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ALMOST EVERYWHERE CONVERGENCE THEOREMS FOR NONLINEAR OPERATORS

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1. INTRODUCTION

There have been many important researches on the study of ergodic theorems for nonlinear operators. However, most researches on this subject dealt only with questions of mean ergodic theorems, while the problem of pointwize ergodic theorems had been ignored. In the 1980's the study on this problem was started by Krengel and Lin etc.

Let (E,μ) be a σ-finite measure space and all the L^p spaces are with respect to this measure space. The classical pointwise ergodic theorem for linear operators is the following theorem of Hopf:

Theorem 1.1. *Let T be a positive linear contraction on L^1 with $\mu(E)<\infty$ such that $T1=1$. Then for any $f\in L^1$, the averages*

$$A_nf=\frac{1}{n+1}\sum_{i=0}^{n}T^if$$

converge almost everywhere.

In the nonlinear situation, instead of being a contraction we assume that T is nonexpansive, and instead of positivity we assume that T is order preserving. Krengel and Lin [3] obtained a result for this class of nonlinear operators:

Theorem 1.2. *Let T be an order preserving, L^1 nonexpansive and L^∞ norm decreasing mapping on L^1 with $\mu(E)<\infty$. Then for any $f\in L^1$, the averages*

$$A_nf=\frac{1}{n+1}\sum_{i=0}^{n}T^if \tag{1}$$

converge weakly in L^1.

Further Krengel [2] gave an example that almost everywhere convergence of the averages (1) fails. Namely, we can not expect the almost everywhere convergence for A_nf defined by (1).

In the linear case, the partial sums $S_nf=\sum_{i=0}^{n}T^if$ satisfy

$$S_0f=f,\qquad S_{n+1}f=f+T(S_nf)\quad(n\geq 0). \tag{2}$$

However in the nonlinear case, the partial sums $\{S_n f\}$, of course, do not satisfy the recursive relations (2). Then Lin and Wittmann [5, 7] adopted the definition (2) for $\{S_n f\}$ and put

$$A_n f = \frac{1}{n+1} S_n f \tag{3}$$

to get the almost everywhere convergence. Wittmann [7] obtained the following:

Theorem 1.3. *Let T be an order preserving, integral preserving, positively homogeneous and L^∞ nonexpansive mapping on L^1 with $\mu(E) < \infty$. Then $A_n f$ defined by (3) converges almost everywhere for any $f \in L^1$.*

Recently Wittmann [6] improved his own result:

Theorem 1.4. *Let T be an order preserving and L^1 and L^∞ nonexpansive mapping on L^1. Then $A_n f$ converges almost everywhere for any $f \in L^1$.*

In this paper, we prove another extention of Theorem 1.3 by assuming the existence of a kind of strictly positive invariant functions. The method of the proof is due to that of Wittmann [6]. Further we give some properties of the limit point of $A_n f$.

2. Preliminaries

Throughout this paper, (E, μ) is a σ-finite measure space and all the L^p spaces are with respect to this measure space. And expressions involving measurable functions or sets have to be understood in the almost everywhere sense.

Let T be an operator on L^1. The nonlinear ergodic average $A_n f$ $(f \in L^1)$ is defined by

$$S_0 f = f, \quad S_{n+1} f = f + T(S_n f) \quad (n \geq 0)$$

and

$$A_n f = \frac{1}{n+1} S_n f \quad (n \geq 0).$$

In the linear case, $A_n f$ is equal to $\frac{1}{n+1} \sum_{i=0}^{n} T^i f$.

T is said to be order preserving if $f \leq g \Rightarrow Tf \leq Tg$ $(f, g \in L^1)$. T is said to be L^p nonexpansive if $\|Tf - Tg\|_p \leq \|f - g\|_p$ $(f, g \in L^1 \cap L^p, 1 \leq p \leq \infty)$. T is called positively homogeneous if $T(\alpha f) = \alpha T f$ $(f \in L^1, \alpha \geq 0)$. T is called integral preserving if $\int T f d\mu = \int f d\mu$ $(f \in L^1)$. Note that an order preserving and integral preserving mapping is L^1 nonexpansive (See Krengel and Lin [3]).

We denote by $F(T)$ the set of invariant functions of T, i.e. $F(T) = \{f \in L^1 : Tf = f\}$. We define $G(T)_+$ by $G(T)_+ = \{k \in L^1 : k > 0, T(f + tk) = Tf + tk, \forall f \in L^1, \forall t \in \mathbf{R}\}$. Obviously if $T0 = 0$, then $G(T)_+ \subset F(T)$. We know from Lin and Wittmann [5] that if $\mu(E) < \infty$ and T is order preserving, integral preserving and L^∞ nonexpansive, then $G(T)_+ \neq \emptyset$.

3. Almost Everywhere Convergence Theorems for Nonlinear Operators

For a sequence $\{f_n\} \subset L^1$, we define f_n^* by

$$f_n^* = \sup_{0 \le i < n} f_i \quad (1 \le n \le \infty).$$

Our first aim is to prove the nonlinear version of the maximal ergodic theorem. The maximal ergodic theorem plays important role in the proof of pointwize ergodic theorems. Wittmann [6] proved the following theorem and lemma.

Theorem 3.1. *Let T be an order preserving mapping on L^1. And let $K \in \mathbf{R}$, and let h, h' be two measurable functions with value in $(-\infty, \infty]$ (both functions may attain the value ∞ on a set of positive measure) such that*

$$\int (Tg - h')_+ d\mu + \le \int (g - h)_+ d\mu + K < \infty \quad (g \in L^1). \tag{4}$$

Further, let $\{f_n\}$ be a sequence in L^1 and let $f \in L^1$ such that

$$f_0 \le h, \quad 1_{\{f_n > h\}} f_n \le 1_{\{f_n > h\}} (f + T f_{n-1}) \quad (n \ge 1). \tag{5}$$

Then $1_{\{f_n^* > h\}}(f + h - h')_-$ is integrable (since $\{h = \infty\} \cap \{f_n^* > h\} = \emptyset$, this is well defined) and

$$\int_{\{f_n^* > h\}} (f + h - h') d\mu \ge -K \quad (1 \le n < \infty). \tag{6}$$

If $1_{\{f_\infty^* > h\}}(f + h' - h)_+$ or $1_{\{f_\infty^* > h\}}(f + h' - h)_-$ is integrable, then (6) holds also for $n = \infty$.

