mathbf{R}^{n+1})$, defined by $\gamma(x) = \hat{T}_xM$ satisfies $\mathrm{rank}(\gamma) < \dim M$.

If M is developable and $\mathrm{rank}(\gamma) = r$, then there exists an $(m-r)$-dimensional foliation on $\Omega = \{x \in M \mid \mathrm{rank}_x(\gamma) = r\}$, which we call Monge-Ampère foliation [D]. Moreover in this case, $M^\vee$ is ruled by r-parameter $(n-m-1)$-planes, and $\mathrm{rank}(\rho') < \dim \widetilde{M} = n-1$.

Remark that, if $m = n-1$, then ρ is diffeomorphic and the Gauss map is decomposed as $\gamma = \rho' \circ \rho^{-1}$.

Let $g : W \to \mathbf{R}P^{n*}$ be an immersion. For $x \in W$, the **second fundamental form** of g at x is a linear family of quadratic forms (Hessians) on T_xW parametrized by conormal vector space $N^* = (T_{g(x)}\mathbf{R}P^{n*}/g_*(T_xW))^*$ to g at x:

$$II^* : N^* \to S^2(T_x^*M) \text{ (the symmetric product).}$$

Then we recall the following fundamental result [IM], which we are going to use for showing Theorem 2:

Lemma 3. *For an immersed submanifold W of $\mathbf{R}P^{n*}$ of codim ≥ 2, the following conditions are equivalent to each other:*

(i) *W is a projective dual of a properly immersed hypersurface in $\mathbf{R}P^n$.*

(ii) *The second fundamental form at each point of W does not contain any singular quadratic forms.*

(iii) *For any projective hyperplane $H \subset \mathbf{R}P^{n*}$, each singular point of the hyperplane section $W \cap H$ on W is non-degenerate.*

Proof. The condition (i) is equivalent to that $\rho : \widetilde{W} \to \mathbf{R}P^n$ is an immersion. For a local equation

$$y_{r+1} = \varphi_{r+1}(y_1, \ldots, y_r), \ \ldots, \ y_n = \varphi_n(y_1, \ldots, y_r)$$

of W, $\widetilde{W}$ is defined by $F = \partial F/\partial y_1 = \cdots = \partial F/\partial y_r = 0$, where

$$F(X; y_1, \ldots, y_r) = X_0\varphi_n + \cdots + X_{n-r-1}\varphi_{r+1} + X_{n-r}y_r + \cdots + X_{n+1}y_1 + X_n,$$

for a homogeneous coordinates $(X_0, X_1, \ldots, X_n)$ of $\mathbf{R}P^n$. Then ρ is an immersion on $\widetilde{W}$ if and only if the second fundamental form

$$II^*(X_0, \ldots, X_{n-r-1}) = \sum_{k=0}^{n-r-1} X_k \left(\frac{\partial^2 \varphi_{n-k}}{\partial y_i \partial y_j} \right)_{1 \leq i,j \leq r}$$

does not represent a singular matrix, provided $(X_0, \ldots, X_{n-r-1}) \neq (0, \ldots, 0)$. This condition is equivalent to (ii). The equivalence (ii) $\Leftrightarrow$ (iii) is clear.

1. Proof of Theorem 2

First we show the construction of a cubic non-sungular developable hypersurface M in $\mathbf{R}P^4$. For this, first we construct the projective dual $M^\vee$, then M is obtained as the dual of $M^\vee$.

Define $\varphi : \mathbf{R}P^2 \to \mathbf{R}P^{4*}$ by

$$\varphi([u,v,w]) = [\frac{1}{2}(u^2-v^2), \frac{1}{2}(v^2-w^2), uv, vw, wu],$$

which is an embedding obtained after a linear projection of the Veronese embedding ψ : $\mathbf{R}P^2 \to \mathbf{R}P^{5*}$ defined by

$$\psi([u,v]) = [\frac{1}{2}u^2, \frac{1}{2}v^2, \frac{1}{2}w^2, uv, vw, wu].$$

Then we set $M^\vee = \varphi(\mathbf{R}P^2)$. Further we set

$$F = X_0\frac{1}{2}(u^2-v^2) + X_1\frac{1}{2}(v^2-w^2) + X_2uv + X_3vw + X_4wu.$$

