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Contact structure and nonlinear problems

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Abstract

We will consider the Cauchy problem for nonlinear hyperbolic equations of second order. It is well known that the Cauchy problem does not generally have a classical solution in the large. This means that singularities generally appear in finite time. Our principal problem is to study the structure of singularities of global solutions. The method we use is to lift the solution surfaces into cotangent space, and to construct geometric solutions defined in the whole cotangent space. Next we will project it to the base space. In this procedure we will meet the singularities of smooth mappings. One of typical singularities is "caustics". The principal subject of this note is how to lift the solution surfaces into the cotangent space.

1 Introduction

In this note we will consider the Cauchy problem for nonlinear partial differential equations of hyperbolic type. We are interested in the global theory. Then one of important phenomena is the appearance of singularities in their solutions. But, even if singularities may appear in solutions, physical phenomena can exist with the singularities. Moreover it seems to us that the singularities might cause various kinds of interesting phenomena. Hence we would like to extend the solutions beyond their singularities. To do so, we are obliged to study the structure of their singularities. This means that we must construct exact solutions in neighbourhoods of singularities. In §2, we will study the method of integration for second order partial differential equations. The principal idea of the method is to express the solution surface by a family of smooth curves. Historically, it is D. Darboux [2] and E. Goursat [3, 4] who investigated this subject for the first time. We will explain their theory a little in §2 "from our point of view". Then we will see that we must assume strong conditions so that the equations might be "*integrable in the sense of Darboux and Goursat*". Therefore many important examples are not contained in the class of equations which are integrable in the sense of Darboux and Goursat. If we may

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not assume their integrability conditions, the family of characteristic strips is obtained as a solution of certain nonlinear system of first order partial differential equations. This topic will be discussed in §2. In §3 we will construct solutions of nonlinear wave equations in cotangent space, and of systems of conservation law in §4. The author thanks to V. Zakalyukin for his interesting remarks on singularities of Riemann invariants. He will give a remark on this subject at the end of §4. The author will soon publish a detailed paper on this subject, in which he will mention about the properties of solutions obtained by the projection of the geometric solutions into the base space. This note is almost same to the one which the author sent to "Banach Center Publications" (Warsaw, Poland) in September, 1998.

2 Contact structure and integration of Monge-Ampère equations

In this section we will study the method of integration of second order nonlinear partial differential equations. As the equations of second order are too general, we will treat the equations of Monge-Ampère type as follows:

$$F(x,y,z,p,q,r,s,t) = Ar + Bs + Ct + D(rt - s^2) - E = 0 \tag{2.1}$$

where $p = \partial z/\partial x, q = \partial z/\partial y, r = \partial^2 z/\partial x^2, s = \partial^2 z/\partial x \partial y$, and $t = \partial^2 z/\partial y^2$. Here we assume that A, B, C, D and E are real smooth functions of (x,y,z,p,q). Partial differential equations of second order which appear in physics and geometry are often written in the above form.

Let us consider equation (2.1) from the geometrical point of view. Equation (2.1) is regarded as a smooth surface defined in eight dimensional space $\mathbf{R}^8 = \{(x,y,z,p,q,r,s,t)\}$. As p and q are first order derivatives of $z = z(x,y)$, we put the relation $dz = pdx + qdy$. Moreover, as r, s and t are second order derivatioves of $z = z(x,y)$, we introduce the relations $dp = rdx + sdy$ and $dq = sdx + tdy$. Let us call $\{dz = pdx + qdy,\ dp = rdx + sdy,\ dq = sdx + tdy\ \}$ as the "contact structure of second order". We define a solution of (2.1) as a maximal integral submanifold of the contact structure of second order in the surface $\{(x,y,z,p,q,r,s,t) \in \mathbf{R}^8; f(x,y,z,p,q,r,s,t) = 0\}$. We will use this geometric formulation to solve equation (2.1) in exact form.