Lemma 3.2. *Let T be an L^1 nonexpansive mapping on L^1. Further, let h, h' be measurable functions such that*

$$f \le h \Rightarrow Tf \le h'$$

for any $f \in L^1$. Then

$$\int (Tf - h')_+ d\mu \le \int (f - h)_+ d\mu < \infty \quad (f \in L^1).$$

We prepare one more lemma.

Lemma 3.3. *Let T be an order preserving and L^1 nonexpansive mapping on L^1 such that $G(T)_+$ is nonempty. Let $K \in \mathbf{R}$, and let h, h' be two measurable functions such that*

$$\int (Tg - h')_+ d\mu \le \int (g - h)_+ d\mu + K < \infty \quad (g \in L^1).$$

Then

$$\int (Tf - (h' + \alpha k))_+ d\mu \le$$
$$\int (f - (h + \alpha k))_+ d\mu + K < \infty \quad (f \in L^1, k \in G(T)_+, \alpha \ge 0).$$

Proof. We fix $\alpha \ge 0, k \in G(T)_+$ and $f \in L^1$. Since $\int (g - h)_+ d\mu < \infty$, we obtain $h_- \in L^1$ by putting $g = 0$. Setting $f_\alpha = \sup(f - \alpha k, -h_-)$, we can show $f_\alpha \in L^1$. In fact,

$$\begin{aligned} \int |f_\alpha| d\mu &= \int |\sup(f - \alpha k, -h_-)| d\mu \\ &= \int_{\{f \ge \alpha k\}} (f - \alpha k) d\mu - \int_{\{f < \alpha k\}} \sup(f - \alpha k, -h_-) d\mu \\ &\le \int_{\{f \ge \alpha k\}} (f - \alpha k) d\mu + \int_{\{f < \alpha k\}} h_- d\mu < \infty. \end{aligned}$$

Further we obtain

$$\begin{aligned} (f_\alpha - h)_+ &= (\sup(f - \alpha k, -h_-) - h)_+ \\ &= (\sup(f - (h + \alpha k), -h_- - h))_+ \\ &= (f - (h + \alpha k))_+ \end{aligned}$$

and

$$\sup(f, -h_-) - f_\alpha = \sup(f, -h_-) - \sup(f - \alpha k, -h_-) \le \alpha k.$$

Since T is order preserving and $k \in G(T)_+$, $T \sup(f, -h_-) \le T(f_\alpha + \alpha k) = Tf_\alpha + \alpha k$. This implies $Tf_\alpha + \alpha k \ge T \sup(f, -h_-) \ge Tf$. The assertion follows from

$$\begin{aligned} \int (Tf - (h' + \alpha k))_+ d\mu &\le \int (Tf_\alpha + \alpha k - (h' + \alpha k))_+ d\mu \\ &= \int (Tf_\alpha - h')_+ d\mu \\ &\le \int (f_\alpha - h)_+ d\mu + K \\ &= \int (f - (h + \alpha k))_+ d\mu + K < \infty. \end{aligned}$$

□

Now we can prove the nonlinear maximal ergodic theorem.

Theorem 3.4. *Let T be an order preserving mapping on L^1 such that $G(T)_+$ is nonempty. Let $K \in \mathbf{R}$ and let h, h' be two measurable functions with values $(-\infty, \infty]$ such that $1_{\{h < \infty\}}(h' - h)_+$ is integrable and*

$$\int (Tg - h')_+ d\mu \le \int (g - h)_+ d\mu + K < \infty \quad (g \in L^1). \tag{7}$$

Further, let $\{f_n\}$ be a sequence in L^1 and $f \in L^1$ such that

$$1_{\{f_n>0\}}f_n \leq 1_{\{f_n>0\}}(f + Tf_{n-1}) \quad (n \geq 1). \tag{8}$$

Then we have

$$\int_{\{f^*_\infty=\infty\}\setminus\{h=\infty\}}(f + h' - h)d\mu \geq -K. \tag{9}$$

Proof. We assume $k \in G(T)_+$. As observed in the last proof, we have $h_- \in L^1$. Thus for a given $\epsilon > 0$, there exists $\alpha_0 > 0$ such that $\int_{\{h_->\alpha_0 k\}} h_- d\mu + \int_{\{f_0>\alpha_0 k\}} f_0 \leq \epsilon$. Hence, setting $\tilde{h} = \sup(h, -\alpha_0 k) + (f_0 - \alpha_0 k)_+$, we have $\tilde{h} = \sup(h, -\alpha_0 k) + (f_0 - \alpha_0 k)_+ = (-h - \alpha_0 k)_+ + (f_0 - \alpha_0 k)_+ + h \geq h$ and

$$\begin{aligned}
\int |\tilde{h} - h| d\mu &= \int (-h - \alpha_0 k)_+ d\mu + \int (f_0 - \alpha_o k)_+ d\mu \\
&= \int_{\{-h-\alpha_0 k>0\}} (-h - \alpha_0 k)_+ d\mu + \int_{\{f_0-\alpha_0 k>0\}} (f_0 - \alpha_0 k)_+ d\mu \\
&\leq \int_{\{-h>\alpha_0 k\}} h_- d\mu + \int_{\{f_0>\alpha_0 k\}} f_0 d\mu \\
&\leq \int_{\{h_->\alpha_0 k\}} h_- d\mu + \int_{\{f_0>\alpha_0 k\}} f_0 d\mu \leq \epsilon.
\end{aligned}$$

