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STRONG CONVERGENCE TO FIXED POINTS OF NON-LIPSCHITZIAN MAPPINGS IN BANACH SPACES

GANG-EUN KIM

ABSTRACT. In this paper, we study the strong convergence of the modified Ishikawa and Das-Debata iteration process of non-Lipschitzian mappings which satisfies the property (K) type in a Banach spaces.

1. INTRODUCTION

Let C be a nonempty bounded closed convex subset of a Banach space E and let T be a mapping of C into itself. Then T is said to be *asymptotically nonexpansive* [5] if there exists a sequence $\{k_n\}$ of real numbers with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|$$

for $x, y \in C$ and $n = 1, 2, \cdots$. In particular, if $k_n = 1$ for all $n \geq 1$, T is said to be *nonexpansive.* The weaker definition (cf., Kirk [10]) requires that

$$\limsup_{n\to\infty} \sup_{y\in C} (\|T^n x - T^n y\| - \|x - y\|) \leq 0$$

for each $x \in C$, and that T^N be continuous for some $N \geq 1$. Consider a definition somewhere between these two: T is said to be *weakly asymptotically nonexpansive* provided T is continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \leq 0.$$

Compare with the definition of asymptotically nonexpansive mappings in the intermediate sense initiated by Bruck et al. [1]. For two mappings S, T of C into itself, we consider the following modified Das-Debata iteration scheme (cf. Das-Debata [3]): $x_1 \in C$,

$$x_{n+1} = \alpha_n S^n[\beta_n T^n x_n + (1 - \beta_n)x_n] + (1 - \alpha_n)x_n \tag{$*$}$$

for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ in [0,1]. In this case of $S = T$, such an iteration scheme was considered by Tan-Xu [17]; see also Ishikawa [7], Mann [11], Schu [14]. Reich [12], using Mann iteration procedure in a uniformly convex Banach space whose norm is Fréchet differentiable, proved that the iterates $\{x_n\}$ defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n,$$

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for all $n \geq 1$, converge weakly to a fixed point of nonexpansive mappings $T : C \to C$ under $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$. Tan-Xu [16] improved a result of Reich [12] to the case of the Ishikawa type iteration. On the other hand, Takahashi-Tamura [15] studied the weak convergence of iterates $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n S[\beta_n T x_n + (1-\beta_n)x_n] + (1-\alpha_n)x_n$$

for all $n \geq 1$, in a uniformly convex Banach space which satisfies Opial's condition or whose norm is Fréchet differentiable. Recently Verma [18] proved the following interesting result using modified iterative algorithm: Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Let $T : C \to C$ be a relaxed Lipschitz (see Definition below) and Lipschitz continuous operator on C. Let $r \geq 0$ and $s \geq 1$ be constants for relaxed Lipschitzity and Lipschitz continuity of T, respectively. Let $F = \{x \in C : Tx = x\}$ be nonempty, and let $\{\alpha_n\}$ be a sequence in $[0,1]$ such that $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then for any x_0 in C the sequence $\{x_n\}$ defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n[(1-t)x_n + tTx_n]$$

for $n \geq 0$, $0 < k = ((1-t)^2 - 2t(1-t)r + t^2 s^2)^{\frac{1}{2}} < 1$ for all t such that $0 < t < \frac{2(1+r)}{(1+2r+s^2)}$ and $r \leq s$, converges to a fixed point of T.

In this paper, we first show how to construct (in a uniformly convex Banach space which neither satisfies the Opial property nor has a Fréchet differentiable norm) a unique fixed point of a non-Lipschitzian mapping $T : C \to C$ which satisfies the property (K) type (see Definition 2.2 below) as the strong limt of a sequence $\{x_n\}$ defined by a modified Ishikawa iteration of the form