Then the ρ-projection of the projective conormal bundle $\widetilde{M^\vee}$ of $M^\vee$ is obtained by eliminating u, v, w from

$$F = \frac{\partial F}{\partial u} = \frac{\partial F}{\partial v} = \frac{\partial F}{\partial w} = 0.$$

Then we have

$$\begin{vmatrix} X_0 & X_2 & X_4 \\ X_2 & -X_0+X_1 & X_3 \\ X_4 & X_3 & -X_1 \end{vmatrix} = 0,$$

which is the equation of required $M \subset \mathbf{R}P^4$.

In fact, M is the projectivization of the set of real symmetric matrices of determinant zero and of trace zero. Since $SO(3)$ acts on M transitively, we see M is non-singular and $M \cong SO(3)/H$, where H is the subgroup of $SO(3)$ of order 8:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

In general, we set $\mathbf{K} = \mathbf{R}, \mathbf{C}, \mathbf{H}, \mathbf{O}$. Then $\dim_{\mathbf{R}} \mathbf{K} = 2^{i-1}$, $i = 1, 2, 3, 4$. Consider

$$\mathcal{J} = \{A \in M_3(\mathbf{K}) \mid A^* = A\},$$

the space of "Hermitian" matrices of size 3 ([H][Y]). Each element A of $\mathcal{J}$ has the form

$$A = \begin{pmatrix} \xi_1 & z_1 & z_3 \\ \bar{z_1} & \xi_2 & z_2 \\ \bar{z_3} & \bar{z_2} & \xi_3 \end{pmatrix}, \ \xi_j \in \mathbf{R}, \ z_j \in \mathbf{K}, \ j = 1, 2, 3.$$

We see

$$\dim_{\mathbf{R}} \mathcal{J} = 3 \cdot 2^{i-1} + 3 = 6, 9, 15, 27.$$

For $A, B \in \mathcal{J}$, we define the **Jordan product**

$$A \circ B = \frac{1}{2}(AB + BA) \in \mathcal{J}.$$

Moreover we set $\mathrm{tr} A = \xi_1 + \xi_2 + \xi_3 \in \mathbf{R}$ and

$$\det A = \xi_1 \xi_2 \xi_3 + 2\mathrm{Re}((z_2 \bar{z}_3) z_1) - \xi_1 z_2 \bar{z}_2 - \xi_2 z_3 \bar{z}_3 - \xi_3 z_1 \bar{z}_1 \in \mathbf{R},$$

for $A \in \mathcal{J}$. The bilinear form $\mathrm{tr}(A \circ B)$ on the real vector space $\mathcal{J}$ is positive definite and induces the isomorphism between $\mathcal{J}$ and its dual vector space $\mathcal{J}^*$.

Set

$$\Sigma = \{A \in \mathcal{J} \mid \det A = 0\}.$$

Then the projectivization $P\Sigma \subset P\mathcal{J} = (\mathcal{J} - 0)/\mathbf{R}^\times \cong \mathbf{R}P^{3 \cdot 2^{i-1}+2}$ is a real cubic hypersurface. Setting

$$\mathcal{J}_0 = \{A \in \mathcal{J} \mid \mathrm{tr} A = 0\},$$

we will see

$$M = P\mathcal{J}_0 \cap P\Sigma \subset P\mathcal{J}_0 = \mathbf{R}P^4, \mathbf{R}P^7, \mathbf{R}P^{13}, \mathbf{R}P^{25},$$

is a non-singular real cubic developable hypersurface. The projective dual $M^\vee = \mathbf{K}P^2$ is embedded in $P\mathcal{J}_0^* = \mathbf{R}P^{4*}, \mathbf{R}P^{7*}, \mathbf{R}P^{13*}, \mathbf{R}P^{25*}$, as a linear projection of the Veronese embedding of $\mathbf{K}P^2$ in $P\mathcal{J} \cong P\mathcal{J}^*$. Remak that $\mathrm{rank}(\gamma) = 2, 4, 8, 16$ and the dimension of the Monge-Ampère foliation is $1, 2, 4, 8$, respectively.