Together with (7), this implies

$$\begin{aligned}
\int (Tg - h')_+ d\mu &\leq \int (g - h)_+ d\mu + K \\
&\leq \int (g - \tilde{h})_+ d\mu + \int (\tilde{h} - h)_+ d\mu + K \\
&\leq \int (g - \tilde{h})_+ d\mu + K + \epsilon < \infty \quad (g \in L^1).
\end{aligned}$$

Combining this with Lemma 3.3, we obtain

$$\begin{aligned}
&\int (Tg - (h' + \alpha k))_+ d\mu \\
&\qquad \leq \int (g - (\tilde{h} + \alpha k))_+ d\mu + K + \epsilon < \infty \quad (g \in L^1, k \in G(T)_+, \alpha > 0).
\end{aligned}$$

From the definition of $\tilde{h}$, $\tilde{h} - f_0 = \sup(h, -\alpha_0 k) + (f_0 - \alpha_0)_+ - f_0 \geq -\alpha_0 + (-\alpha_0 k) \vee (-f_0) \geq -2\alpha_0 k$. This implies

$$f_0 \leq \tilde{h} + \alpha k \quad (\alpha \geq 2\alpha_0).$$

Further we also have $\tilde{h} + 2\alpha_0 k = \sup(h, -\alpha_0 k) + (f_0 - \alpha_0 k)_+ + 2\alpha_0 k = \sup(h + 2\alpha_0, \alpha_0 k) + (f_0 - \alpha_0 k)_+ \geq 0$. So we obtain $\tilde{h} + \alpha k \geq 0$ $(\alpha \geq 2\alpha_0)$. Together with (8), this implies

$$1_{\{f_n>\tilde{h}+\alpha k\}} \leq 1_{\{f_n\geq\tilde{h}+\alpha k\}}(f + Tf_{n-1}) \quad (n \geq 1, \alpha \geq 2\alpha_0).$$

Thus the assumption of Theorem 3.1 are satisfied for any $\alpha \geq 2\alpha_0$, if we replace h, h' by $\tilde{h} + \alpha k, h' + \alpha k$ and K by $K + \epsilon$. Hence we obtain, from Theorem 3.1 and $h \leq \tilde{h}$, that $1_{\{f^*_\infty > \tilde{h} + \alpha k\}}(f + h' - h)$ is integrable and that

$$\begin{aligned}\int_{\{f^*_\infty > \tilde{h} + \alpha k\}} (f + h' - h) d\mu &\geq \int_{\{f^*_\infty > \tilde{h} + \alpha k\}} (f + (h' + \alpha k) - (\tilde{h} + \alpha k)) d\mu \\ &\geq -K - \epsilon \quad (\alpha \geq 2\alpha_0).\end{aligned}$$

Since $\{f^*_\infty > \tilde{h} + \alpha k\} \downarrow \{f^*_\infty = \infty\} \backslash \{h = \infty\}$ as α tends to ∞, we may let α tend to ∞ in the above inequalities to obtain

$$\int_{\{f^*_\infty = \infty\} \backslash \{h = \infty\}} (f + h' - h) d\mu \geq -K - \epsilon.$$

Since $\epsilon > 0$ is arbitrary, the assertion (9) follows. $\square$

Using the above theorem, we obtain the following lemma which is crucial for our main result.

Lemma 3.5. *Let T be an order preserving and L^1 nonexpansive mapping on L^1 such that $G(T)_+$ is nonempty. Then for any $f \in L^1$*

$$\mu(\{\limsup_{n \to \infty} A_n f > 0\} \cap \{\liminf_{n \to \infty} A_n f < 0\}) = 0.$$

Proof. We assume $k \in G(T)_+$. It suffices to show that $\mu(A_\epsilon \cap \tilde{A}_\epsilon) = 0$ for any $\epsilon > 0$, where

$$A_\epsilon = \{\limsup_{n \to \infty} A_n f > \epsilon k\} \quad \textit{and} \quad \tilde{A}_\epsilon = \{\liminf_{n \to \infty} A_n f < -\epsilon k\}.$$

We fix $\epsilon > 0$. We define a mapping $\tilde{T}$ on L^1 by setting $\tilde{T} f = -T(-f)$ $(f \in L^1)$. Let $\tilde{S}_n$ be defined as S_n but with $\tilde{T}$ instead of T. Putting $f_n = (S_n f - (n+1)\epsilon k)_+$ and $\tilde{f}_n = (\tilde{S}_n(-f) - (n+1)\epsilon k)_+ = (S_n f + (n+1)\epsilon k)_-$, we have

$$A_\epsilon \subset A := \{f^*_\infty = \infty\} \quad \textit{and} \quad \tilde{A}_\epsilon \subset \tilde{A} := \{\tilde{f}^*_\infty = \infty\}.$$

In fact, if $x \notin A$, there exists $M \in \mathbf{R}$ such that $f_n(x) \leq M$ $(0 \leq n < \infty)$. Then we have $S_n f(x) \leq (n+1)\epsilon k(x) + M$ and $A_n(x) \leq \epsilon k(x) + \frac{M}{n+1}$. This implies $\limsup_{n \to \infty} A_n f(x) \leq \epsilon k(x)$, and hence $x \notin A_\epsilon$. So we obtain $A_\epsilon \subset A$. Similarly we can show $\tilde{A}_\epsilon \subset \tilde{A}$. Thus the proof is complete if we can show that $\mu(A \cap \tilde{A}) = 0$.

