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Radon-Nikodym sets, Pettis sets and uniform Gateaux differentiability of support functions

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1. Introduction

Let X be an arbitrary real Banach space, X^* and X^{**} its topological dual space and bidual space, respectively. In dual Banach spaces, we have two notions of weak*-compact sets called Pettis sets and Radon-Nikodym (RN in brief) sets, which are generalizations of weak*-compact convex sets with the weak RNP and the RNP, respectively. Now, let us (re)define two notions, generalizations of equivalent notions of Pettis sets and RN sets, in the following form.

Definition 1. Let A be a bounded subset of X and K a weak*-compact (not necessarily convex) subset of X^*. Then

(1) K is said to be an A-Pettis set if every weak*-compact subset D of K has the following property : For every $x^{**} \in \bar{A}^*$ (the weak*-closure of A in X^{**}) and every $\varepsilon > 0$, there exists a weak*-open subset U such that $U \cap D \neq \phi$ and $O(x^{**}|U \cap D)$ (= $\sup\{ (u^*, x^{**}) : u^* \in U \cap D \} - \inf\{ (v^*, x^{**}) : v^* \in U \cap D \}$, the oscillation of x^{**} on $U \cap D$) $< \varepsilon$.

(2) K is said to be an A-RN set if every weak*-compact subset D of K has the following property : For every $\varepsilon > 0$, there exists a weak*-open subset U such that $U \cap D \neq \phi$ and $\mathrm{diam}_A(U \cap D)$ (= $\sup\{ q_A(u^* - v^*) : u^*, v^* \in U \cap D \}$, the q_A-diameter of $U \cap D$) $< \varepsilon$.

Here q_A is the seminorm given by $q_A(x^*) = \sup_{x \in A} |(x, x^*)|$ for every $x^* \in X^*$.

Note that if $A = B(X)$ (the closed unit ball of X) in (1) (resp. (2)) of Definition 1, then we have an equivalent notion of Pettis (resp. RN) sets.

Definition 2. Let $g : X \to R$ be a continuous convex function and A a bounded subset of X. Then g is said to be A-uniformly Gateaux differentiable at $x \in X$ if $Dg(x, y)$ exists uniformly in $y \in A$, where $Dg(x, y)$ is defined by $\lim_{t \to 0} \{ g(x + ty) - g(x) \}/t$ provided that this limit exists.

Now, in a series of our papers [4], [5], and [6], we have made a study of K-weakly precompact sets A, an equivalent notion of A-Pettis sets K, by the effective use of K-valued weak*-measurable functions constructed in the case where A is not K-weakly precompact. In this paper as well, by following the same ideas of the best use of K-valued functions constructed in the case where K is a non-A-Pettis set or a non-A-RN set, we wish to give characterizations of A-Pettis sets and A-RN sets in terms of uniform Gateaux differentiability of support functions. This result provides us with an information to recognize not only the similarity but also the subtle difference between these two notions in the convex analytic phenomenon.

Theorem. Let A be a bounded subset of X and K a weak*-compact subset of X^*. Then

(1) K is an A-RN set if and only if for every nonempty subset G of K and every sequence $\{x_n\}_{n\geq 1}$ in A, there exists a point y of Y (closed linear span of $\Psi = \{ x_n : n \geqq 1 \}$) such that $Ds_G(y, x_n)$ exists uniformly in n (that is, s_G is Ψ-uniformly Gateaux differentiable at y), where s_G (: $Y \to R$) is the support function of G defined by $s_G(y) = \sup_{x^* \in G} (y, x^*)$ for each $y \in Y$.

(2) K is an A-Pettis set if and only if for every nonempty subset G of K and every sequence $\{x_n\}_{n\geq 1}$ in A, there exists a point y of Y and a subsequence $\{x_{n(k)}\}_{k\geq 1}$ of $\{x_n\}_{n\geq 1}$ such that $Ds_G(y, x_{n(k)})$ exists uniformly in k (that is, s_G is Φ-uniformly Gateaux differentiable at y, where $\Phi = \{ x_{n(k)} : k \geqq 1 \}$.