$$x_{n+1} = \alpha_n T^n[\beta_n T^n x_n + (1-\beta_n)x_n] + (1-\alpha_n)x_n,$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ in [0,1] are chosen so that $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$ and $0 \leq \beta_n < b$ for some b with $0 < b < 1$. Next, we consider the sequence $\{x_n\}$ defined by $(*)$ converges strongly to a common fixed point of T and S under another conditions, that is, in cases when $\{\alpha_n\}$ and $\{\beta_n\}$ are chosen so that $\alpha_n \in [a,b]$ and $\beta_n \in [0,b]$ or $\alpha_n \in [a,1]$ and $\beta_n \in [a,b]$ for some a, b with $0 < a \leq b < 1$. Finally, we consider the sequence $\{x_n\}$ defined by $(*)$ converges strongly to a common fixed point of T and S under another parameter conditions, that is, in cases when $\{\alpha_n\}$ is a sequence in $[0,1]$ such that $\alpha_n \to 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $0 \leq \beta_n \leq 1$ for all $n \geq 1$.

2. Preliminaries and some examples

Let H be a real Hilbert space. We denote by $\langle x, y \rangle$ and $\|x\|$ the inner product and the norm on H for $x, y \in H$, respectively. An operator $T : H \to H$ is said to be *relaxed Lipschitz* [18] if, for all $x, y \in H$, there exists a constant $r > 0$ such that

$$\langle Tx - Ty, x - y \rangle \leq -r\|x-y\|^2.$$

Throughout this paper, let E be a Banach space. Recall that E is said to be *uniformly convex* if the modulus of convexity $\delta_E = \delta_E(\epsilon)$, $0 < \epsilon \leq 2$, of E defined by

$$\delta_E(\epsilon) = \inf\{1 - \frac{\|x+y\|}{2} : x, y \in E, \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \epsilon\}$$

satisfies the inequality $\delta_E(\epsilon) > 0$ for every $\epsilon \in (0,2]$. With each $x \in E$, we associate the set

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\},$$

where $\langle x, x^* \rangle$ denotes the value of x^* at x. Then J is said to be the *duality* mapping of E.

Let C be a nonempty closed convex subset of E and let T be a mapping from C into itself. Then we denote by $F(T)$ the set of all fixed points of T, i.e., $F(T) = \{x \in C : Tx = x\}$. When $\{x_n\}$ is a sequence in E, then $x_n \to x$ ($x_n \rightharpoonup x$) will denote strong (weak) convergence of the sequence $\{x_n\}$ to x. We denote by $\mathbb{R}$ the set of all real numbers.

Let C be a nonempty closed convex subset of E. If $F(T) \neq \emptyset$, the mapping $T : C \to E$ is said to be *strictly hemicontractive* [2] if there exists $t > 1$ such that for all $x \in C$ and $y \in F(T)$ there exists $j \in J(x - y)$ such that

$$\operatorname{Re}\langle Tx - y, j \rangle \leq \frac{1}{t}\|x - y\|^2.$$

Definition 2.1 [8]. Let C be a nonempty subset of E. Let T be a mappings of C into itself with $F(T) \neq \emptyset$. Then T is said to be of (H) type if there exists $t > 1$ such that for each $x \in C$ and $y \in F(T)$, there exists $j \in J(x - y)$ such that

$$\limsup_{n\to\infty} \operatorname{Re}\langle T^n x - y, j \rangle \leq \frac{1}{t}\|x - y\|^2.$$

Here we need the following stronger concept than (H) type for constructing an approximating fixed point of a non-Lipschitzian self-mapping in a Banach space.

Definition 2.2. Let C be a nonempty subset of E. Let T be a mappings of C into itself with $F(T) \neq \emptyset$. Then T is said to be of (K) type if, for each $x \in C$ and $y \in F(T)$, there exists $j \in J(x - y)$ such that

$$\limsup_{n\to\infty} \operatorname{Re}\langle T^n x - y, j \rangle \leq 0.$$

It is obvious that if $T : C \to C$ is mapping with $F(T) = \{y\}$ and $T^n x \rightharpoonup y$ as $n \to \infty$ for each $x \in C$, then T is of (K) type. Every relaxed Lipschitz mappings are obviously of (K) type.