Next we will show

$$1_{\{f_n > 0\}} \leq 1_{\{f_n > 0\}}((f - \epsilon k) + T f_{n-1}) \quad (n \geq 1)$$

and

$$1_{\{\tilde{f}_n > 0\}} \leq 1_{\{\tilde{f}_n > 0\}}((-f - \epsilon k) + \tilde{T} \tilde{f}_{n-1}) \quad (n \geq 1).$$

Since T is order preserving, $k \in G(T)_+$ and $(S_{n-1} f)_+ - f_{n-1} = (S_{n-1} f)_+ - (S_{n-1} f - n\epsilon k)_+ \leq n\epsilon k$, we have $T((S_{n-1} f)_+) \leq T f_{n-1} + n\epsilon k$. This implies $T f_{n-1} + n\epsilon k \geq$

$T((S_{n-1}f)_+) \geq T(S_{n-1}f)$ and therefore $(f-\epsilon k)+Tf_{n-1} \geq f+T(S_{n-1}f)-(n+1)\epsilon k = S_nf-(n+1)\epsilon k$. Hence we obtain

$$\begin{aligned} 1_{\{f_n>0\}}((f-\epsilon k)+Tf_{n-1}) &\geq 1_{\{f_n>0\}}(S_nf-(n+1)\epsilon k) \\ &= 1_{\{f_n>0\}}(S_nf-(n+1)\epsilon k)_+ \\ &= 1_{\{f_n>0\}}f_n \quad (n \geq 1). \end{aligned}$$

Similarly we can obtain $1_{\{\tilde{f}_n>0\}} \leq 1_{\{\tilde{f}_n>0\}}((-f-\epsilon k)+\tilde{T}\tilde{f}_{n-1})$.

Putting

$$h_{\alpha,\beta} = -\alpha k 1_{\tilde{A}\setminus A} + \beta k 1_{A\setminus\tilde{A}} \quad \text{and} \quad \tilde{h}_{\alpha,\beta} = -\alpha k 1_{A\setminus\tilde{A}} + \beta k 1_{\tilde{A}\setminus A}$$
$$(0 \leq \alpha < \infty, 0 \leq \beta < \infty),$$

we define

$$h_\alpha = \lim_{\beta\to\infty} h_{\alpha,\beta} \quad \text{and} \quad \tilde{h}_\alpha = \lim_{\beta\to\infty} \tilde{h}_{\alpha,\beta}.$$

Further, since $h_{\alpha,\beta}, \tilde{h}_{\alpha,\beta}$ belong to L^1, we can define

$$h'_\alpha = \sup_{n\in\mathbf{N}} Th_{\alpha,n} \quad \text{and} \quad \tilde{h}'_\alpha = \sup_{n\in\mathbf{N}} \tilde{T}\tilde{h}_{\alpha,n} \quad (\alpha > 0).$$

Since $f \leq h_\alpha\,(f \in L^1) \Rightarrow Tf \leq Th_\alpha = \sup_{n\in\mathbf{N}} Th_{\alpha,n} = h'_\alpha$, we get from Lemma 3.2 that

$$\int (Tg-h'_\alpha)_+d\mu \leq \int (g-h_\alpha)_+d\mu \quad (g \in L^1, \alpha \geq 0). \tag{10}$$

Similarly we can get

$$\int (\tilde{T}g-\tilde{h}'_\alpha)_+d\mu \leq \int (g-\tilde{h}_\alpha)_+d\mu \quad (g \in L^1, \alpha \geq 0). \tag{11}$$

To see the principal idea, we postpone the proof of the following statements until later,

$$1_{\{h_\alpha<\infty\}}h'_\alpha \text{ is integrable for any } \alpha \geq 0, \tag{12}$$

$$1_{\{\tilde{h}_\alpha<\infty\}}\tilde{h}'_\alpha \text{ is integrable for any } \alpha \geq 0, \tag{13}$$

$$-\infty < \inf_{\alpha>0} \int_{A\cap\tilde{A}} h'_\alpha d\mu \leq \sup_{\alpha>0} \int_{A\cap\tilde{A}} h'_\alpha d\mu < \infty, \tag{14}$$

and

$$\lim_{\alpha\to\infty}\lim_{\beta\to\infty} \int_{A\cap\tilde{A}} Th_{\alpha,\beta}d\mu = \lim_{\beta\to\infty}\lim_{\alpha\to\infty} \int_{A\cap\tilde{A}} Th_{\alpha,\beta}d\mu. \tag{15}$$

Because of (10) and (12), we can apply Theorem 3.4 to $\{f_n\}$ with $K=0$ and h_α, h'_α instead of h, h'. Then we obtain

$$
\begin{aligned}
0 &\leq \int_{\{f^*_\infty=\infty\}\setminus\{h_\alpha=\infty\}}((f-\epsilon k)+h'_\alpha-h_\alpha)d\mu \\
&= \int_{A\cap\tilde{A}}((f-\epsilon k)+h'_\alpha)d\mu
\end{aligned} \tag{16}
$$

since $h_\alpha=0$ on $A\cap\tilde{A}$. Analogously, we can show that

$$
0 \leq \int_{A\cap\tilde{A}}((-f-\epsilon k)+\tilde{h}'_\alpha)d\mu. \tag{17}
$$

Since $Th_{\alpha,\beta}=T(\beta k 1_{\tilde{A}\setminus A}-\alpha k 1_{A\setminus\tilde{A}})=T(-\tilde{h}_{\alpha,\beta})=-\tilde{T}\tilde{h}_{\alpha,\beta}$, (15) implies

$$
\begin{aligned}
\lim_{\alpha\to\infty}\int_{A\cap\tilde{A}}h'_\alpha d\mu &= \lim_{\alpha\to\infty}\lim_{\beta\to\infty}\int_{A\cap\tilde{A}}Th_{\alpha,\beta}d\mu \\
&= \lim_{\beta\to\infty}\lim_{\alpha\to\infty}\int_{A\cap\tilde{A}}Th_{\alpha,\beta}d\mu \\
&= \lim_{\beta\to\infty}\lim_{\alpha\to\infty}\int_{A\cap\tilde{A}}-\tilde{T}\tilde{h}_{\beta,\alpha}d\mu \\
&= -\lim_{\beta\to\infty}\int_{A\cap\tilde{A}}\tilde{h}'_\beta d\mu.
\end{aligned} \tag{18}
$$

Adding (16) and (17), we obtain

$$
\begin{aligned}
0 &\leq \int_{A\cap\tilde{A}}((f-\epsilon k)+h'_\alpha)d\mu+\int_{A\cap\tilde{A}}((-f-\epsilon k)+\tilde{h}'_\alpha)d\mu \\
&= \int_{A\cap\tilde{A}}(-2\epsilon k+h'_\alpha+\tilde{h}'_\alpha)d\mu.
\end{aligned}
$$

Letting α tend to ∞ and using (14) and (18), we obtain

$$
0 \leq \int_{A\cap\tilde{A}}(-2\epsilon k)d\mu
$$

and therefore $\mu(A\cap\tilde{A})=0$. Thus the proof is completed if we can show (12)– (15).

