数理解析研究所講究録
1071 巻 1998 年 82-87

Generalized supremum in ordered linear space and facial structure of a convex set

北海道教育大学旭川校　小室　直人　(Naoto Komuro)

§1 Definitions and basic results

Let E be a linear space over $\mathbb{R}$, and P be a convex cone in E satisfying

(P1) $E=P-P$,

(P2) $P\cap(-P)=\{0\}$.

An order relation in E can be defined by $x\leq y \Longleftrightarrow y-x\in P$. It can easily be seen that

(1) $x\leq y$ and $y\leq x\Longrightarrow x=y$,

(2) $x\leq y$ and $y\leq z\Longrightarrow x\leq z$,

(3) $x\leq y\Longrightarrow x+z\leq y+z$ for all $z\in E$,

(4) $0\leq x$ and $0\leq\lambda\in\mathbb{R}\Longrightarrow 0\leq\lambda x$,

(5) For every $x\in E$, there exists $x_1, x_2\in E$ such that $x=x_1-x_2$, and $0\leq x_1, x_2$.

Conversely, if an order in E satisfies (1) $\sim$ (5), then $P=\{x\in E\mid 0\leq x\}$ is a convex cone satisfying (P1) and (P2). A linear space E equipped with such a positive cone P is called a partially ordered linear space, and is sometimes denoted by (E, P).

Definition. *For a subset A of E, the generalized supremum* $\operatorname{Sup} A$ *is defined to be the set of all minimal elements of $U(A)$, where $U(A)$ is the set of all upper bound of A.*

We say in other words that $a\in\operatorname{Sup}A$ if and only if $a\leq b$ whenever $b\in U(A)$ and a, b are comparable. The generalized infimum $\operatorname{Inf} A$ can be defined similarly. In order to distinguish this notion from the least upper bound and the greatest lower bound, we denote the latter ones by $\sup A$ and $\inf A$ respectively. If E is order complete, then $\operatorname{Sup}A=\{\sup A\}$ holds whenever the subset A is upper bounded (i.e.,$U(A)\neq\emptyset$). When $E=\mathbb{R}^n$ and P is closed and not a lattice cone, $\operatorname{Sup}A$ becomes a infinite set in most cases. However, it is possibly empty, even when A is upper bounded.

Proposition 1. *For $a\in E$ and $\lambda>0$, we have*

(1) $\operatorname{Sup}(A+a)=\operatorname{Sup}A+a$,

(2) $\operatorname{Sup}\lambda A=\lambda\operatorname{Sup}A$,

(3) $\operatorname{Sup}A=-\operatorname{Inf}(-A)$.

Proposition 2. *For an arbitrary set $A\subset E$ with $U(A)\neq\emptyset$,*

$$\operatorname{Sup}A=\operatorname{Sup}(coA)$$

holds where coA is the convex hull of A.

proof. It suffices to show that $U(A)=U(coA)$. Take $x_0\in U(A)$ arbitrarily. For $x\in coA$ there exist some points $x_1, x_2, \cdots, x_n$ in A such that $x=\sum_{i=1}^n\lambda_i x_i$ with $0\leq\lambda_i\leq 1$ and $\sum_{i=1}^n\lambda_i=1$. Hence $x_0-x=\sum_{i=1}^n\lambda_i(x_0-x_i)\geq 0$ and we have $x_0\in U(coA)$.

When A is a finite set of the form $\{a_1, \cdots a_n\}$, we denote the set of the upper bound of A by $U(a_1, \cdots, a_n)$ instead of $U(A)$. With this notation, we define $a\vee b$ $(a, b\in E)$ to be the set of all minimal elements of $U(a, b)$. Also $a\wedge b$ can be defined similarly. When (E, P) is a lattice, $a\vee b$ is always a single element which is the minimum of $U(a, b)$.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Proposition 3. *For every $a, b, c \in E$ and $\lambda \in \mathbb{R}$,*

(1) $(a+c) \vee (b+c) = (a \vee b) + c$,

(2) $\lambda a \vee \lambda b = \lambda(a \vee b)$.

Theorem 1. *For $a, b \in E$, $a \vee b \neq \emptyset$ implies $a \wedge b \neq \emptyset$ and the converse is also true. Moreover,*

$$a + b - (a \vee b) = a \wedge b$$

holds and in particular we have $a \in a_+ + a_-$ where $a_+ = a \vee 0$ and $a_- = a \wedge 0$.

The proof of Theorem 1 can be seen in [6]. Also, some examples in which $a \vee b$ can be empty are shown.

A partially ordered linear space (E, P) is said to be monotone order complete (m.o.c. for short) if every upper bounded totally ordered subset of E has the least upper bound in E. The followings are known.

Proposition 4. *In the case $E = \mathbb{R}^d$, (E, P) is m.o.c. if and only if P is closed.*

Proposition 5. *Suppose that E is a Banach space and P is closed. Let E^* be the topological dual of E and let $P^* = \{x^* \in E^* \mid x^*(x) \geq 0,\ x \in P\}$. If $P^* - P^* = E^*$, then (E^*, P^*) is m.o.c.*

The proof can be done by using Banach Steinhaus theorem, and in [2], one can see some conditions under which $P^* - P^* = E^*$ holds.

A linear topology of (E, P) is called an order continuous topology if every decreasing net $\{a_\lambda\} \subset E$ with $\inf a_\lambda = 0$ converges to 0 by the topology. We consider some further conditions for P;

(P3) P is closed with respect to an order continuous topology,

(P4) For every decreasing net $\{a_\lambda\}$ in P, $\inf a_\lambda = a$ implies $a \in P$.