The part (2) of Theorem (essentially given in [7]) may be regarded as a slight generalization and improvement of results due to Bator and Lewis[1]. The thing to be emphasized is the proof of the sufficiency of statements (1) and (2) of Theorem, which readers should appreciate well. In Section 2, we introduce a result showing that the construction of certain K-valued weak*-measurable functions can be done under some assumption of weak*-compact subsets K of X^* (For further details of this result, refer to [4] and [6]). Making use of this result, in Section 3, we can present basic functions to study A-Pettis sets K and A-RN sets K and further, applying these functions, we state the proof of the sufficiency of statements (1) and (2) of Theorem. Indeed, these functions are useful for us to study other various properties of A-Pettis sets and A-RN sets simultaneously. In what follows, all notations and terminology, unless otherwise stated, are as in [2], [4] and [6].

2. A brief on the construction of certain weak*-measurable functions

In order to proceed our argument concerning the construction of certain weak*-measurable functions with various desired properties, we first need :

Definition 3 ([9]). A sequence $(A_n, B_n)_{n\geq 1}$ of pairs of subsets of some set is called independent provided $A_n \cap B_n = \phi$ for every n and for every $\{\varepsilon_j\}_{1\leq j\leq k}$ with $\varepsilon_j = 1$ or -1, $\cap\{\varepsilon_j A_j : 1 \leqq j \leqq k\} \neq \phi$, where $\varepsilon_j A_j = A_j$ if $\varepsilon_j = 1$ and $\varepsilon_j A_j = B_j$ if $\varepsilon_j = -1$.

Let D be a weak*-compact subset of X^*. Suppose that there exists a system $\{ V(n,i) : n = 0, 1, \cdots ; i = 0, \cdots , 2^n - 1 \}$ of nonempty weak*-closed subsets of D such that $V(n+1,2i)\cup V(n+1,2i+1) \subset V(n,i)$ and $V(n+1,2i)\cap V(n+1,2i+1) = \phi$ for $n = 0, 1, \cdots$ and $i = 0, \cdots , 2^n - 1$. Then, letting $A_n = \cup\{ V(n,2i+1) : i = 0, \cdots , 2^{n-1} - 1 \}$ and $B_n = \cup\{ V(n,2i) : i = 0, \cdots , 2^{n-1} - 1 \}$ for every $n \geqq 1$, $(A_n, B_n)_{n\geq 1}$ is an independent sequence of pairs of weak*-closed subsets of D. Then $\Gamma = \cap_{n\geq 1}(A_n \cup B_n)$ is a nonempty weak*-compact subset of D, since $(A_n, B_n)_{n\geq 1}$ is independent. Now, define $\phi : \Gamma \to \mathcal{P}(N)$ (Cantor space, with its usual compact metric topology) by $\phi(x^*) = \{ j : A_j \ni x^* \} \in \mathcal{P}(N)$. Then ϕ is a continuous surjection and so we have a Radon probability measure γ on Γ such that $\phi(\gamma) = \nu$ (the normalized Haar measure if we identify $\mathcal{P}(N)$ with $\{0,1\}^N$) and $\{ u\circ\phi : u \in L_1(\mathcal{P}(N), \Sigma_\nu, \nu) \} = L_1(\Gamma, \Sigma_\gamma, \gamma)$, where Σ_ν (resp. Σ_γ) is the family of all ν (resp. γ)-measurable subsets of $\mathcal{P}(N)$ (resp. Γ). Further, consider a function $\tau : \mathcal{P}(N) \to I$ defined by $\tau(J) = \sum_{j\in J} 1/2^j$ for every $J \in \mathcal{P}(N)$. Then τ is a continuous surjection such that $\tau(\nu) = \lambda$ and $\{ v\circ\tau : v \in L_1 \} = L_1(\mathcal{P}(N), \Sigma_\nu, \nu)$. Then, making use of the lifting theory, we have a weak*-measurable function $k : I \to \Gamma(\subset D)$ such that

(a) $\rho(f\circ k)(t) = f(k(t))$ for every $f \in C(\Gamma)$ and every $t \in I$,

(b) $\displaystyle\int_E f(k(t))d\lambda(t) = \int_{\phi^{-1}(\tau^{-1}(E))} f(x^*)d\gamma(x^*)$

for every $E \in \Lambda$ and every $f \in C(\Gamma)$. Here ρ is a lifting of L_∞. Further we should remark here that $\tau(\phi(\gamma)) = \lambda$, $\cup\{\phi^{-1}(\tau^{-1}(I(n,2i))) : 0 \leqq i \leqq 2^{n-1} - 1\} \equiv \Gamma \cap B_n$, $\cup\{\phi^{-1}(\tau^{-1}(I(n,2i+1))) : 0 \leqq i \leqq 2^{n-1} - 1\} \equiv \Gamma \cap A_n$ (with respect to γ) for $n = 1, 2, \cdots$, and it also holds that $\phi^{-1}(\tau^{-1}(O(n,2i))) \subset V(n,2i)$ and $\phi^{-1}(\tau^{-1}(O(n,2i+1))) \subset V(n,2i+1)$ for $n = 1, 2, \cdots$ and $i = 0, \cdots, 2^{n-1} - 1$ (Here $O(n,i)$ is the set of all interior points of $I(n,i)$). Hence this function $k : I \to D$ satisfies the following :