First we will give some fundamental notions and the defintion of "hyperbolicity". Let

$$\Gamma : (x,y,z,p,q) = (x(\alpha), y(\alpha), z(\alpha), p(\alpha), q(\alpha))\,, \quad \alpha \in \mathbf{R}^1$$

be a smooth curve in $\mathbf{R}^5$. It is called a "strip" if it satisfies the following

$$\frac{dz}{d\alpha}(\alpha) = p(\alpha)\frac{dx}{d\alpha}(\alpha) + q(\alpha)\frac{dy}{d\alpha}(\alpha)\,. \tag{2.2}$$

Let Γ be any strip in $\mathbf{R}^5$, and consider equation (2.1) in its open neighbourhood. As a "characteristic" strip means that one can not determine the values of the second order derivatives of solution along the strip, we have the following

Definition 2.1 *A curve Γ in $\mathbf{R}^5 = \{(x,y,z,p,q)\}$ is a "characteristic strip" if it satisfies* (2.2) *and*

$$\det \begin{bmatrix} F_r & F_s & F_t \\ \dot{x} & \dot{y} & 0 \\ 0 & \dot{x} & \dot{y} \end{bmatrix} = F_t\dot{x}^2 - F_s\dot{x}\dot{y} + F_r\dot{y}^2 = 0 \tag{2.3}$$

where $F_t = \partial F/\partial t, F_s = \partial F/\partial s, F_r = \partial F/\partial r, \dot{x} = dx/d\alpha$ and $\dot{y} = dy/d\alpha$.

Denote the discriminant of (2.3) by Δ, then it holds

$$\Delta = F_s^2 - 4F_rF_t = B^2 - 4(AC + DE).$$

If $\Delta < 0$, equation (2.1) is called to be elliptic. If $\Delta > 0$, equation (2.1) is hyperbolic. In this note, we will treat the equations of hyperbolic type. Let λ_1 and λ_2 be the solutions of $\lambda^2 + B\lambda + (AC + DE) = 0$. Then the characteristic strip satisfies the following equations:

$$\begin{cases} dz - pdx - qdy = 0 \\ Ddp + Cdx + \lambda_1 dy = 0 \\ Ddq + \lambda_2 dx + Ady = 0 \end{cases}, \tag{2.4}$$

or

$$\begin{cases} dz - pdx - qdy = 0 \\ Ddp + Cdx + \lambda_2 dy = 0 \\ Ddq + \lambda_1 dx + Ady = 0 \end{cases}. \tag{2.5}$$

Let us denote $\omega_0 = dz - pdx - qdy$, $\omega_1 = Ddp + Cdx + \lambda_1 dy$ and $\omega_2 = Ddq + \lambda_2 dx + Ady$. Take an exterior product of ω_1 and ω_2, and substitute into their product the relations of the contact structure $\omega_0 = 0$, $dp = rdx + sdy$ and $dq = sdx + rdy$. Then we get

$$\omega_1 \wedge \omega_2 = D\Big\{Ar + Bs + Ct + D(rt - s^2) - E\Big\}\, dx \wedge dy\,. \tag{2.6}$$

The decomposition as (2.6) is a small idea, but it will be shown that it works effectively to solve equayion (2.1) in exact form. In a space whose dimension is greater than two, it is generally impossible to do so. Here we recall a little the characteristic method developed principally by D. Darboux [2] and E. Goursat [4, 5] "from our point of view", because their theory is not familiar today. The principal idea of Darboux and Goursat is how to reduce the solvability of (2.1) to the integration of first order partial differential equations. First they introduce the notion of "first integral".