Recall that the projective plane over $\mathbf{K}$ is defined by

$$\begin{aligned}\mathbf{K}P^2 &= \{X \in \mathcal{J} \mid X^2 = X,\ \mathrm{tr}X = 1\},\\ &= \{\mathbf{x}\mathbf{x}^* \mid {}^t\mathbf{x} = (x_1, x_2, x_3) \in \mathbf{K}^3 - \mathbf{0}, \|\mathbf{x}\| = 1, x_1(x_2x_3) = (x_1x_2)x_3\}\end{aligned}$$

which is embedded in $P\mathcal{J}$. The embedding $\mathbf{K}P^2 \hookrightarrow P\mathcal{J}$ is called the **Verenose embedding** [F][H, Lemma 14.90][L2][Z]. This definition fits with the ordinary one in cases $\mathbf{K} = \mathbf{R}, \mathbf{C}, \mathbf{H}$ by the correspondence

$$\mathbf{K}P^2 \ni [x_1, x_2, x_3] = [{}^t\mathbf{x}] \mapsto \frac{1}{\|\mathbf{x}\|^2}\mathbf{x}\mathbf{x}^*.$$

In cases $\mathbf{K} = \mathbf{R}, \mathbf{C}, \mathbf{H}$, we set $G = \mathrm{O}(3), \mathrm{U}(3), \mathrm{Sp}(3)$. Then G acts on $\mathcal{J}$ by $f(A) = P^{-1}AP, (f = P \in G)$. In the case $\mathbf{K} = \mathbf{O}$, we take as G the exeptional simple Lie group

$$F_4 = \{f : \mathcal{J} \to \mathcal{J},\ \mathbf{R}\text{-linear isomorphism} \mid f(A \circ B) = f(A) \circ f(B)\}.$$

Then G preserves the Jordan product, the trace and the determinant, so G naturally acts on $P\mathcal{J}_0$, $P\Sigma$, so on $M = P\mathcal{J}_0 \cap P\Sigma$, as well as it acts on $\mathbf{O}P^2$. Furthemore G acts on M transitively. In fact, for $A \in \mathcal{J}$, there exists a $f \in G$ such that $f(A) = \begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & 0 \\ 0 & 0 & \xi_3 \end{pmatrix}$, for some $\xi_1, \xi_2, \xi_3 \in \mathbf{R}$. Moreover the diagnals are permuted freely by an element of G. Then, an $A \in \mathcal{J}_0 \cap \Sigma$ is transformed into $f(A) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi & 0 \\ 0 & 0 & -\xi \end{pmatrix}$, by some $f \in G$, for some $\xi \in \mathbf{R}$. (See, for $\mathbf{K} = \mathbf{O}$, [H] Page 313, [Y] Page 35). Also the action of G on $\mathbf{K}P^2$ is transitive. ([H] Theorem 14.99, [Y] Theorem 2.21).

Now set

$$Q = \{([A], [B]) \in P\mathcal{J} \times P\mathcal{J} \mid \mathrm{tr}(A \circ B) = 0\},$$

the incident hypersurface of projective duality ([Sch][IM]). Then G acts on Q naturally by $f([A], [B]) = ([f(A)], [f(B)])$. Then, since the action on M is transitive, the action on $\widetilde{M}$ is also transitive. Here we remark that $\widetilde{M}$ projects diffeomorphically to M by ρ. Then the key fact is the following:

Lemma 4. *The projective conormal bundle of* $\mathbf{K}P^2 \subset P\mathcal{J}^*$ *is described by*

$$\widetilde{\mathbf{K}P^2} = PT^*_{\mathbf{K}P^2}P\mathcal{J}^* = \{([A], [X]) \in Q \mid X \in \mathbf{K}P^2,\ A \circ X = O\}.$$