$$\int_{I(n,2i)} f(k(t))d\lambda(t) = \int_{\phi^{-1}(\tau^{-1}(I(n,2i)))} f(x^*)d\gamma(x^*)$$

$$= \int_{\Gamma \cap V(n,2i)} f(x^*)d\gamma(x^*)$$

and

$$\int_{I(n,2i+1)} f(k(t))d\lambda(t) = \int_{\phi^{-1}(\tau^{-1}(I(n,2i+1)))} f(x^*)d\gamma(x^*)$$

$$= \int_{\Gamma \cap V(n,2i+1)} f(x^*)d\gamma(x^*)$$

for $f \in C(\Gamma)$, $n = 1, 2, \cdots$ and $i = 0, \cdots, 2^{n-1} - 1$.

3. Basic functions associated with non-A-RN sets and non-A-Pettis sets, and proof of the sufficiency of statements (1) and (2) of Theorem

Here we present a fundamental result (Proposition) giving the sufficiency of statements (1) and (2) of Theorem simultaneously.

Proposition. Let A be a bounded subset of X and K a weak*-compact subset of X^*.

(i) Assume that there exists a weak*-compact subset D of K with the property : For an $\varepsilon > 0$, it holds that $\mathrm{diam}_A(U \cap D) > \varepsilon$ whenever U is a weak*-open subset with $U \cap D \neq \phi$. Then the following statements hold.

(a) There exist a system $\{ x(n,i) : n = 0, 1, \cdots ; i = 0, \cdots, 2^n - 1 \}$ in

A and a system $\{V(n,i) : n = 0, 1, \cdots ; i = 0, \cdots, 2^n - 1\}$ of nonempty weak*-closed subsets of D such that

(1) $V(n+1,2i) \cup V(n+1,2i+1) \subset V(n,i)$,

(2) $x^* \in V(n+1,2i)$ and $y^* \in V(n+1,2i+1)$ imply $(x(n,i), x^* - y^*) \geqq \varepsilon$ for $n = 0, 1, \cdots$ and $i = 0, \cdots, 2^n - 1$.

Consequently,

(b) We have a weak*-measurable function $g : I \to D$ such that for an appropriate sequence $\{y_n\}_{n\geq 1}$ in A, s_G is nowhere Ψ-uniformly Gateaux differentiable in Y, where $G = g(I)$, $\Psi = \{ y_n : n \geqq 1 \}$ and Y denotes the closed linear span of Ψ, (and further, s_G is nowhere A-differentiable in X if $A = -A$).

(ii) Suppose that there exists a weak*-compact subset D of K with the property : For an adequate element $a^{**} \in \bar{A}^*$ and an $\varepsilon > 0$, it holds that $O(a^{**}|U \cap D) > \varepsilon$ whenever U is a weak*-open subset with $U \cap D \neq \phi$. Then the following statements hold.

(c) There exist a sequence $\{x_n\}_{n\geq 1}$ in A and a system $\{ W(n,i) : n = 0, 1, \cdots ; i = 0, \cdots, 2^n - 1 \}$ of nonempty weak*-closed subsets of D such that

(1) $W(n+1,2i) \cup W(n+1,2i+1) \subset W(n,i)$,

(2) $x^* \in W(n+1,2i)$ and $y^* \in W(n+1,2i+1)$ imply $(x_{n+1}, x^* - y^*) \geqq \varepsilon$ for $n = 0, 1, \cdots$ and $i = 0, \cdots, 2^n - 1$.

Consequently,

(d) We have a weak*-measurable function $h : I \to D$ satisfying that for every subsequence $\{x_{n(k)}\}_{k\geq 1}$ of $\{x_n\}_{n\geq 1}$, s_H is nowhere Φ-uniformly Gateaux differentiable in Z, where $H = h(I)$, $\Phi = \{ x_{n(k)} : k \geqq 1 \}$ and Z denotes the closed linear span of $\{ x_n : n \geqq 1 \}$.