Definition 2.2 *A function $V = V(x,y,z,p,q)$ is called "first integral" of $\{\omega_0, \omega_1, \omega_2\}$ if $dV \equiv 0 \mod\{\omega_0, \omega_1, \omega_2\}$.*

Proposition 2.3 *Assume that $\lambda_1 \neq \lambda_2$, and that* (2.4), *or* (2.5), *has two independent first integrals $\{u, v\}$. Then there exists a function $k = k(x,y,z,p,q) \neq 0$ satisfying*

$$du \wedge dv = k\, \omega_1 \wedge \omega_2 = kD\{Ar + Bs + Ct + D(rt - s^2) - E\} dx \wedge dy\,. \tag{2.7}$$

If equation (2.1) is written as (2.7), it would be obvious that (2.3), or (2.4), has two independent first integrals $\{u, v\}$. If (2.4), or (2.5), has at least two independent first integrals, equation (2.1) is called to be *integrable in the sense of Monge.* But, if we may follow G. Darboux (p. 263 of [2]), it seems to us that we had better call it to be *integrable in the sense of Darboux.* Moreover, as E. Goursat had profoundly studied equations (2.1) satisfying the above condition, we would like to add the name of Goursat. By these reasons, we will call equations (2.1) with two independent first integrals to be *integrable in the sense of Darboux and Goursat.* Then the representation (2.7) gives the characterization of "Monge-Ampère equations which is integrable in the sense of Darboux and Goursat". Next we advance to the integration of the Cauchy problem for (2.1). Let $\{u, v\}$ be two independent first integrals of (2.4). For any function g of two variables whose gradient does not vanish, $g(u, v) = 0$ is called an "intermediate integral" of (2.1). Let C_0 be an initial strip defined in $\mathbf{R}^5 = \{(x, y, z, p, q)\}$. If the strip C_0 is not characteristic, we can find an "intermediate integral" $g(u, v)$ which vanishes on C_0. Here we put $g(u, v) = f(x, y, z, p, q)$. The Cauchy problem for (2.1) satisfying the initial condition C_0 is to look for a solution $z = z(x, y)$ of (2.1) which contains the strip C_0, i.e., the two dimensional surface $\{(x, y, z(x, y), \partial z/\partial x(x, y), \partial z \partial y(x, y))\}$ in $\mathbf{R}^5$ contains the strip C_0. The representation (2.7) assures that, as $du \wedge dv = 0$ on a surface $g(u, v) = 0$, a smooth solution of $f(x, y, z, \partial z/\partial x, \partial z/\partial y) = 0$ satisfies equation (2.1). Therefore we get the following

Theorem 2.4 ([2], [4, 5]) *Assume that the initial strip C_0 is not characteristic. Then a function $z = z(x, y)$ is a solution of the Cauchy problem for* (2.1) *with the initial condition C_0 if and only if it is a solution of $f(x, y, z, \partial z/\partial x, \partial z \partial y) = 0$ satisfying the same initial condition C_0.*

In the following of this section, we will study the method of integration of (2.1) in the case where (2.4), and (2.5) also, has not two independent first integrals. We start from the point at which equation (2.1) is represented as a product of one forms as (2.7). We suppose $D \neq 0$ for simplicity, though it is not indispensable for our study. The essential condition for our following discussion is $\Delta \neq 0$. We will here take heuristic approach to get solutions of (2.1). As the preparation, we will give another representation of (2.1) which is similar to (2.7).

Exchanging λ_1 and λ_2 in ω_1 and ω_2, we define ϖ_1 and ϖ_2 by

$$\varpi_1 = Ddp + Cdx + \lambda_2 dy, \quad \varpi_2 = Ddq + \lambda_1 dx + Ady.$$

Then we get the following identity:

$$\omega_1 \wedge \omega_2 = \varpi_1 \wedge \varpi_2 = D\, F(x, y, z, p, q, r, s, t)\, dx \wedge dy. \tag{2.8}$$

The left hand sides of (2.8), that is to say $\omega_1 \wedge \omega_2$ and $\varpi_1 \wedge \varpi_2$, are defined in $\mathbf{R}^5 = \{(x, y, z, p, q)\}$. This is the characteristic property of equations (2.1) of Monge-Ampère type. Here we introduce a notion of "geometric solution".

