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Some Two-Person Zero-Sum Dynamic Game

by

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1 A Two-person Zero-Sum Dynamic Game with a Parameter

We give a two-person zero-sum dynamic game with a parameter (DPG_θ) by a sequence of the following objects

$$(S_n, A_n, B_n, t_{n+1}, u_n, v_n, \theta; n \in N) \tag{1.1}$$

where

1. S_n is the state space at time $n \in N$ and is assumed to be a Borel space, that is, a nonempty Borel subset of a complete separable metric space.

2. A_n and B_n are the action spaces at time $n \in N$ of players I and II, respectively. It is assumed that A_n and B_n are Borel spaces.

3. $\{t_{n+1}\}$ is the law of motion of the system; t_{n+1} is a Borel measurable transition probability from $H_nA_nB_n$ to S_{n+1}, $n \in N$. Here, $H_1 = S_1, H_n = S_1A_1B_1\cdots S_{n-1}A_{n-1}B_{n-1}S_n$, $H_\infty = S_1A_1B_1S_2A_2B_2S_3\cdots$. Then, H_n is the set of histories of the game for horizon $n \in N$, while H_∞ is the set of all infinite histories of the game.

4. $u_n : H_nA_nB_n \to \mathbb{R}$, is a Borel measurable function and $v_n : H_nA_nB_n \to \mathbb{R}_+$, is a nonnegative bounded Borel measurable function, where $\mathbb{R}_+ = (0, \infty)$. Of course, u_n and v_n may be recognized as functions on H_∞. Doing so, we assume that

$$\lim_{n\to\infty} u_n = u \in \mathbb{R}, \ \lim_{n\to\infty} v_n = v \in \mathbb{R}_+.$$

5. $\theta : S_1 \to \mathbb{R}$ is a real valued function , which is called a parameter function of the game.

6. $T_\theta^n = u_n - \theta v_n : H_nA_nB_n \to \mathbb{R}$, is a loss function of player I at stage $n \in N$ and $-T_\theta^n$, is a loss function of player II.

Let $F_n(G_n)$ be the set of all universally measurable transition probabilities from $H_n(H_n)$ to $A_n(B_n)$. A universally measurable strategy of player I(II) is a sequence $f = \{f_n\}(g = \{g_n\})$ such that $f_n \in F_n(g_n \in G_n)$ for each $n \in N$. Denote by $F(G)$ the set of all strategies for player I(II).

Let $E_{f_n}, E_{g_n}, E_{t_{n+1}}$ denote the conditional expectation operator with respect to $f_n \in F_n, g_n \in G_n, t_{n+1}$, respectively. Then, each pair of strategies $f =$

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$\{f_n\}(g = \{g_n\})$, together with the law of motion $\{t_{n+1}\}$, defines uniquely a universally measurable transition probability $P_{fg}(\cdot|\cdot)$ from S_1 to $A_1B_1S_2A_2B_2S_3\cdots$ such that, for two bounded Borel measurable functions u_n, v_n defined on $H_nA_nB_n$ $(n \in N)$, we have for $s_1 \in S_1$ and $h \in H_\infty$,

$$\begin{aligned} E(u_n, f, g)(s_1) &= \int u_n(h)P_{fg}(dh|s_1) \\ &= E_{f_1}E_{g_1}E_{t_2}\cdots E_{f_{n-1}}E_{g_{n-1}}E_{t_n}E_{f_n}E_{g_n}u_n(s_1) \end{aligned}$$

and

$$\begin{aligned} E(v_n, f, g)(s_1) &= \int v_n(h)P_{fg}(dh|s_1) \\ &= E_{f_1}E_{g_1}E_{t_2}\cdots E_{f_{n-1}}E_{g_{n-1}}E_{t_n}E_{f_n}E_{g_n}v_n(s_1) \end{aligned}$$

where u_n and v_n are also regarded as functions on H_∞.