Proof. (I) To show the statement (a) of (i), replacing the unit ball by a bounded subset A of X in the proof of Proposition 5.6 in [8], we can construct a system $\{ x(n,i) : n = 0, 1, \cdots ; i = 0, \cdots, 2^n - 1 \}$ in A and a system $\{ U(n,i) : n = 0, 1, \cdots ; i = 0, \cdots, 2^n - 1 \}$ of weak*-open subsets such that

(a) $U(n,i) \cap D \neq \phi$,

(b) $(U(n+1,2i) \cap D) \cup (U(n+1,2i+1) \cap D) \subset U(n,i) \cap D$,

(c) $x^* \in U(n+1,2i) \cap D$ and $y^* \in U(n+1,2i+1) \cap D$ imply $(x(n,i), x^* - y^*) \geqq \varepsilon$ for $n = 0, 1, \cdots$ and $i = 0, \cdots, 2^n - 1$.

Let $V(n,i) = w^*\text{-cl}(U(n,i) \cap D)$ (the weak*-closure of $U(n,i) \cap D$). Then we have desired systems $\{ x(n,i) : n = 0, 1, \cdots ; i = 0, \cdots , 2^n - 1 \}$ and $\{ V(n,i) : n = 0, 1, \cdots ; i = 0, \cdots , 2^n - 1 \}$.

(II) Let us prove the statement (c) of (ii). This also can be proved by an argument analogous to (I). In virtue of the assumption, there exist an element $a^{**} \in \bar{A}^*$ and a positive number ε such that $O(a^{**}|U\cap D) > \varepsilon$ whenever U is a nonempty weak*-open subset with $U\cap D \neq \phi$. Let $U(0,0) = X$. Suppose that for some positive integer k, $\{ U(n,i) : n = 0, 1, \cdots , k ; i = 0, \cdots , 2^k - 1 \}$ and $\{x_n\}_{1\leq n\leq k}$ have already been defined so that proprties (a), (b) and (c) hold.

(a) $U(n,i)\cap D \neq \phi$ for $n = 0, 1, \cdots , k$ and $i = 0, \cdots , 2^k - 1$

(b) $(U(n+1,2i)\cap D)\cup(U(n+1,2i+1)\cap D) \subset U(n,i)\cap D$ for $n = 0, 1, \cdots , k - 1$ and $i = 0, \cdots , 2^{k-1} - 1$,

(c) $x^* \in U(n+1,2i)\cap D$ and $y^* \in U(n+1,2i+1)\cap D$ imply $(x_{n+1}, x^* - y^*) \geqq \varepsilon$ for $n = 0, 1, \cdots , k - 1$ and $i = 0, \cdots , 2^{k-1} - 1$.

Then, by assumption, we have $O(a^{**}|U(k,i)\cap D) > \varepsilon$ for $i = 0, \cdots , 2^k - 1$, and hence, for every such i there exist elements $x^*(k+1,2i)$ and $x^*(k+1,2i+1)$ of $U(k,i)\cap D$ such that $(x^*(k+1,2i) - x^*(k+1,2i+1), a^{**}) > \varepsilon$. Since A is weak*-dense in $\bar{A}^*$, we can choose an element $x_{k+1} \in A$ such that for every i with $0 \leqq i \leqq 2^k - 1$, $(x_{k+1}, x^*(k+1,2i) - x^*(k+1,2i+1)) > \varepsilon$. Take a positive number δ such that $(x_{k+1}, x^*(k+1,2i) - x^*(k+1,2i+1)) > \varepsilon + \delta$ for every i with $0 \leqq i \leqq 2^k - 1$, and let $U(k+1,2i) = \{ z^* \in U(k,i) : (x_{k+1}, z^*) > (x_{k+1}, x^*(k+1,2i)) - \delta/2 \}$ and $U(k+1,2i+1) = \{ z^* \in U(k,i) : (x_{k+1}, z^*) < (x_{k+1}, x^*(k+1,2i+1)) + \delta/2 \}$ for every i with $0 \leqq i \leqq 2^k - 1$. Then they are nonempty weak*-open subsets with $U(k+1,i)\cap D \neq \phi$ for every i with $0 \leqq i \leqq 2^{k+1} - 1$. Furthermore, we easily get that $x^* \in U(k+1,2i)\cap D$ and $y^* \in U(k+1,2i+1)\cap D$ imply $(x_{k+1}, x^* - y^*) \geqq \varepsilon$ for every i with $0 \leqq i \leqq 2^k - 1$. Hence, letting $W(n,i) = w^*\text{-cl}(U(n,i)\cap D)$, we have desired systems $\{x_n\}_{n\geq1}$ and $\{ W(n,i) : n = 0, 1, \cdots ; i = 0, \cdots , 2^n - 1 \}$.