At first, we will prove (15) . Because of Lemma 3.2, we can apply Theorem 3.4 to $\{f_n\}$ with $h_{\alpha,0}, Th_{\alpha,0}$ instead of h, h' and $K=0$. Then we obtain

$$
\begin{aligned}
0 &\leq \int_{\{f^*_\infty=\infty\}}(f-\epsilon k+Th_{\alpha,0}-h_{\alpha,0})d\mu \\
&= \int_A(f-\epsilon k+Th_{\alpha,0})d\mu,
\end{aligned}
$$

because $h_{\alpha,0}=0$ on A. It follows that

$$
-\infty < \int_A(\epsilon k-f)d\mu \leq \int_A Th_{\alpha,0}d\mu \quad (\forall\alpha\geq 0). \tag{19}
$$

Since $h_{\alpha,0} \leq 0$ and T is order preserving, we have $Th_{\alpha,0} - T0 \leq 0$. Together with $Th_{\alpha,0} \leq h'_\alpha$, this implies $\int_A (Th_{\alpha,0} - T0)d\mu \leq \int_{A\cap\tilde{A}} (Th_{\alpha,0} - T0)d\mu \leq \int_{A\cap\tilde{A}} (h'_\alpha - T0)d\mu$. Therefore $\int_A Th_{\alpha,0} d\mu \leq \int_{A\cap\tilde{A}} h'_\alpha d\mu + \int_A T0 d\mu - \int_{A\cap\tilde{A}} T0 d\mu \leq \int_{A\cap\tilde{A}} h'_\alpha d\mu + \|T0\|_1$. Because of (19), the "$-\infty$ half" of (14) follows from

$$-\infty < \int_A (\epsilon k - f)d\mu \leq \int_A Th_{\alpha,0} d\mu \leq \int_{A\cap\tilde{A}} h'_\alpha d\mu + \|T0\|_1 \quad (\forall \alpha \geq 0).$$

Replacing T by $\tilde{T}$ and $h_{\beta,0}, Th_{\beta,0}$ by $\tilde{h}_{\beta,0}, \tilde{T}\tilde{h}_{\beta,0}$, we obtain analogously

$$-\infty < \int_{\tilde{A}} (\epsilon k + f)d\mu \leq \int_{\tilde{A}} \tilde{T}\tilde{h}_{\beta,0} d\mu \quad (\forall \beta \geq 0).$$

Since $\tilde{T}\tilde{h}_{\beta,0} = -Th_{0,\beta}$ and $Th_{0,\beta} - T0 \geq 0$, this implies

$$\begin{aligned} \int_{A\cap\tilde{A}} Th_{0,\beta} d\mu - \|T0\|_1 &\leq \int_{\tilde{A}} (-\tilde{T}\tilde{h}_{\beta,0})d\mu \\ &\leq \int_{\tilde{A}} (-f - \epsilon k)d\mu < \infty \quad (\forall \beta \geq 0), \end{aligned} \tag{20}$$

and therefore the "$+\infty$ half" of (14) follows from

$$\sup_{\alpha\geq 0} \int_{A\cap\tilde{A}} h'_\alpha d\mu = \int_{A\cap\tilde{A}} h'_0 d\mu = \sup_{\beta\geq 0} \int_{A\cap\tilde{A}} Th_{0,\beta} d\mu < \infty.$$

Our next aim is to show that

$$C_1 = \sup_{\alpha,\beta\geq 0} \int_{A\setminus\tilde{A}} (h_{\alpha,\beta} - Th_{\alpha,\beta})d\mu < \infty. \tag{21}$$

To prove this, we put $g_{\alpha,\beta} = -\alpha k 1_{\tilde{A}\setminus A} + \beta k 1_A$ and apply Theorem 3.4 to $\{f_n\}$ with $K = 0$ and $g_{\alpha,\beta}$, $Tg_{\alpha,\beta}$ instead of h, h'. Then we obtain

$$\int_{\{f^*_\infty = \infty\}} ((f - \epsilon k) + Tg_{\alpha,\beta} - g_{\alpha,\beta}) \geq 0$$

and therefore

$$\sup_{\alpha,\beta\geq 0} \int_A (g_{\alpha,\beta} - Tg_{\alpha,\beta})d\mu \leq \int_A (f - \epsilon k)d\mu < \infty. \tag{22}$$

Since $g_{\alpha,\beta} - g_{\alpha,0} = \beta k 1_A \leq \beta k$, $g_{\alpha,0} \leq 0$ and $k \in G(T)_+$, we have $Tg_{\alpha,\beta} \leq Tg_{\alpha,0} + \beta k \leq T0 + \beta k$. Therefore $g_{\alpha,\beta} - Tg_{\alpha,\beta} = \beta k - Tg_{\alpha,\beta} \geq -T0$ on A. It follows that $\int_A (g_{\alpha,\beta} - Tg_{\alpha,\beta})_- d\mu \leq \int_A |T0| d\mu \leq \|T0\|_1$. Together with (22), this implies

$$\begin{aligned} C_2 = \sup_{\alpha,\beta\geq 0} &\int_A (g_{\alpha,\beta} - Tg_{\alpha,\beta})_+ d\mu \\ &\leq \int_A (f - \epsilon k)d\mu + \|T0\|_1 < \infty. \end{aligned} \tag{23}$$