Under our assumptions, we infer that, for each $s_1 \in S_1$, $f = \{f_n\} \in F$, $g = \{g_n\} \in G$, from the dominated convergence theorem and Fubini's theorem

$$\begin{aligned} U(f,g)(s_1) &= \lim_{n\to\infty} E(u_n, f, g)(s_1) \\ &= \lim_{n\to\infty} E_{f_1}E_{g_1}E_{t_2}\cdots E_{f_{n-1}}E_{g_{n-1}}E_{t_n}E_{f_n}E_{g_n}u_n(s_1) \\ &= \lim_{n\to\infty} E_{g_1}E_{f_1}E_{t_2}\cdots E_{g_{n-1}}E_{f_{n-1}}E_{t_n}E_{g_n}E_{f_n}u_n(s_1) \end{aligned}$$

and

$$\begin{aligned} V(f,g)(s_1) &= \lim_{n\to\infty} E(v_n, f, g)(s_1) \\ &= \lim_{n\to\infty} E_{f_1}E_{g_1}E_{t_2}\cdots E_{f_{n-1}}E_{g_{n-1}}E_{t_n}E_{f_n}E_{g_n}v_n(s_1) \\ &= \lim_{n\to\infty} E_{g_1}E_{f_1}E_{t_2}\cdots E_{g_{n-1}}E_{f_{n-1}}E_{t_n}E_{g_n}E_{f_n}v_n(s_1). \end{aligned}$$

For the loss function with the parameter function θ;

$$T_\theta^n = u_n - \theta v_n,$$

we have for each $s_1 \in S_1$, $f = \{f_n\} \in F$, $g = \{g_n\} \in G$,

$$\begin{aligned} T_\theta(f,g)(s_1) &= \lim_{n\to\infty} E_{fg}T_\theta^n(f,g)(s_1) \\ &= U(f,g)(s_1) - \theta(s_1)V(f,g)(s_1). \end{aligned}$$

We define for initial state $s_1 \in S_1$,

$$\overline{T}_\theta(s_1) = \inf_{f\in F}\sup_{g\in G} T_\theta(f,g)(s_1), \ \underline{T}_\theta(s_1) = \sup_{g\in G}\inf_{f\in F} T_\theta(f,g)(s_1).$$

Then, $\overline{T}_\theta(s_1)(\underline{T}_\theta(s_1))$ is called the upper (the lower) value function of the parametric game. In general, it holds that $\overline{T}_\theta(s_1) \geq \underline{T}_\theta(s_1)$ for all $s_1 \in S_1$. Further, we call the **duality gap** the interval $[\underline{T}_\theta(s_1), \overline{T}_\theta(s_1)]$ for all $s_1 \in S_1$.

Definition 1.1 *We shall say that the two-person zero-sum game* (DPG_θ) *has a* **saddle value function** *(in short, a* **value function** *), if*

$$\overline{T}_\theta(s_1) = \underline{T}_\theta(s_1) = T_\theta^*(s_1)$$

and this common function is called the value function of the game and is denoted by $T_\theta^*(s_1)$.

Definition 1.2 *A strategy $\bar{f} \in F$ is said to be a* **mini-sup** *of the game (DPG_θ) if*

$$\sup_{g \in G} T_\theta(\bar{f}, g)(s_1) = \underline{T}_\theta(s_1)$$

and a strategy $\bar{g} \in G$ is said to be a **max-inf** *of the game (DPG_θ) if*

$$\inf_{f \in F} T_\theta(f, \bar{g})(s_1) = \overline{T}_\theta(s_1).$$

Definition 1.3 *A pair strategies $(\bar{f}, \bar{g}) \in F \times G$ is said to be a* **saddle point** *of the game (DPG_θ) if*

$$\inf_{f \in F} T_\theta(f, \bar{g})(s_1) = T_\theta(\bar{f}, \bar{g})(s_1) = \sup_{g \in G} T_\theta(\bar{f}, g)(s_1).$$

2 A Two-Person Zero-Sum Dynamic Fractional Game

We define a two-person zero-sum dynamic fractional game (DFG) as follows :

$$(S_n, A_n, B_n, t_{n+1}, u_n, v_n, \overline{\theta}, \underline{\theta}; n \in N) \tag{2.1}$$

where S_n is the state space and A_n and B_n are the action spaces at time $n \in N$ of players I and II, respectively. $\{t_{n+1}\}$ is the law of motion of the system. These terms are defined like as the game (DPG_θ). Further, $u_n : H_n A_n B_n \to \mathbb{R}$, is a bounded Borel measurable function and $v_n : H_n A_n B_n \to \mathbb{R}_+$, is a nonnegative bounded Borel measurable function, $\mathbb{R}_+ = (0, \infty)$. We assume that

$$\lim_{n \to \infty} u_n = u \in \mathbb{R}, \ \lim_{n \to \infty} v_n = v \in \mathbb{R}_+.$$