(III) (Construction of functions) In order to obtain g (resp. h) in (b) of (i) (resp. (d) of (ii)), take $\Gamma_1 = \bigcap_{n\geq1}(A_n\cup B_n)$ (resp. $\Gamma_2 = \bigcap_{n\geq1}(C_n\cup D_n)$), where $A_n = \bigcup\{ V(n,2i+1) : i = 0, \cdots , 2^{n-1} - 1 \}$ (resp. $C_n = \bigcup\{ W(n,2i+1) : i = 0, \cdots , 2^{n-1} - 1 \}$) and $B_n = \bigcup\{ V(n,2i) : i = 0, \cdots , 2^{n-1} - 1 \}$ (resp. $D_n = \bigcup\{ W(n,2i) : i = 0, \cdots , 2^{n-1} - 1 \}$). Then, by the result in Section 2, we have a weak*-measurable function g (resp. h) $: I \to D$ such that

(a) $\rho(f\circ g)(t) = f(g(t))$ (resp. $\rho(f\circ h)(t) = f(h(t))$ for every $f \in C(\Gamma_1)$ (resp. $C(\Gamma_2)$) and every $t \in I$,

$$\text{(b)} \int_E f(g(t))d\lambda(t) = \int_{\psi_1^{-1}(\tau^{-1}(E))} f(x^*)d\gamma_1(x^*)$$

$$\left(\text{resp.} \int_E f(h(t))d\lambda(t) = \int_{\psi_2^{-1}(\tau^{-1}(E))} f(x^*)d\gamma_2(x^*)\right)$$

for every $E \in \Lambda$ and every $f \in C(\Gamma_1)$ (resp. $C(\Gamma_2)$). Here ψ_1 (resp. ψ_2) is the function defined by $\psi_1(x^*)$ (resp. $\psi_2(x^*)$) = $\{ j : A_j \ni x^* \}$ (resp. $\{ j : C_j \ni x^* \}$) $\in \mathcal{P}(N)$ for each $x^* \in \Gamma_1$ (resp. Γ_2) and γ_1 (resp. γ_2) is the Radon probability measure on Γ_1 (resp. Γ_2) such that $\psi_1(\gamma_1)$ (resp. $\psi_2(\gamma_2)$) = ν.

(Ⅳ) We intend to show that this function g (resp. h) has the property in the statement (b) of (i) (resp. the statement (d) of (ii)). To this end, we note a following elementary fact used to show such properties of g and h.

Lemma (Lemma 2 in [3]). Let $E_1, \cdots, E_m$ be arbitrary members of Λ^+. Then there exist a natural number p and a finite collection $\{ i_1, \cdots, i_m \}$ of non-negative integers such that

(1) $0 \leqq 2\cdot i_1, \cdots, 2\cdot i_m < 2^p - 1$,

(2) Both $E_k \cap I(p, 2\cdot i_k)$ and $E_k \cap I(p, 2\cdot i_k+1)$ are in Λ^+ for $k = 1, \cdots, m$.

In the following, let $a(n,i)$ (resp. $c(n,i)$) = $\inf\{ (x(n,i), x^*) : x^* \in V(n+1,2i) \}$ (resp. $\inf\{ (x_{n+1}, x^*) : x^* \in W(n+1,2i) \}$) and $b(n,i)$ (resp. $d(n,i)$) = $\sup\{ (x(n,i), x^*) : x^* \in V(n+1,2i+1) \}$ (resp. $\sup\{ (x_{n+1}, x^*) : x^* \in W(n+1,2i+1) \}$) for every (n,i). Then it holds that $a(n,i) - b(n,i)$ (resp. $c(n,i) - d(n,i)$) $\geqq \varepsilon$ for all (n,i).