Definition 2.5 *A geometric solution of* (2.1) *is a two-dimensional submanifold in* $\mathbf{R}^5$ *on which* $dz = pdx + qdy$ *and* $\omega_1 \wedge \omega_2 = \varpi_1 \wedge \varpi_2 = 0$.

Let us suppose that a geometric solution can be represented by two parameters, i.e.,

$$x = x(\alpha,\beta), y = y(\alpha,\beta), z = z(\alpha,\beta), p = p(\alpha,\beta), q = q(\alpha,\beta). \tag{2.9}$$

Then ω_i and ϖ_i $(i = 1,2)$ are written as

$$\omega_i = c_{i1}d\alpha + c_{i2}d\beta, \quad \varpi_i = d_{i1}d\alpha + d_{i2}d\beta \quad (i = 1,2).$$

Hence we have $\omega_1 \wedge \omega_2 = (c_{11}c_{22} - c_{12}c_{21})\, d\alpha \wedge d\beta$ and $\varpi_1 \wedge \varpi_2 = (d_{11}d_{22} - d_{12}d_{21})\, d\alpha \wedge d\beta$. Since the geometric solution is obtained as a two-dimensional submanifold on which $\omega_1 \wedge \omega_2 = \varpi_1 \wedge \varpi_2 = 0$, a sufficient condition so that (2.9) is a geometric solution of (2.1) is

$$c_{11} = c_{21} = d_{12} = d_{22} = 0. \tag{2.10}$$

Adding the contact relation $dz = pdx + qdy$ to (2.10), we get a system of first order partial differential equation as follows:

$$\left\{\begin{array}{l} \dfrac{\partial z}{\partial \alpha} - p\dfrac{\partial x}{\partial \alpha} - q\dfrac{\partial y}{\partial \alpha} = 0 \\[2ex] D\dfrac{\partial p}{\partial \alpha} + C\dfrac{\partial x}{\partial \alpha} + \lambda_1\dfrac{\partial y}{\partial \alpha} = 0 \\[2ex] D\dfrac{\partial q}{\partial \alpha} + \lambda_2\dfrac{\partial x}{\partial \alpha} + A\dfrac{\partial y}{\partial \alpha} = 0 \\[2ex] D\dfrac{\partial p}{\partial \beta} + C\dfrac{\partial x}{\partial \beta} + \lambda_2\dfrac{\partial y}{\partial \beta} = 0 \\[2ex] D\dfrac{\partial q}{\partial \beta} + \lambda_1\dfrac{\partial x}{\partial \beta} + A\dfrac{\partial y}{\partial \beta} = 0 \end{array}\right. . \tag{2.11}$$

If $(x(\alpha,\beta), y(\alpha,\beta), z(\alpha,\beta), p(\alpha,\beta), q(\alpha,\beta))$ satisfies system (2.11), we can prove $\partial z/\partial\beta - p\partial x/\partial\beta - q\partial y/\partial\beta = 0$. Therefore we do not need to add this to (2.11). This means that (2.11) is just the "determined" system. The local solvability of (2.11) is already proved at first by H. Lewy [14] and afterward by J. Hadamard [7].

Let us denote the solution of the Cauchy problem for (2.11) by $(x(\alpha,\beta), y(\alpha,\beta), z(\alpha,\beta), p(\alpha,\beta), q(\alpha,\beta))$. We can prove that, if the initial strip is not characteristic, the Jacobian $D(x,y)/D(\alpha,\beta)$ does not vanish in a neighbourhood of the initial strip. Therefore we can uniquely solve the system of equations $x = x(\alpha,\beta)$, $y = y(\alpha,\beta)$ with respect to (α,β). Then the solution of (2.1) with the initial condition is obtained by $z(x,y) = z(\alpha(x,y), \beta(x,y))$.