Under our assumptions, we infer that, for each $s_1 \in S_1$, $f = \{f_n\} \in F$, $g = \{g_n\} \in G$,

$$U(f,g)(s_1) = \lim_{n \to \infty} E(u_n, f, g)(s_1), V(f,g)(s_1) = \lim_{n \to \infty} E(v_n, f, g)(s_1) > 0.$$

Using the notations $U(f,g)(s_1)$ and $V(f,g)(s_1)$, we give

$$W(f,g)(s_1) = \frac{U(f,g)(s_1)}{V(f,g)(s_1)}$$

and we define for an initial state $s_1 \in S_1$,

$$\overline{\theta}(s_1) = \inf_{f \in F} \sup_{g \in G} W(f,g)(s_1), \ \underline{\theta}(s_1) = \sup_{g \in G} \inf_{f \in F} W(f.g)(s_1).$$

Then, $\overline{\theta}(s_1)(\underline{\theta}(s_1))$ is called the upper (the lower) value function of the game (DFG). In general, it holds that $\overline{\theta}(s_1) \geq \underline{\theta}(s_1)$ for all $s_1 \in S_1$ and the interval $[\underline{\theta}(s_1), \overline{\theta}(s_1)]$ is called the **duality gap** of the game (DFG).

Definition 2.1 *The game (DFG) is said to have a* **value function** *if the duality gap is equal to zero. We shall call the value function of the game (DFG) the common value function*

$$\overline{\theta}(s_1) = \underline{\theta}(s_1) = \theta^*(s_1).$$

Further, $g^* \in G$ *is said to be a* **max-inf** *of the game* (DFG) *if*

$$\overline{\theta}(s_1) = \inf_{f \in F} \sup_{g \in G} W(f,g)(s_1) = \inf_{f \in F} W(f,g^*)(s_1). \tag{2.2}$$

Similarly, $f^* \in F$ *is said to be a* **mini-sup** *of the game* (DFG) *if*

$$\underline{\theta}(s_1) = \sup_{g \in G} \inf_{f \in F} W(f,g)(s_1) = \sup_{g \in G} W(f^*,g)(s_1). \tag{2.3}$$

Lemma 2.1 $\overline{T}_\theta(s_1)$ *has the following properties.*

(1) *If two parameter functions* $\theta_1(s_1)$ *and* $\theta_2(s_1)$ *satisfy that* $\theta_1(s_1) > \theta_2(s_1) \geq 0$, *it follows that*

$$\overline{T}_{\theta_1}(s_1) \leq \overline{T}_{\theta_2}(s_1).$$

(2) *If* $\overline{T}_\theta(s_1) < 0$, *it holds that* $\theta(s_1) \geq \overline{\theta}(s_1)$.

(3) *If* $\overline{T}_\theta(s_1) > 0$, *it holds that* $\theta(s_1) \leq \overline{\theta}(s_1)$.

(4) *If* $\theta(s_1) > \overline{\theta}(s_1)$, *it holds that* $\overline{T}_\theta(s_1) \leq 0$.

(5) *If* $\theta(s_1) < \overline{\theta}(s_1)$, *it holds that* $\overline{T}_\theta(s_1) \geq 0$.

Proof. (1) If $\theta_1(s_1) > \theta_2(s_1)$, then, we get $\theta_1(s_1)U(f,g)(s_1) > \theta_2(s_1)U(f,g)(s_1)$, because $U(f,g)(s_1)$ is positive for all $(f,g) \in F \times G$. Then, it follows that for all $(f,g) \in F \times G$,

$$T_{\theta_1}(f,g)(s_1) < T_{\theta_2}(f,g)(s_1).$$

Therefore, we get that

$$\begin{aligned} \overline{T}_{\theta_1}(s_1) &= \inf_{f \in F} \sup_{g \in G} T_{\theta_1}(f,g)(s_1) \\ &\leq \inf_{f \in F} \sup_{g \in G} T_{\theta_2}(f,g)(s_1) \\ &= \overline{T}_{\theta_2}(s_1). \end{aligned}$$

Thus, the proof of (1) in the lemma is complete.