(1) Let us prove that g has the property in (b) of (i). Let $\{y_n\}_{n\geq 1}$ be a sequence defined by $y_n = x(m,i)$ for $n = 2^m + i$ with $m = 0, 1, \cdots$ and $i = 0, \cdots, 2^m - 1$. Take any point y of Y and consider a family of weak*-open slices of M_g (= $\overline{co}^*(j^*(T_g^*(\Delta(I))))$) : $\{ S(y, \varepsilon/3n, M_g) : n \geqq 1 \}$, where j^* is the dual mapping of the inclusion map $j : Y \to X$. Then we have that

$$S(y, \varepsilon/3n, M_g) = \{ y^* \in M_g : (y, y^*) > \sup_{z^* \in M_g} (y, z^*) - \varepsilon/3n \}$$

$$= \{ y^* \in M_g : (y, y^*) > \operatorname{ess-sup}_{t \in I} (j(y), g(t)) - \varepsilon/3n \}$$

$$= \{ y^* \in M_g : (y, y^*) > s_G(y) - \varepsilon / 3n \}.$$

So, letting $E_n = \{ t \in I : (j(y), g(t)) > s_G(y) - \varepsilon / 3n \}$, we know that $E_n \in \Lambda^+$ and $j^*(g(E_n)) \subset S(y, \varepsilon / 3n, M_g)$ for every n. Hence, in virtue of Lemma (and its proof in [3]), there exists a strictly increasing sequence $\{p_n\}_{n \geq 1}$ of natural numbers and a sequence $\{i_n\}_{n \geq 1}$ of non-negative integers such that $0 \leqq 2 \cdot i_n < 2^{p_n} - 1$, $E_n \cap I(p_n, 2 \cdot i_n) \in \Lambda^+$ and $E_n \cap I(p_n, 2 \cdot i_n + 1) \in \Lambda^+$ for every $n \geqq 1$. Let $F_n = E_n \cap O(p_n, 2 \cdot i_n)$ and $G_n = E_n \cap O(p_n, 2 \cdot i_n + 1)$, and define $u_n^* = j^*(T_g^*(\chi_{F_n} / \lambda(F_n)))$ and $v_n^* = j^*(T_g^*(\chi_{G_n} / \lambda(G_n)))$ for every $n \geqq 1$. Then we have that for every n

(a) $(y, u_n^*) > s_G(y) - \varepsilon / 3n$ and $(y, v_n^*) > s_G(y) - \varepsilon / 3n$,

(b) $(z_n, u_n^* - v_n^*) \geqq \varepsilon$ (Here, $z_n = x(p_n - 1, i_n)$, and so $\{z_n\}_{n \geq 1}$ is a subsequence of $\{y_n\}_{n \geq 1}$),

(c) $s_G(y + z_n/n) \geqq (y + z_n/n, u_n^*)$ and $s_G(y - z_n/n) \geqq (y - z_n/n, v_n^*)$.

Indeed, we have that

$$(y, u_n^*) = (j(y), T_g^*(\chi_{F_n} / \lambda(F_n)))$$

$$= \left\{ \int_{F_n} (j(y), g(t)) d\lambda(t) \right\} / \lambda(F_n) > s_G(y) - \varepsilon / 3n,$$

since $j^*(g(F_n)) \subset S(y, \varepsilon / 3n, M_g)$. Similarly, $(y, v_n^*) > s_G(y) - \varepsilon / 3n$. Thus we have (a). Also we can prove (b) as follows. We have that for every n

$$(z_n, u_n^* - v_n^*) = (j(z_n), T_g^*(\chi_{F_n} / \lambda(F_n))) - (j(z_n), T_g^*(\chi_{G_n} / \lambda(G_n)))$$

$$= (j(x(p_n-1, i_n)), T_g^*(\chi_{F_n} / \lambda(F_n))) - (j(x(p_n-1, i_n)), T_g^*(\chi_{G_n} / \lambda(G_n)))$$

$$= \left\{ \int_{F_n} (j(x(p_n-1, i_n)), g(t)) d\lambda(t) \right\} / \lambda(F_n)$$

$$- \left\{ \int_{G_n} (j(x(p_n-1, i_n)), g(t)) d\lambda(t) \right\} / \lambda(G_n)$$

$$= \left\{ \int_{\psi_1^{-1}(\tau^{-1}(F_n))} (j(x(p_n-1, i_n)), x^*) d\gamma_1(x^*) \right\} / \lambda(F_n)$$

$$- \{ \int_{\phi_1^{-1}(\tau^{-1}(G_n))} (j(x(p_n-1, i_n)), x^*) d\gamma_1(x^*) \} / \lambda(G_n)$$

$$\geqq a(p_n-1, i_n) - b(p_n-1, i_n) \geqq \varepsilon.$$

As to (c), we have that for every n

$$s_G(y + z_n/n) = \sup_{t \in I} (j(y + z_n/n), g(t))$$

$$\geqq \{ \int_{F_n} (j(y + z_n/n), g(t)) d\lambda(t) \} / \lambda(F_n) = (y + z_n/n, u_n^*).$$