3 Nonlinear wave equations

Let us consider the Cauchy problem for nonlinear wave equations as follows:

$$F(q,r,t) = \frac{\partial^2 z}{\partial x^2} - \frac{\partial}{\partial y} f(\frac{\partial z}{\partial y}) = r - f'(q)t = 0 \quad \text{in} \quad \{x > 0, y \in \mathbf{R}^1\} \equiv \mathbf{R}^2_+, \tag{3.1}$$

$$z(0,y) = z_0(y), \quad \frac{\partial z}{\partial x}(0,y) = z_1(y) \qquad on \quad \{x = 0, y \in \mathbf{R}^1\} \tag{3.2}$$

where $f(q)$ is in $C^\infty(\mathbf{R}^1)$ and $f'(q) > 0$. Here $z = z(x,y)$ is an unknown function of $(x,y) \in \mathbf{R}^2$, and we assume that the initial functions $z_i(y)$ $(i = 0,1)$ are sufficiently smooth. Equation (3.1) is also of Monge-Ampère type. For example, if we may put $A = 1, B = D = E = 0$, and $C = -f'(q)$ in (2.1), then we get (3.1).

It is well known that the Cauchy problem (3.1)-(3.2) does not have a classical solution in the large. For example, see N. F. Zabusky [27] for the case where $f'(q) = (1+\varepsilon q)^{2\alpha}$ and P. D. Lax [12] for 2×2 hyperboric systems of conservation law. After them, many people have considered the life-span of classical solutions. As the number of papers on this subject is too many, we do not mention here their contributions.

Let us denote $\omega_1 = dp \pm \lambda(q)dq$ and $\omega_2 = \pm\lambda(q)dx + dy$ where $\lambda(q) = \sqrt{f'(q)}$. Take an exterior product of ω_1 and ω_2, and substitute into their product the relations of the contact structure $\omega_0 = 0$, $dp = rdx + sdy$ and $dq = sdx + rdy$. Then we get

$$\omega_1 \wedge \omega_2 = \{r - f'(q)t\} \; dx \wedge dy \,. \tag{3.3}$$

Therefore equations (2.11) for equation (3.1) are written as

$$\begin{cases} \dfrac{\partial p}{\partial \alpha} + \lambda(q)\dfrac{\partial q}{\partial \alpha} = 0 \\[2ex] \lambda(q)\dfrac{\partial x}{\partial \alpha} + \dfrac{\partial y}{\partial \alpha} = 0 \\[2ex] \dfrac{\partial p}{\partial \beta} - \lambda(q)\dfrac{\partial q}{\partial \beta} = 0 \\[2ex] -\lambda(q)\dfrac{\partial x}{\partial \beta} + \dfrac{\partial y}{\partial \beta} = 0 \end{cases} . \tag{3.4}$$

Adding the initial conditions which are corresponding to (3.2), we can uniquely get the solutions of (3.4) by

$$p + \Lambda(q) = \psi_1(\beta) \quad \text{and} \quad p - \Lambda(q) = \psi_2(\alpha) \tag{3.5}$$

where $\Lambda'(q) = \lambda(q)$, $\psi_1(-\alpha) = z_1(\alpha) + \Lambda(z_0'(\alpha))$ and $\psi_2(\alpha) = z_1(\alpha) - \Lambda(z_0'(\alpha))$.

As $\Lambda'(q) > 0$, we see that an inverse function of $\Lambda(q)$ is smooth. Therefore p and q are obtained as smooth functions of (α, β) defined in the whole space $\mathbf{R}^2$. On the other hand, x and y are solutions of the system as follows:

$$\begin{cases} \lambda(q)\dfrac{\partial x}{\partial \alpha} + \dfrac{\partial y}{\partial \alpha} = 0 \\ \\ -\lambda(q)\dfrac{\partial x}{\partial \beta} + \dfrac{\partial y}{\partial \beta} = 0 \end{cases} . \tag{3.6}$$