(2) Since $\overline{T}_\theta(s_1) < 0$, from the definition of $\overline{T}_\theta(s_1)$, there exists $\overline{f} \in F$ such that $\sup_{g \in G} T_\theta(\overline{f},g)(s_1) < 0$, that is, for all $g \in G$,

$$T_\theta(\overline{f},g)(s_1) = U(\overline{f},g)(s_1) - \theta(s_1)V(\overline{f},g)(s_1) < 0. \tag{2.4}$$

From (2.4), this shows that for all $g \in G$,

$$W(\overline{f},g)(s_1) = \frac{U(\overline{f},g)(s_1)}{V(\overline{f},g)(s_1)} < \theta(s_1) \tag{2.5}$$

that is,

$$\sup_{g \in G} W(\overline{f},g)(s_1) \leq \theta(s_1). \tag{2.6}$$

From the definition of $\overline{\theta}(s_1)$ and (2.6), it follows that $\theta(s_1) \geq \overline{\theta}(s_1)$.

(3) Since $\overline{T}_\theta(s_1) > 0$, that is, for all $f \in F$, $\sup_{g \in G} T_\theta(f,g)(s_1) > 0$, there exists $g_f \in G$, which depends on f, such that

$$T_\theta(f,g_f)(s_1) = U(f,g_f)(s_1) - \theta(s_1)V(f,g_f)(s_1) > 0. \tag{2.7}$$

From (2.7), it follows that for all $f \in F$, $W(f, g_f)(s_1) = U(f, g_f)(s_1)/V(f, g_f)(s_1) > \theta(s_1)$. This shows that $\overline{\theta}(s_1) \geq \theta(s_1)$.

(4) Since $\theta(s_1) > \overline{\theta}(s_1)$, from the definition of $\overline{\theta}(s_1)$, there exists $\overline{f} \in F$ such that for all $g \in G$,

$$\theta(s_1) > \sup_{g \in G} W(\overline{f}, g)(s_1).$$

This shows that for all $g \in G$, $T_\theta(\overline{f}, g)(s_1) < 0$. Hence, we get that

$$\begin{aligned} 0 &\geq \sup_{g \in G} T_\theta(\overline{f}, g)(s_1) \\ &\geq \inf_{f \in F} \sup_{g \in G} T_\theta(f, g)(s_1) \\ &= \overline{T}_\theta(s_1). \end{aligned}$$

(5) Since $\overline{\theta}(s_1) > \theta(s_1)$, from the definition of $\overline{\theta}(s_1)$, it follows that for all $f \in F$,

$$\sup_{g \in G} W(f, g)(s_1) > \theta(s_1).$$

Thus, there exists $g_f \in G$, which depends on f, such that $W(f, g_f)(s_1) > \theta(s_1)$, that is, for all $f \in F$,

$$\begin{aligned} \sup_{g \in G} T_\theta(f, g)(s_1) &\geq T_\theta(f, g_f)(s_1) \\ &> 0. \end{aligned}$$

Hence, we get that

$$\overline{T}_\theta(s_1) = \inf_{f \in F} \sup_{g \in G} T_\theta(f, g)(s_1) \geq 0.$$

□

Lemma 2.2 *$\underline{T}_\theta(s_1)$ has the following properties.*

(1) *If two parameter functions $\theta_1(s_1)$ and $\theta_2(s_1)$ satisfy that $\theta_1(s_1) > \theta_2(s_1) \geq 0$, it follows that*

$$\underline{T}_{\theta_1}(s_1) \leq \underline{T}_{\theta_2}(s_1).$$

(2) *If $\underline{T}_\theta(s_1) < 0$, it holds that $\theta(s_1) \geq \underline{\theta}(s_1)$.*

(3) *If $\underline{T}_\theta(s_1) > 0$, it holds that $\theta(s_1) \leq \underline{\theta}(s_1)$.*

(4) *If $\theta(s_1) > \underline{\theta}(s_1)$, it holds that $\underline{T}_\theta(s_1) \leq 0$.*

(5) *If $\theta(s_1) < \underline{\theta}(s_1)$, it holds that $\underline{T}_\theta(s_1) \geq 0$.*

Proof. Using $\underline{T}_\theta(s_1)$ and $\underline{\theta}(s_1)$ instead of $\overline{T}_\theta(s_1)$ and $\overline{\theta}(s_1)$, respectively. we can prove this lemma by similar arguments to the previous one. □

We have the following relations between the game (DFG) and (DPG_θ).

Theorem 2.1 *Suppose that $g^* \in G$ is a max-inf of the game (DFG). Then, it holds that*

(1) $\overline{\theta}(s_1) = \underline{\theta}(s_1) = \theta^*(s_1)$.