Similarly, $s_G(y - z_n/n) \geqq (y - z_n/n, v_n^*)$. Now, making use of (a), (b) and (c), let us show that s_G is not Ψ-uniformly Gateaux differentiable at y. We easily get from these properties that for every n

$$s_G(y + z_n/n) + s_G(y - z_n/n) - 2 \cdot s_G(y)$$

$$> (y + z_n/n, u_n^*) + (y - z_n/n, v_n^*) - \{ (y, u_n^* + v_n^*) + 2\varepsilon/3n \}$$

$$= (z_n, u_n^* - v_n^*)/n - 2\varepsilon/3n \geqq \varepsilon/3n,$$

whence $\{ s_G(y + z_n/n) + s_G(y - z_n/n) - 2 \cdot s_G(y) \}/(1/n) > \varepsilon/3$ for every n. This means that $Ds_G(y, z_n)$ does not exist uniformly in n and so s_G is not Ψ-uniformly Gateaux differentiable at y. Further, assume that there exists a point $x \in X$ such that s_G (: $X \to R$) is A-differentiable at x, and let x^* be its A-differential. Well, since $A = -A$, by a slight modification of the argument above, we have sequences $\{w_n\}_{n \geq 1}$ in A and $\{u_n^*\}_{n \geq 1}$ in X^* such that for every $n \geqq 1$, $(x, u_n^*) > s_G(x) - \varepsilon/3n$, $(w_n, u_n^* - x^*) \geqq \varepsilon/2$ and $s_G(x + w_n/n) \geqq (x + w_n/n, u_n^*)$. Then we have that for every $n \geqq 1$,

$$s_G(x + w_n/n) - s_G(x) - (w_n/n, x^*)$$

$$\geqq (x + w_n/n, u_n^*) - \{(x, u_n^*) + \varepsilon/3n\} - (w_n/n, x^*)$$

$= (w_n/n,\ u_n^* - x^*) - \varepsilon/3n > \varepsilon/6n.$

But, this is contradictory to the fact that x^* is an A-differential of s_G at x. Hence we complete the proof of properties concerning the function g.

(2) Let us prove that h has the property in (d) of (ii). We note a following elementary fact : Let $E \in \Lambda^+$ and $\{n(i)\}_{i \geq 1}$ be a strictly increasing sequence of natural numbers. Then there exists a natural number i and a non-negative number q with $0 \leqq 2q < 2^{n(i)} - 1$ such that both $E \cap O(n(i), 2q)$ and $E \cap O(n(i), 2q+1)$ are in Λ^+, which can be easily shown by an argument used in Lemma 2 of [3].

Now, let us show that for every subsequence $\{x_{n(k)}\}_{k \geq 1}$ of $\{x_n\}_{n \geq 1}$ and every $z \in Z$, $Ds_H(z,\ x_{n(k)})$ does not exist uniformly in k. To this end, take any point z in Z and any subsequence $\{x_{n(k)}\}_{k \geq 1}$ of $\{x_n\}_{n \geq 1}$, and set $z_k = x_{n(k)}$ for every k. Consider a family of weak*-open slices of $\overline{co}^*(j^*(T_h^*(\Delta(I))))$ $(= M_h)$: $\{\ S(z, \varepsilon/3i, M_h) : i \geqq 1\ \}$, where j^* is the dual mapping of the inclusion map $j : Z \to X$. Then we have that for every i

$$S(z, \varepsilon/3i, M_h) = \{\ z^* \in M_h : (z,\ z^*) > \sup_{v^* \in M_h} (z,\ v^*) - \varepsilon/3i\ \}$$

$$= \{\ z^* \in M_h : (z,\ z^*) > \operatorname{ess-sup}_{t \in I} (j(z),\ h(t)) - \varepsilon/3i\ \}$$