As this means that x and y satisfy linear wave equations, we can get $x = x(\alpha, \beta)$ and $y = y(\alpha, \beta)$ as smooth functions of (α, β) defined in the whole space $\mathbf{R}^2 = \{(\alpha, \beta)\}$. The function $z = z(\alpha, \beta)$ is uniquely determined by the contact relation $dz = pdx + qdy$ and the initial conditions (3.2), and it is defined in the whole space $\mathbf{R}^2 = \{(\alpha, \beta)\}$. Next, projecting the surface $\{(x(\alpha,\beta), y(\alpha,\beta), z(\alpha,\beta), p(\alpha,\beta), q(\alpha,\beta)); (\alpha,\beta) \in \mathbf{R}^2\}$ into the base space $\mathbf{R}^3 = \{(x, y, z)\}$, we get the solution of (3.1) with singularities. We will explain a little the situation how the singularities may appear.

Let us suppose that there exists a solution of (2.1) in the space C^2, then it holds

$$\begin{cases} \dfrac{\partial p}{\partial \alpha} = r\dfrac{\partial x}{\partial \alpha} + s\dfrac{\partial y}{\partial \alpha}, & \dfrac{\partial p}{\partial \beta} = r\dfrac{\partial x}{\partial \beta} + s\dfrac{\partial y}{\partial \beta} \\ \\ \dfrac{\partial q}{\partial \alpha} = s\dfrac{\partial x}{\partial \alpha} + t\dfrac{\partial y}{\partial \alpha}, & \dfrac{\partial q}{\partial \beta} = s\dfrac{\partial x}{\partial \beta} + t\dfrac{\partial y}{\partial \beta} \end{cases} , \tag{3.7}$$

where r, s and t are the second order derivatives of $z = z(x, y)$. See the notations of equation (2.1). If the Jacobian $D(x,y)/D(\alpha,\beta) = 2\lambda(q)(\partial x/\partial\alpha)(\partial x/\partial\beta)$ does not vanish, we can solve the system of equations (3.7) with respect to (r, s, t). Using equations (3.4) and (3.5), we can write r, s and t as follows:

$$r = \frac{1}{4}\left\{\frac{\psi_1{}'(\beta)}{\frac{\partial x}{\partial \beta}(\alpha,\beta)} + \frac{\psi_2{}'(\alpha)}{\frac{\partial x}{\partial \alpha}(\alpha,\beta)}\right\}, \quad s = \frac{1}{4\lambda(q)}\left\{\frac{\psi_1{}'(\beta)}{\frac{\partial x}{\partial \beta}(\alpha,\beta)} - \frac{\psi_2{}'(\alpha)}{\frac{\partial x}{\partial \alpha}(\alpha,\beta)}\right\}, \quad t = \frac{1}{\lambda(q)^2} r . \tag{3.8}$$

We assume that the Jacobian $D(x,y)/(D(\alpha,\beta)$ vanishes at a point (α^0, β^0) where $(\psi_1{}'(\beta^0) \cdot \psi_2{}'(\alpha^0)) \neq 0$. Then (3.8) says that, if a point (x, y) tends to a point $(x(\alpha^0, \beta^0),$ $y(\alpha^0, \beta^0))$ along a curve on which the Jacobian does not vanish, (r, s, t) goes to infinity. Therefore we can exactly determine a life-span of the classical solution by getting zeros of the Jacobian. Concerning the life-span of classical solutions, various kinds of results have been published, for example [27], [12], [26], etc, etc.

Our principal problem is how to extend the solution beyond the singularites. This depends on the definition of weak solutions of equations (3.1). We do not yet arrive at the final decision on this subject. In the following paper, we will discuss what kinds of solutions we could get by the projection of the geometric solutions into the base space.