(2) *If* $\overline{T}_{\theta^*}(s_1) \leq 0$, g^* *is a max-inf of the game* (DPG_{θ^*}).

Proof. (1) From the definition of $\overline{\theta}(s_1)$ and $\underline{\theta}(s_1)$, in general it holds that $\overline{\theta}(s_1) \geq \underline{\theta}(s_1)$.

On the other hand, since $g^* \in G$ is a max-inf of the game (DFG), it follows that

$$\begin{aligned}\overline{\theta}(s_1) &= \inf_{f \in F} W(f, g^*)(s_1) \\ &\leq \sup_{g \in G} \inf_{f \in F} W(f, g)(s_1) \\ &= \underline{\theta}(s_1).\end{aligned}$$

Thus, the game (DFG) has a value function, that is, $\overline{\theta} = \underline{\theta}$ on S_1.

(2) Since $g^* \in G$ is a max-inf of the game (DFG), it holds that for all $f \in F$,

$$\theta^*(s_1) = \inf_{f \in F} W(f, g^*)(s_1) \leq W(f, g^*)(s_1)$$

that is, for all $f \in F$,

$$0 \leq T_{\theta^*}(f, g^*)(s_1) \leq \sup_{g \in G} T_{\theta^*}(f, g)(s_1). \tag{2.8}$$

Thus, from (2.8) and (2) of the theorem, we get the following :

$$\begin{aligned}0 &\leq \inf_{f \in F} T_{\theta^*}(f, g^*)(s_1) \\ &\leq \inf_{f \in F} \sup_{g \in G} T_{\theta^*}(f, g)(s_1) \\ &= \overline{T}_{\theta^*}(s_1) \leq 0.\end{aligned}$$

This shows that

$$\inf_{f \in F} T_{\theta^*}(f, g^*)(s_1) = \inf_{f \in F} \sup_{g \in G} T_{\theta^*}(f, g)(s_1).$$

That is, g^* is a max-inf of the game (DPG_{θ^*}). □

Corollary 2.1 *Suppose that* $(f^*, g^*) \in F \times G$ *is a saddle point of the game* (DFG). *Then, it holds that*

(1) $T_{\theta^*}(f^*, g^*)(s_1) = 0$.

(2) (f^*, g^*) *is a saddle point of the game* (DPG_{θ^*}).

The proof of the corollary is easily given by Theorem 2.1.

Theorem 2.2 *Under* $\overline{\theta}(s_1) = \underline{\theta}(s_1) = \theta^*(s_1)$, *suppose that* $g^* \in G$ *is a max-inf of the game* (DPG_{θ^*}) *and*

$$\inf_{f \in F} T_{\theta^*}(f, g^*)(s_1) = \overline{T}_{\theta^*}(s_1) \geq 0.$$

Then, g^* *is a max-inf of the game* (DFG).

Proof. Since $\overline{T}_{\theta^*}(s_1) \geq 0$ and g^* is a max-inf of the game (DPG_{θ^*}), it follows that

$$\begin{aligned} 0 &\leq \inf_{f \in F} \sup_{g \in G} T_{\theta^*}(f,g)(s_1) \\ &= \inf_{f \in F} T_{\theta^*}(f,g^*)(s_1) \\ &\leq T_{\theta^*}(f,g^*)(s_1) \text{ for all} f \in F, \end{aligned}$$

which implies that for all $f \in F$,

$$\theta^*(s_1) \leq W(f,g^*)(s_1) \leq \sup_{g \in G} W(f,g)(s_1).$$

Therefore, we get that

$$\begin{aligned} \theta^*(s_1) &\leq \inf_{f \in F} W(f,g^*)(s_1) \\ &\leq \inf_{f \in F} \sup_{g \in G} W(f,g)(s_1) \\ &= \theta^*(s_1). \end{aligned}$$

This shows that g^* is a max-inf of the game (DFG). □

Corollary 2.2 *Under* $\overline{\theta}(s_1) = \underline{\theta}(s_1) = \theta^*(s_1)$, *suppose that* $(f^*, g^*) \in F \times G$ *is a saddle point of the game* (DPG_{θ^*}) *and* $T_{\theta^*}(f^*, g^*)(s_1) = 0$ *holds. Then,* (f^*, g^*) *is a saddle point of the game* (DFG).

The proof of the corollary is easily given by Theorem 2.2.

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