$$= \{\ z^* \in M_h : (z,\ z^*) > s_H(z) - \varepsilon/3i\ \}.$$

So, letting $E_i = \{\ t \in I : (j(z),\ h(t)) > s_H(z) - \varepsilon/3i\ \}$, we get that $E_i \in \Lambda^+$ and $j^*(h(E_i)) \subset S(z, \varepsilon/3i, M_h)$ for every i. Hence, by the elementary fact stated above, there exists a natural number $k(i)$ and a non-negative number $q(i)$ with $0 \leqq 2q(i) < 2^{n(k(i))} - 1$ such that both $E_i \cap O(n(k(i)), 2q(i))$ and $E_i \cap O(n(k(i)), 2q(i)+1)$ are in Λ^+. For every i, let $F_i = E_i \cap O(n(k(i)), 2q(i))$ and $G_i = E_i \cap O(n(k(i)), 2q(i)+1)$, and let $u_i^* = j^*(T_h^*(\chi_{F_i}/\lambda(F_i)))$ and $v_i^* = j^*(T_h^*(\chi_{G_i}/\lambda(G_i)))$. Then we have that for every i

(a) $(z,\ u_i^*) > s_H(z) - \varepsilon/3i$ and $(z,\ v_i^*) > s_H(z) - \varepsilon/3i$,

(b) $(z_{k(i)},\ u_i^* - v_i^*) \geqq \delta$,

(c) $s_H(z + z_{k(i)}/i) \geqq (z + z_{k(i)}/i,\ u_i^*)$ and $s_H(z - z_{k(i)}/i) \geqq (z - z_{k(i)}/i,\ v_i^*)$.

Indeed, we have that

$$(z,\ u_i^*) = (j(z),\ T_h^*(\chi_{F_i}/\lambda(F_i)))$$

$$= \{\int_{F_i}(j(z),\ h(t))d\lambda(t)\}/\lambda(F_i) > s_H(z) - \varepsilon/3i,$$

since $j^*(h(F_i)) \subset S(z, \varepsilon/3i, M_h)$. Similarly, $(z, v_i^*) > s_H(z) - \varepsilon/3i$. Thus we have (a). Also we can prove (b) as follows. We have that for every i

$$(z_{k(i)},\ u_i^* - v_i^*)$$

$$= (j(z_{k(i)}),\ T_h^*(\chi_{F_i}/\lambda(F_i))) - (j(z_{k(i)}),\ T_h^*(\chi_{G_i}/\lambda(G_i)))$$

$$= (j(x_{n(k(i))}),\ T_h^*(\chi_{F_i}/\lambda(F_i))) - (j(x_{n(k(i))}),\ T_h^*(\chi_{G_i}/\lambda(G_i)))$$

$$= \{\int_{F_i}(j(x_{n(k(i))}),\ h(t))d\lambda(t)\}/\lambda(F_i) - \{\int_{G_i}(j(x_{n(k(i))}),\ h(t))d\lambda(t)\}/\lambda(G_i)$$

$$= \{\int_{\varphi_2^{-1}(\tau^{-1}(F_i))}(j(x_{n(k(i))}),\ x^*)d\gamma_2(x^*)\}/\lambda(F_i) - \{\int_{\varphi_2^{-1}(\tau^{-1}(G_i))}(j(x_{n(k(i))}),\ x^*)d\gamma_2(x^*)\}/\lambda(G_i)$$

$$\geqq c(n(k(i)),q(i)) - d(n(k(i)),q(i)) \geqq \varepsilon.$$

As to (c), we have that for every i

$$s_H(z + z_{k(i)}/i) = \sup_{t\in I}(j(z + z_{k(i)}/i),\ h(t))$$

$$\geqq \{\int_{F_i}(j(z + z_{k(i)}/i),\ h(t))d\lambda(t)\}/\lambda(F_i) = (z + z_{k(i)}/i,\ u_i^*).$$

Similarly, $s_H(z - z_{k(i)}/i) \geqq (z - z_{k(i)}/i, v_i^*)$. Then, making use of (a), (b) and (c), we have that for every i

$$s_H(z + z_{k(i)}/i) + s_H(z - z_{k(i)}/i) - 2\cdot s_H(z)$$

$$> (z + z_{k(i)}/i, u_i^*) + (z - z_{k(i)}/i, v_i^*) - \{ (z, u_i^* + v_i^*) + 2\varepsilon/3i\}$$

$$= (z_{k(i)}, u_i^* - v_i^*)/i - 2\varepsilon/3i \geqq \varepsilon/3i,$$

whence $\{ s_H(z + z_{k(i)}/i) + s_H(z - z_{k(i)}/i) - 2\cdot s_H(z)\}/(1/i) > \varepsilon/3$ for every i. This implies that $Ds_H(z, x_{n(k)})$ does not exist uniformly in k and so s_H is not Φ-uniformly Gateaux differentiable at z. Thus the proof is completed.

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