Example 2.1 [2]. Take $E = C = \mathbb{R}$ with the usual norm $|\cdot|$. Let $T : C \to C$ be defined by

$$Tx = \frac{2}{3}x \cos x$$

for all $x \in C$. Clearly $F(T) = \{0\}$ and, since $T^n x \to 0$ for each $x \in C$, T is of (K) type.

Example 2.2. Take $E = C = \mathbb{R}$ with the usual norm $|\cdot|$ and let $0 < k < 1$. Let $T : C \to C$ be defined by

$$Tx = kx$$

for all $x \in C$. Clearly $F(T) = \{0\}$. Since $T^n x \to 0$ for each $x \in C$, T is also of (K) type.

Example 2.3. Take $E = \mathbb{R}$ with the usual norm $|\cdot|$ and let $C = (0,2]$. Let $T : C \to C$ be defined by

$$Tx = \sqrt{x}$$

$\forall\, x \in C$. Clearly $F(T) = \{1\}$ and, since $T^n x \to 1$ as $n \to \infty$ for each $x \in C$, T is weakly asymptotically nonexpansive which is of (K) type but not Lipschitz mapping.

3. Strong convergence theorems

We first begin with the following:

Lemma 3.1 [1]. *Suppose $\{v_n\}$ is a bounded sequence of real numbers and $\{a_{n,m}\}$ is a doubly-indexed sequence of real numbers which satisfy $\limsup_{n\to\infty}\limsup_{m\to\infty} a_{n,m} \leq 0$, $v_{n+m} \leq v_n + a_{n,m}$ for each $n, m \geq 1$. Then $\{v_n\}$ converges to an $v \in \mathbb{R}$; $a_{n,m}$ can be taken to be independent of n, $a_{n,m} = a_m$, then $v \leq v_n$ for each n.*

Lemma 3.2 [6]. *For any $x, y \in E$ and $j \in J(x+y)$, we obtain*

$$\|x+y\|^2 \leq \|x\|^2 + 2Re\langle y, j\rangle.$$

From the proof of Lemma 3 of [16], we note

Lemma 3.3. *Let a_n, $b_n > 0$ for $n \geq 1$. If $\sum_{n=1}^{\infty} a_n = \infty$ and $\sum_{n=1}^{\infty} a_n b_n < \infty$, then $\liminf_{n\to\infty} b_n = 0$.*

Using Lemma 3.1-3.3, we obtain the following Theorem 3.1.

Theorem 3.1 [9]. *Let E be a uniformly convex Banach space and let C be a nonempty bounded closed convex subset of E. Suppose that $T : C \to C$ is both weakly asymptotically nonexpansive and of (K) type. Put*

$$c_n = \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x-y\|) \vee 0,$$

so that $\sum_{n=1}^{\infty} c_n < \infty$. Then for any x_1 in C, the sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n T^n y_n + (1-\alpha_n)x_n, \qquad y_n = \beta_n T^n x_n + (1-\beta_n)x_n,$$

which $\{\alpha_n\}$ and $\{\beta_n\}$ are chosen so that $\alpha_n \in [0,1]$ and $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$ and $0 \leq \beta_n < b < 1$ for all $n \geq 1$, converge strongly to the unique fixed point of T.

Remark. If $\{\alpha_n\}$ is a sequence in $[0,1]$ which is bounded away from 0 and 1, i.e., $a \leq \alpha_n \leq b$ for some a, b with $0 < a \leq b < 1$, then $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$.

As a direct consequence of Theorem 3.1 with $\beta_n = 0$, we have the following result.

Corollary 3.1. *Let E be a uniformly convex Banach space and C be a nonempty bounded closed convex subset of E. Let $T : C \to C$ be both weakly asymptotically nonexpansive and of (K) type. Put*

$$c_n = \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x-y\|) \vee 0,$$

so that $\sum_{n=1}^{\infty} c_n < \infty$. Then for any x_1 in C, the sequence $\{x_n\}$ defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n,$$

which $\{\alpha_n\}$ is chosen so that $\alpha_n \in [0,1]$ and $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$ for all $n \geq 1$, converge strongly to the unique fixed point of T.