Moreover its projection $S = \rho(PT^*_{\mathbf{K}P^2}P\mathcal{J}^*)$ *by* $\rho : PT^*_{\mathbf{K}P^2}P\mathcal{J}^* \to P\mathcal{J}$ *to the first component coinsides with*

$$P\Sigma = \{[A] \in P\mathcal{J} \mid \det A = 0\}.$$

Proof. We show for $\mathbf{K} = \mathbf{O}$; other cases are treated similarly. Let $\mathbf{x} = (x_1, x_2, x_3) \in \mathbf{K}^3 - \mathbf{0}$. Write $x_i = \sum_{j=0}^{7} x_{ij}e_j$, $i = 1, 2, 3$, with the the standard basis $e_0 = 1, e_1, \ldots, e_7$ and $x_{ij} \in \mathbf{R}$. Then the linear subspace $\hat{T}_{\mathbf{x}_0}\mathbf{O}P^2 \subset \mathcal{J}$ of the tangent space to $\mathbf{O}P^2$ at $\mathbf{x}_0 = {}^t(1, 0, 0)$ is generated over $\mathbf{R}$ by

$$\frac{\partial}{\partial x_{10}} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \frac{\partial}{\partial x_{20}} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \frac{\partial}{\partial x_{2i}} = \begin{pmatrix} 0 & -e_i & 0 \\ e_i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\frac{\partial}{\partial x_{30}} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \frac{\partial}{\partial x_{3i}} = \begin{pmatrix} 0 & 0 & -e_i \\ 0 & 0 & 0 \\ e_i & 0 & 0 \end{pmatrix}, \quad 1 \le j \le 7,$$

while $\frac{\partial}{\partial x_{1j}} = O, 1 \le j \le 7$. Set $A = \begin{pmatrix} \xi_1 & w_1 & w_3 \\ \bar{w}_1 & \xi_2 & w_2 \\ \bar{w}_3 & \bar{w}_2 & \xi_3 \end{pmatrix}$. Then the condition that A annihilates $\hat{T}_{\mathbf{x}_0}\mathbf{O}P^2$ via the inner product $\mathrm{tr}(A \circ B)$, namely that $\mathrm{tr}(A \circ \frac{\partial}{\partial x_{ij}}) = O, i = 1, 2, 3, 0 \le j \le 7$, is equivalent to that $\xi_1 = 0, w_1 = 0, w_3 = 0$. This is equivalent to that

$$A \circ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2\xi_1 & w_1 & w_3 \\ \bar{w}_1 & 0 & 0 \\ \bar{w}_3 & 0 & 0 \end{pmatrix}$$

equals to O. By the trasitivity we have the first half. The second half follows from the following Lemma.

Lemma 5. *For* $A \in \mathcal{J}$, (1) $A \circ X = O$, *for some* $X \in \mathbf{K}P^2$, *if and only* (2) $\det A = 0$.

Proof. (1) $\Rightarrow$ (2): Choose $f \in G$ such that $f(X) = X_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then, since $f(A) \circ X_0 = f(A \circ X) = O$, we see $\det A = \det f(A) = 0$. (2) $\Rightarrow$ (1): Take $f \in G$ such that

$$f(A) = \begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & 0 \\ 0 & 0 & \xi_3 \end{pmatrix},$$

for some $\xi_1, \xi_2, \xi_3 \in \mathbf{R}$. Then $\det f(A) = \det A = 0$, so $\xi_1\xi_2\xi_3 = 0$, thus $\xi_i = 0$, for some i. Changing f if necessary, we may assume $\xi_1 = 0$. Then $A \circ f^{-1}(X_0) = f(A) \circ X_0 = O$. □

Thus we see the projective dual of the hyperplane section $M = S \cap P\mathcal{J}_0 \subset P\mathcal{J}_0$ is the linear projection of $\mathbf{K}P^2 \subset P\mathcal{J} \cong P\mathcal{J}^*$ from the point in $P\mathcal{J}^*$ corresponding to the hyperplane $P\mathcal{J}_0 \subset P\mathcal{J}$.