4 Systems of conservation law

Let us recall a well-known relation between equation (3.1) and certain first order system of conservation law. We write $p = \partial z/\partial x$ and $q = \partial z/\partial y$, and put $U(x, y) =^t (p, q)$,

$F(U) =^t (f(q), p)$ and $U_0(y) =^t (z_1(y), z_0'(y))$. Then we get

$$\frac{\partial}{\partial x}U - \frac{\partial}{\partial y}F(U) = 0 \qquad \text{in} \quad \{x > 0, y \in \mathbf{R}^1\}, \tag{4.1}$$

$$U(0, y) = U_0(y) \qquad \text{on} \quad \{x = 0, y \in \mathbf{R}^1\} \quad . \tag{4.2}$$

By the same method as in §3, we get the solution of (4.1) by

$$p + \Lambda(q) = \psi_1(\beta) \quad \textit{and} \quad p - \Lambda(q) = \psi_2(\alpha) \tag{4.3}$$

where we have used the same notations used in §3. It is well known that, even if the initial data are sufficiently smooth, singularities generally appear in the solution of (4.1)-(4.2). P. D. Lax [13] introduced the notion of weak solutions of (4.1)-(4.2) as follows:

Definition 4.1 *A bounded and measurabule 2-vecter function U=U(x,y) is a weak solution of* (4.1)-(4.2) *if it satisfies* (4.1)-(4.2) *in distribution sense, i.e.,*

$$\int_{\mathbf{R}^2_+} \{U(x,y)\frac{\partial \varphi}{\partial x}(x,y) - F\ (U)\frac{\partial \varphi}{\partial y}(x,y)\}dxdy + \int_{\mathbf{R}^1} U_0(y)\varphi(0,y)dy = 0 \tag{4.4}$$

for any two-vector function $\varphi(x, y) \in C_0^\infty(\mathbf{R}^2)$.

If $U = U(x, y)$ is a weak solution of (4.1) which has jump discontinuity along a curve $y = \gamma(x)$, we get the jump condition of Rankine-Hugoniot as follows:

$$[p]\dot{\gamma} + [f(q)] = 0, \tag{4.5}$$

$$[q]\dot{\gamma} + [p] = 0. \tag{4.6}$$

What we would like to insist here is that we can not construct a piecewise smooth weak solution of (4.1) by analysing the solution (4.3) so that it would satisfy (4.5) and (4.6). Concerning single first order partial differential equations, we could construct weak solutions by the above method. For example, see M. Tsuji [20, 21], S. Nakane [16, 17], S. Izumiya and G. T. Kossioris [9, 10], etc. We have written this a little in [24]. In the following paper, we will discuss what kind of solutions we can get by projecting the geometric solutions into the base space.

At the end we will mention some relation between Riemann invariants and characteristic strips. One of the methods to solve system (4.1) is to rewrite it in a diagonal form. The notion of Riemann invariants has been introduced for this purpose. A. Kh. Rakhimov [18] has considered the singularities of Riemann invariants. But it seems to us that he has considered this subject in a little too general framework. Here we will explain the relation between Riemann invariants and characteristic strips, and write down Riemann invariants in exact form so that they are related with the above Cauchy problem (4.1) – (4.2). Now let us recall the definition of Riemann invariants. As $f'(q) > 0$, the Jacobian matrix $F'(U)$ has two real and distinct eigenvalues $\{\lambda_i(U)\}_{i=1,2}$. Corresponding to each $\lambda_i(U)$, we denote a right and left eigenvertor by $\vec{r_i}$ and $\vec{l_i}$ respectively. A function $w = w(U)$ is called to be a k-Riemann invariant of system (4.1) if $\langle \vec{r_k}, \mathrm{grad}\, w(U) \rangle = 0$ where $\langle \ , \ \rangle$ denotes the usual inner product in $\mathbf{R}^2$. In this case, Riemann invariants are given by $\{p + \Lambda(q),\, p - \Lambda(q)\}$ where $\Lambda = \Lambda(q)$ is introduced in (3.5). Therefore, as we can represent the Riemann invariants exactly by using the initial data (4.2), we see that the singularities of the Riemann invariants are caused by those of the smooth mapping $\{x = x(\alpha, \beta), y = y(\alpha, \beta)\}$.

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