From (20) and $Th_{\alpha,\beta} \leq Th_{0,\beta}$, we get

$$C_3 = \sup_{\alpha,\beta\geq 0} \int_{A\cap\tilde{A}} Th_{\alpha,\beta} d\mu < \infty. \tag{24}$$

Since T is order preserving, $h_{\alpha,\beta} \leq g_{\alpha,\beta}$ implies $Th_{\alpha,\beta} \leq Tg_{\alpha,\beta}$. Since $\|g_{\alpha,\beta} - h_{\alpha,\beta}\| = \beta \int_{A\cap\tilde{A}} k d\mu$ and T is L^1 nonexpansive, we have

$$\int (Tg_{\alpha,\beta} - Th_{\alpha,\beta}) d\mu = \|Tg_{\alpha,\beta} - Th_{\alpha,\beta}\|_1 \leq \beta \int_{A\cap\tilde{A}} k d\mu. \tag{25}$$

On the other hand, by (23) and (24),

$$\begin{aligned}
&\int_{A\cap\tilde{A}} (Tg_{\alpha,\beta} - Th_{\alpha,\beta}) d\mu \\
&\qquad = \int_{A\cap\tilde{A}} g_{\alpha,\beta} d\mu - \int_{A\cap\tilde{A}} Th_{\alpha,\beta} d\mu - \int_{A\cap\tilde{A}} (g_{\alpha,\beta} - Tg_{\alpha,\beta}) d\mu \\
&\qquad \geq \beta \int_{A\cap\tilde{A}} k d\mu - C_3 - C_2.
\end{aligned}$$

Together with (25), this implies

$$\begin{aligned}
\int_{A\setminus\tilde{A}} (Tg_{\alpha,\beta} - Th_{\alpha,\beta}) d\mu &\leq \int_{E\setminus(A\cap\tilde{A})} (Tg_{\alpha,\beta} - Th_{\alpha,\beta}) d\mu \\
&= \int_E (Tg_{\alpha,\beta} - Th_{\alpha,\beta}) d\mu - \int_{A\cap\tilde{A}} (Tg_{\alpha,\beta} - Th_{\alpha,\beta}) d\mu \\
&\leq \beta \int_{A\cap\tilde{A}} k d\mu - \beta \int_{A\cap\tilde{A}} k d\mu + C_2 + C_3 \\
&= C_2 + C_3.
\end{aligned} \tag{26}$$

Since $h_{\alpha,\beta} = g_{\alpha,\beta} = \beta k$ on $A\setminus\tilde{A}$, we obtain from (23) and (26) that

$$\begin{aligned}
\int_{A\setminus\tilde{A}} (h_{\alpha,\beta} - Th_{\alpha,\beta}) d\mu &= \int_{A\setminus\tilde{A}} (g_{\alpha,\beta} - Th_{\alpha,\beta}) d\mu \\
&\leq \int_{A\setminus\tilde{A}} (g_{\alpha,\beta} - Tg_{\alpha,\beta}) d\mu + \int_{A\setminus\tilde{A}} (Tg_{\alpha,\beta} - Th_{\alpha,\beta}) d\mu \\
&\leq C_2 + (C_2 + C_3) \quad (\forall \alpha, \beta > 0).
\end{aligned}$$

So we obtain (21).

Our next aim is to show

$$\alpha_1 \leq \alpha_2, \beta_1 \leq \beta_2 \Rightarrow 1_{A\setminus\tilde{A}}(h_{\alpha_1,\beta_1} - Th_{\alpha_1,\beta_2}) \leq 1_{A\setminus\tilde{A}}(h_{\alpha_2,\beta_2} - Th_{\alpha_2,\beta_2}). \tag{27}$$

If $\alpha_1 \leq \alpha_2$, then $h_{\alpha_1,\beta} \geq h_{\alpha_2,\beta}$. Therefore we have $Th_{\alpha_1,\beta} \geq Th_{\alpha_2,\beta}$ $(\alpha_1 \leq \alpha_2)$, since T is order preserving. On the other hand, $h_{\alpha_1,\beta} = h_{\alpha_2,\beta}$ on A, and hence

$$\alpha_1 \leq \alpha_2 \Rightarrow 1_{A\setminus\tilde{A}}(h_{\alpha_1,\beta} - Th_{\alpha_1,\beta}) \leq 1_{A\setminus\tilde{A}}(h_{\alpha_2,\beta} - Th_{\alpha_2,\beta}). \tag{28}$$

If $\beta_1 \leq \beta_2$, then $h_{\alpha,\beta_2} - h_{\alpha,\beta_1} = (\beta_2 - \beta_1)k1_{A\backslash\tilde{A}} \leq (\beta_2 - \beta_1)k$. Therefore we obtain $Th_{\alpha,\beta_2} \leq Th_{\alpha,\beta_1} + (\beta_2 - \beta_1)k$ $(\beta_1 \leq \beta_2)$ since $k \in G(T)_+$. On the other hand, $h_{\alpha,\beta_2} - h_{\alpha,\beta_1} = (\beta_2 - \beta_1)k$ on $A\backslash\tilde{A}$ and therefore

$$\beta_1 \leq \beta_2 \Rightarrow 1_{A\backslash\tilde{A}}(h_{\alpha,\beta_1} - Th_{\alpha,\beta_1}) \leq 1_{A\backslash\tilde{A}}(h_{\alpha,\beta_2} - Th_{\alpha,\beta_2}). \tag{29}$$

Now (27) follows from (28) and (29).