Lemma 3.4 [13]. *Let E be a uniformly convex Banach space, $0 < b \leq t_n \leq c < 1$ for all $n \geq 1$, $a \geq 0$. Suppose that $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are sequences of E such that $\limsup_{n\to\infty} \|x_n\| \leq a$, $\limsup_{n\to\infty} \|y_n\| \leq a$, and $\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = a$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

By using Lemma 3.4, we obtain the following Theorem 3.2.

Theorem 3.2. *Let E be a uniformly convex Banach space and C be a nonempty bounded closed convex subset of E. Let $T, S : C \to C$ be both weakly asymptotically nonexpansive and of (K) type with $F(T) \bigcap F(S) \neq \emptyset$. Put*

$$c_n = \max(0, \sup_{x,y\in C}(\|T^n x - T^n y\| - \|x-y\|), \sup_{x,y\in C}(\|S^n x - S^n y\| - \|x-y\|)),$$

so that $\sum_{n=1}^{\infty} c_n < \infty$. Then for any x_1 in C, the sequence $\{x_n\}$ defined by $(*)$, which $\{\alpha_n\}$ and β_n are chosen so that $\alpha_n \in [a,b]$ and $\beta_n \in [0,b]$ or $\alpha_n \in [a,1]$ and $\beta_n \in [a,b]$ for some a, b with $0 < a \leq b < 1$, converge strongly to a common fixed point of T and S.

The following lemma is very useful to prove the convergence of a sequence to 0. Compare with Lemma 1 due to Dunn [4].

Lemma 3.5 [19]. *Let β_n be a nonnegative sequence satisfying*

$$\beta_{n+1} \leq (1-\delta_n)\beta_n + \sigma_n$$

with $\delta_n \in [0,1]$, $\sum_{i=1}^{\infty} \delta_i = \infty$, and $\sigma_n = o(\delta_n)$. Then $\lim_{n\to\infty} \beta_n = 0$.

Theorem 3.3. *Let C be a nonempty bounded closed convex subset of a Banach space E. Let $T, S : C \to C$ be both weakly asymptotically nonexpansive and of (K) type with $F(T) \bigcap F(S) \neq \emptyset$. Put*

$$c_n = \max(0, \sup_{x,y\in C}(\|T^n x - T^n y\| - \|x-y\|), \sup_{x,y\in C}(\|S^n x - S^n y\| - \|x-y\|)),$$

so that $\sum_{n=1}^{\infty} c_n < \infty$. Then for any x_1 in C, the sequence $\{x_n\}$ defined by $(*)$, which $\{\alpha_n\}$ is a sequence in $[0,1]$ such that $\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $0 \leq \beta_n \leq 1$ for all $n \geq 1$, converge strongly to a common fixed point of T and S.

As a direct consequence of Theorem 3.3 with $\beta_n = 0$, we have the following result.

Corollary 3.2. *Let C be a nonempty bounded closed convex subset of a Banach space E. Let $T, S : C \to C$ be both weakly asymptotically nonexpansive and of (K) type with $F(T) \bigcap F(S) \neq \emptyset$. Put*

$$c_n = \max(0, \sup_{x,y\in C}(\|T^n x - T^n y\| - \|x-y\|), \sup_{x,y\in C}(\|S^n x - S^n y\| - \|x-y\|)),$$

so that $\sum_{n=1}^{\infty} c_n < \infty$. Then for any x_1 in C, the sequence $\{x_n\}$ defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n,$$

which $\{\alpha_n\}$ is a sequence in $[0,1]$ such that $\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$ for all $n \geq 1$, converge strongly to a common fixed point of T and S.

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PUKYONG NATIONAL UNIVERSITY, DEPARTMENT OF APPLIED MATHEMATICS, PUSAN 608-737, KOREA