Set

$$A_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in \mathcal{J}_0 \cap \Sigma, \quad \text{and,} \quad X_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathbf{O}P^2 \subset \mathcal{J}.$$

Then $([A_0], [X_0]) \in \widetilde{M}$. Let $\mathbf{K} = \mathbf{O}$ and $G = F_4$. Then the isotropy group for $[X_0] \in P\mathcal{J}$ of the F_4-action is isomorphic to Spin(9) ([H] Theorem 14.99, [Y] Theorem 2.10). Further the isotropy group for $A_0 \in J$ of the F_4-action on J is

$$\{f \in F_4 \mid f(E_i) = E_i,\ i = 1,2,3\},$$

which is isomorphic to Spin(8). Here E_i is the 3×3 matrix with (i,i)-element 1 which is the only non-zero element. (So $E_1 = X_0, A_0 = E_2 - E_3$). ([H] Page 313, [Y] Theorem 2.7). Then the isotropy group for $[A_0]$ in $\widetilde{M}$ is isomorphic to a $\mathbf{Z}/2\mathbf{Z}$-extension of Spin(8). Thus we see that the Monge-Ampère foliation is in fact a fibration $\gamma : M \to \mathbf{O}P^2$ described as in §0.

Similarly we have, in cases $\mathbf{K} = \mathbf{R}, \mathbf{C}, \mathbf{H}$, that the Monge-Ampère foliation is given by the fibration $\gamma : M \to \mathbf{K}P^2$ which is described as the fiberwise $\mathbf{Z}/2\mathbf{Z}$-quotient (with respect to the antipodal involution of $\mathbf{K}P^1 \cong S^{2^{i-1}} (i = 1,2,3)$ of the following fibration: For $\mathbf{K} = \mathbf{R}$,

$$\mathbf{R}P^1 \cong O(2)/O(1) \times O(1) \to O(3)/O(1) \times O(1) \times O(1) \to O(3)/O(2) \times O(1) \cong \mathbf{R}P^2,$$

For $\mathbf{K} = \mathbf{C}$,

$$\mathbf{C}P^1 \cong U(2)/U(1) \times U(1) \to U(3)/U(1) \times U(1) \times U(1) \to U(3)/U(2) \times U(1) \cong \mathbf{C}P^2,$$

and for $\mathbf{K} = \mathbf{H}$,

$$\mathbf{H}P^1 \cong \mathrm{Sp}(2)/\mathrm{Sp}(1)\times\mathrm{Sp}(1) \to \mathrm{Sp}(3)/\mathrm{Sp}(1)\times\mathrm{Sp}(1)\times\mathrm{Sp}(1) \to \mathrm{Sp}(3)/\mathrm{Sp}(2)\times\mathrm{Sp}(1) \cong \mathbf{H}P^2.$$

In particular, $M \in \mathbf{R}P^n$, $(n = 3 \cdot 2^{i-1} + 1,\ i = 1,2,3,4)$ is a homogeneous space of $SO(3), SU(3), \mathrm{Sp}(3)$ and F_4, respectively.

The last statement of Theorem 2 is clear, since the condition of Lemma 3 is an open condition for immersions $\mathbf{K}P^2 \to \mathbf{R}P^{n*}$.

References

[Ab] K. Abe, *A complex analogue of Hartman-Nirenberg cylinder theorem*, J. Diff. Geom. **7** (1972), 453–460.

[Ak] M.A. Akivis, *On multidimensional strongly parabolic surfaces*, Izv. Vyssh. Uchebn. Zaved. Mat. **5** (1987), 3–10 = Soviet Math. **31–5** (1987), 1–11.

[AG] M.A. Akivis, V.V. Goldberg, *Projective Differential Geometry of Submanifolds*, North-Holland Math. Lib. **49**, North-Holland, Amsterdam, 1993.

[B] A.A. Borisenko, *Extrinsic geometry of storongly parabolic multidimensional submanifolds*, Russ. Math. Surveys **52–6** (1997), 1141–1190.

[Car] É. Cartan, *Sur des familles remarquables d'hypersurfaces isoparamétriques dans les espaces sphériques*, Math. Z. **45** (1939), 335–367.

[Cay] A. Cayley, *Mémoire sur les coubes à double courbure et les surfaces développables*, Journal de Mathématique Pure et Appliquées (Liouville) **10** (1845), 245–250. = The Collected Mathematical Papers vol.I, pp. 207–211.