Analogously to (21) and (27), we can show

$$\sup_{\alpha,\beta\geq 0} \int_{\tilde{A}\backslash A} (\tilde{h}_{\alpha,\beta} - \tilde{T}\tilde{h}_{\alpha,\beta})d\mu < \infty$$

and

$$\alpha_1 \leq \alpha_2, \beta_1 \leq \beta_2 \Rightarrow 1_{\tilde{A}\backslash A}(\tilde{h}_{\alpha_1,\beta_1} - \tilde{T}\tilde{h}_{\alpha_1,\beta_1}) \leq 1_{\tilde{A}\backslash A}(\tilde{h}_{\alpha_2,\beta_2} - \tilde{T}\tilde{h}_{\alpha_2,\beta_2}).$$

Since $\tilde{h}_{\alpha,\beta} = -h_{\beta,\alpha}$ and $\tilde{T}\tilde{h}_{\alpha,\beta} = -Th_{\beta,\alpha}$, we obtain

$$C_4 = \sup_{\alpha,\beta\geq 0} \int_{\tilde{A}\backslash A} (Th_{\alpha,\beta} - h_{\alpha,\beta})d\mu < \infty \tag{30}$$

and

$$\alpha_1 \leq \alpha_2, \beta_1 \leq \beta_2 \Rightarrow 1_{\tilde{A}\backslash A}(h_{\alpha_1,\beta_1} - Th_{\alpha_1,\beta_1}) \leq 1_{\tilde{A}\backslash A}(h_{\alpha_2,\beta_2} - Th_{\alpha_2,\beta_2}). \tag{31}$$

Let $\delta > 0$ be given. By (21) and (27) (resp. (30), and (31)) there exists $\alpha_\delta \geq 0$ such that

$$\int_{A\backslash\tilde{A}} (h_{\alpha,\beta} - Th_{\alpha,\beta})d\mu \geq C_1 - \delta \quad (\alpha, \beta \geq \alpha_\delta) \tag{32}$$

and

$$\int_{\tilde{A}\backslash A} (Th_{\alpha,\beta} - h_{\alpha,\beta})d\mu \geq C_4 - \delta \quad (\alpha, \beta \geq \alpha_\delta). \tag{33}$$

Let $\alpha_1, \alpha_2, \beta_1, \beta_2 \geq \alpha_\delta$ be given. Then we have $\|h_{\alpha_1,\beta_1} - h_{\alpha_2,\beta_2}\| \leq |\alpha_1 - \alpha_2| \int_{\tilde{A}\backslash A} kd\mu + |\beta_1 - \beta_2| \int_{A\backslash\tilde{A}} kd\mu$ and therefore

$$\int |Th_{\alpha_1,\beta_1} - Th_{\alpha_2,\beta_2}|d\mu \leq |\alpha_1 - \alpha_2| \int_{\tilde{A}\backslash A} kd\mu + |\beta_1 - \beta_2| \int_{A\backslash\tilde{A}} kd\mu. \tag{34}$$

Using (32), we obtain

$$\begin{aligned}
&\int_{A\setminus\tilde{A}}(Th_{\alpha_1,\beta_1} - Th_{\alpha_2,\beta_2})d\mu \\
&\qquad = \int_{A\setminus\tilde{A}}(h_{\alpha_1,\beta_1} - h_{\alpha_2,\beta_2})d\mu - \int_{A\setminus\tilde{A}}(h_{\alpha_1,\beta_1} - Th_{\alpha_1,\beta_1})d\mu \\
&\qquad\qquad + \int_{A\setminus\tilde{A}}(h_{\alpha_2,\beta_2} - Th_{\alpha_2,\beta_2})d\mu \\
&\qquad \geq (\beta_2 - \beta_1)\int_{A\setminus\tilde{A}} k d\mu - C_1 + (C_1 - \delta) \\
&\qquad \geq (\beta_2 - \beta_1)\int_{A\setminus\tilde{A}} k d\mu - \delta. \qquad (35)
\end{aligned}$$

Interchanging α_1, β_1 with α_2, β_2, we also get

$$\int_{A\setminus\tilde{A}}(Th_{\alpha_2,\beta_2} - Th_{\alpha_1,\beta_1})d\mu \geq (\beta_1 - \beta_2)\int_{A\setminus\tilde{A}} k d\mu - \delta. \qquad (36)$$

From (35) and (36), we obtain

$$\int_{A\setminus\tilde{A}} |Th_{\alpha_1,\beta_1} - Th_{\alpha_2,\beta_2}|d\mu \geq |\beta_1 - \beta_2|\int_{A\setminus\tilde{A}} k d\mu - \delta. \qquad (37)$$

Analogously, using (33) instead of (32), we can show that

$$\int_{\tilde{A}\setminus A} |Th_{\alpha_1,\beta_1} - Th_{\alpha_2,\beta_2}|d\mu \geq |\alpha_1 - \alpha_2|\int_{\tilde{A}\setminus A} k d\mu - \delta. \qquad (38)$$

Combining (37) and (38) with (34), we obtain

$$\begin{aligned}
&\int |Th_{\alpha_1,\beta_1} - Th_{\alpha_2,\beta_2}|d\mu \\
&\qquad \leq 2\delta + \int_{A\setminus\tilde{A}} |Th_{\alpha_1,\beta_1} - Th_{\alpha_2,\beta_2}|d\mu + \int_{\tilde{A}\setminus A} |Th_{\alpha_1,\beta_1} - Th_{\alpha_2,\beta_2}|d\mu
\end{aligned}$$

and therefore

$$\int_{A\cap\tilde{A}} |Th_{\alpha_1,\beta_1} - Th_{\alpha_2,\beta_2}|d\mu \leq 2\delta \quad (\alpha_2, \alpha_2, \beta_1, \beta_2 \geq \alpha_\delta).$$

Since $\delta > 0$ is arbitrarily, we have (15).

Using (21), we obtain

$$\begin{aligned}\int_{E\setminus(A\setminus\tilde{A})}|Th_{0,\beta}|d\mu &\le \|Th_{0,\beta}\|_1 - \int_{A\setminus\tilde{A}} Th_{0,\beta}d\mu \\ &\le \|T0\|_1 + \|Th_{0,\beta} - T0\|_1 - \int_{A\setminus\tilde{A}} Th_{0,\beta}d\mu \\ &\le \|T0\|_1 + \|h_{0,\beta}\|_1 - \int_{A\setminus\tilde{A}} Th_{0,\beta}d\mu \\ &= \|T0\|_1 + \int_{A\setminus\tilde{A}}(h_{0,\beta} - Th_{0,\beta})d\mu \\ &\le \|T0\|_1 + C_1\end{aligned}$$

Therefore we obtain $Th_{0,\beta} \le h'_\alpha \le \sup_{\beta\ge 0} Th_{0,\beta}$,

$$\begin{aligned}\int_{\{h_\alpha<\infty\}}|h'_\alpha|d\mu &\le \|Th_{0,\beta}\|_1 + \sup_{\beta\ge 0}\int_{E\setminus(A\setminus\tilde{A})}\|Th_{0,\beta}|d\mu \\ &\le \|Th_{0,\beta}\|_1 + \|T0\|_1 + C_1.\end{aligned}$$

Thus we have (12). The proof of (13) is analogous. □

The following theorem is our main result.

Theorem 3.6. *Let T be an order preserving and L^1 nonexpansive mapping on L^1 such that $G(T)_+$ is nonempty. Then $A_n f$ converges a.e. to an element f^* of L^1 for any $f \in L^1$. Further if $F(T)$ is nonempty, then we have $f^* \in L^1$. In paticular if $0 \in F(T)$, then we also have $\lim_{n\to\infty}\|A_n f - f^*\|_1 = 0$. And further if T is positively homogeneous, then $f^* \in F(T)$.*

Proof. Let $f \in L^1$ and $k \in G(T)_+$ be given. We assume

$$\mu(\{\liminf_{n\to\infty} A_n f < \limsup_{n\to\infty} A_n f\}) > 0.$$

Since $k > 0$, there exists $\alpha \in \mathbf{R}$ such that

$$\mu(\{\liminf_{n\to\infty} A_n f < \alpha k\} \cap \{\limsup_{n\to\infty} A_n f > \alpha k\}) > 0.$$

Since $k \in G(T)_+$, we have $A_n(f - \alpha k) = A_n f - \alpha k$ and therefore

$$\mu(\{\liminf_{n\to\infty} A_n(f - \alpha k) < 0\} \cap \{\limsup_{n\to\infty} A_n(f - \alpha k) > 0\}) > 0.$$

But this contradicts Lemma 3.5. So $A_n f$ converges a.e..

We assume $l \in F(T)$. And we set $(f_+)^* = \lim_{n\to\infty} A_n(f_+)$, $(f_-)^* = \lim_{n\to\infty} A_n(f_-)$. Since A_n is order preserving, we have $(-f_-)^* \le f^* \le (f_+)^*$. Using Fatou's lemma, since A_n is L^1 nonexpansive and $A_n l = l$, we obtain

$$\int (f^*)_+ d\mu \le \int (f_+)^* d\mu$$

$$
\begin{aligned}
&\leq \int \liminf_{n\to\infty} A_n(f_+)d\mu \\
&\leq \liminf_{n\to\infty} \int A_n(f_+)d\mu \\
&\leq \liminf_{n\to\infty} \|A_n(f_+) - l\|_1 + \|l\|_1 \\
&\leq \|f_+ - l\|_1 + \|l\|_1 < \infty
\end{aligned}
$$

and

$$
\begin{aligned}
\int (f^*)_- d\mu &\leq -\int (-f_-)d\mu \\
&\leq -\int \limsup_{n\to\infty} A_n(-f_-)d\mu \\
&\leq -\limsup_{n\to\infty} \int A_n(-f_-)d\mu \\
&= \liminf_{n\to\infty} \int (-A_n(-f_-))d\mu \\
&= \liminf_{n\to\infty} \|A_n(-f_-)\|_1 \\
&\leq \liminf_{n\to\infty} \|A_n(-f_-) - l\|_1 + \|l\|_1 \\
&\leq \|f_- + l\|_1 + \|l\|_1 < \infty.
\end{aligned}
$$

Therefore f^* belongs to L^1. Next we will show that if $0 \in F(T)$, then

$$\lim_{n\to\infty} \|A_n f - f^*\|_1 = 0.$$

We set $f_m = (f \wedge mk) \vee (-mk)$ $(m \in \mathbf{N})$ and $\lim_{n\to\infty} A_n f_m = f_m^*$. Then we obtain $\lim_{n\to\infty} \|A_n f_m - f_m^*\|_1 = 0$ by Lebesgue's convergence theorem, because $A_n 0 = 0$ and $k \in G(T)_+$ implie $|A_n f_m| \leq mk$ $(n \in \mathbf{N})$. Further we have

$$
\begin{aligned}
\|A_n f - f^*\|_1 &\leq \|A_n f - A_n f_m\|_1 + \|A_n f_m - f_m^*\|_1 + \|f_m^* - f^*\|_1 \\
&\leq 2\|f - f_m\|_1 + \|A_n f_m - f_m^*\|_1.
\end{aligned}
$$

Letting n tend to ∞, we obtain

$$\lim_{n\to\infty} \|A_n f - f^*\|_1 \leq 2\|f - f_m\|_1.$$

Therefore, since $\lim_{m\to\infty} \|f - f_m\|_1 = 0$, we get

$$\lim_{m\to\infty} \|A_n f - f^*\|_1 = 0.$$

Moreover if T is positively homogeneous, we have

$$T(A_n f) = \frac{n+2}{n+1} A_{n+1} f - \frac{1}{n+1} f.$$

Hence we obtain $Tf^* = f^*$ by letting n tend to ∞. $\square$

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