[CR] T. Cecil, P. Ryan, *Tight and Taut Immersions of Manifolds*, Research Notes in Math. **107**, Pitman, London, 1985.

[D] Ph. Delanoë, *L'opérateur de Monge-Ampère réel et la géométrie des sous-variétés*, in Geometry and Topology of Submanifolds, ed. by J.-M. Morvan and L. Verstraelen, World Scientific, 1989, pp. 49–72.

[FW] G. Fischer, H. Wu, *Developable complex analytic submanifolds*, International J. Math. **6–2** (1995), 229–272.

[F] H. Freudenthal, *Zur ebenen Oktavengeometrie*, Proc. Kon. Ned. Ak. v. Wet., Ser. A **56–3** (1953), 195–200.

[FL] W. Fulton, R. Lazarsfeld, *Connectivity and its applications in algebraic geometry*, Lecture Notes in Math., **862**, Springer-Verlag, Berlin and New York, 1981, pp. 26–92.

[GH] P. Griffiths, J. Harris, *Algebraic geometry and local differentiable geometry*, Ann. sci. Éc. Norm. Sup. (4) **12** (1979), 355–432.

[HN] P. Hartman, L. Nirenberg, *On spherical image whose Jacobians do not change sign*, Amer. J. Math. **81** (1959), 901–920.

[H] F.R. Harvey, *Spinors and Calibrations*, Perspectives in Math. **9**, Academic Press, 1990.

[I] G. Ishikawa, *Singularities of Developable Surfaces*, to appear in London Math. Soc. Lecture Note Series, Cambridge Univ. Press..

[IM] G. Ishikawa, T. Morimoto, *Solution surfaces of Monge-Ampère equations*, Hokkaido Univ. Preprint Series **376** (1997).

[K] H. Kaji, *Homogeneous projective varieties with degenerate secants*, Preprint..

[L1] J.M. Landsberg, *On second fundamental forms of projective varieties*, Invent. Math. **117** (1994), 303–315.

[L2] J.M. Landsberg, *On degenerate secant and tangential varieties and local differential geometry*, Duke Math. Journal **85–3** (1996), 605–634.

[LV] R. Lazarsfeld, A. Van de Ven, *Topics in the Geometry of Projective Space, Recent Work of F.L. Zak*, Birkhäuser Verlag, 1984.

[M] H. Mori, *Remarks on complete deformable hypersurfaces in* $\mathbf{R}^4$, J. Diff. Geom. **40** (1994), 1–6.

[M1] T. Morimoto, *La géométrie des équations de Monge-Ampère*, C. R. Acad. Sci. Paris **289** (2 juillet 1979), A-25 –A-28.

[M2] T. Morimoto, *Monge-Ampère equations viewed from contact geometry*, to appear in Banach Center Publications vol. **39** "Symplectic Singularities and Geometry of Gauge Fields", pp. 105–120.

[Sac] R. Sacksteder, *On hypersurfaces with no negative sectional curvature* **82–3** (1960), Amer. J. Math., 609–630.

[Sas] T. Sasaki, *On the Veronese embedding and related system of differential equations*, Global Differential Geometry and Global Analysis, Lecture Notes in Math., vol. 1481, 1991, pp. 210–247.

[Sch] O.P. Scherbak, *Projectively dual space curves and Legendre singularities*, Trudy Tbiliss. Univ. **232-233** (1982), 280–336.

[Ste] S. Sternberg, *Lectures on Differential Geometry*, 2nd ed., Printice-Hall, N.J., 1983.

[Sto] J.J. Stoker, *Differential Geometry*, Pure and Applied Math. 20, Wiley-Interscience, 1969.

[W] H. Wu, *Complete developable submanifolds in real and complex Euclidean spaces*, International J. Math. **6–3** (1995), 461–489.

[Y] I. Yokota, *Exceptional Simple Lie Groups*, (in Japanese), Gendai-Suugaku-Sya Co., 1992.

[Z] F.L. Zak, *Tangents and Secants of Algebraic Varieties*, Translations of Math. Monographs **127**, Amer. Math. Soc., Providence, 1993.