Note that (P3) imlpies (P4).

Theorem 2. *Suppose that a partially ordered linear space (E, P) is monotone order complete and P satisfies (P3) or (P4). Then for every subset A of E,*

$$U(A) = (\operatorname{Sup} A) + P$$

holds. In particular, $a \vee b \neq \emptyset, a \wedge b \neq \emptyset$ for every $a, b \in E$, and $U(a, b) = (a \vee b) + P$.

proof. It suffices to show that $U(A) \subset (\operatorname{Sup} A) + P$. For an arbitrary $x \in U(A)$, the section $U(A)_x = \{\ y \in U(A) \mid y \leq x\ \}$ is a nonempty convex set in E. If $T \subset U(A)_x$ is a totally ordered subset, then by monotone order completeness, there exists a greatest lower bound z_0 of T. Since $T \subset U(A) = \cap_{y \in A}(y+P)$, (P4) yields $z_0 \in U(A)_x$. Hence by Zorn's lemma, $U(A)_x$ has at least a minimal element y_0 . It is easy to see that y_0 is also a minimal element of $U(A)$, and it means that $x \in (\operatorname{Sup} A) + P$. The second statement of the theorem is obvious. Indeed, $U(a, b)$ is always nonempty because $P - P = E$. Hence it is sufficient to use the first statement. Q.E.D.

Corollary 1. *Suppose that (E, P) satisfies the hypotheses in Theorem 2 and let A be a subset of E. If $\operatorname{Sup} A$ consists of a single element a, then a is the least upper bound of A.*

Corollary 2. *For every subset A of E, $U(L(U(A))) = U(A)$ holds where $L(U(A))$ denotes the lower bound of $U(A)$. Moreover, if (E, P) satisfies the hypotheses in Theorem 2, then we have* $\mathrm{Sup}\,\mathrm{Inf}\,\mathrm{Sup}\,A = \mathrm{Sup}\,A$.

Next we give another sufficient condition for the same results by considering the faces of P. Moreover, we will give an example which shows that each of the two conditions does not imply the other.

§2 Faces of the positive cone

Let (E, P) be a partially ordered linear space, and suppose that P is algebraically closed, that is, every straight line of E meets P by a closed interval. A point x of a convex subset $A \subset E$ is called an algebraic interior point of A if for every $z \in E$, there exists $\lambda > 0$ such that $x + \lambda z \in A$. Algebraic exterior points are defined similarly, and we denote the algebraic interior (exterior) of A by $\mathrm{int}A$ $(\mathrm{ext}A)$ respectively. Moreover, $\partial A = (\mathrm{int}A \cup \mathrm{ext}A)^c$ is called the algebraic boundary of A. A convex subset C of P is called an exposed face of P if there exists a supporting hyperplane H of P such that $C = P \cap H$. By $\mathfrak{F}(P)$, we denote the set of all exposed faces of P. For $C \in \mathfrak{F}(P)$, $\dim C$ is defined as the dimension of $\mathrm{aff}C$ where $\mathrm{aff}C$ denotes the affine hull of C. The following theorem is a fundamental result, and is also useful when we intend to determine the set $a \vee b$ explicitly.

Theorem 3. *Suppose that P is algebraically closed and* $\mathrm{int}\,P \neq \emptyset$. *If* $\dim C \leq 1$ *for every $C \in \mathfrak{F}(P)$, then*

$$a \vee b = \partial U(a) \cap \partial U(b)$$

holds for every incomparable pair $a, b \in E$.

In the case when a linear topology is given in E, the assertion of Theorem 3 can be translated into the terms of topology and is still valid.

Lemma 1. *If $0 \leq x \leq y$ and $y \in \partial P$, then $x \in \partial P$.*

proof. Suppose that $x \in \mathrm{int}\,P$ and put $z = 2y - x$, then $z = y + (y - x) \in P + P = P$. Since P is convex and $x \in \mathrm{int}\,P$, $y = \frac{1}{2}(x + z) \in \mathrm{int}\,P$. This contradicts the assumption.

proof of Theorem 3. Let x_0 be an element of $a \vee b$, and suppose that $x_0 \in \mathrm{int}\,U(a)$. Then there exists $\lambda > 0$ such that $c \underset{\mathrm{def}}{=} (1 - \lambda)x_0 + \lambda b \in U(a)$. It is easy to see that $c \in U(a) \cap U(b) = U(a, b)$ and $c \lneqq x_0$. This contradicts the fact that x_0 is a minimal element of $U(a, b)$, and hence $a \vee b \subset \partial U(a) \cap \partial U(b)$.

Conversely, take $x_0 \in \partial U(a) \cap \partial U(b)$ arbitrarily and suppose that $y_0 \leq x_0$, $y_0 \in U(a, b)$. Since $a \leq y_0 \leq x_0$, it follows by Lemma 1 that

$$y_0 \in [a, x_0] \subset \partial U(a),$$

where $[a, x_0] = \{x \in E |\ a \leq x \leq x_0\}$ is an order interval. Obviously every order interval is a convex set. Similarly we have

$$y_0 \in [b, x_0] \subset \partial U(b),$$

and hence

$$[a, x_0] \cap \mathrm{int}\,U(a) = \emptyset, \quad [b, x_0] \cap \mathrm{int}\,U(b) = \emptyset,$$

while $\operatorname{int} U(a)$ and $\operatorname{int} U(b)$ are both assumed to be nonempty . Applying the separation theorem, we can find hyperplanes H_1, H_2 of E such that

(1) H_1 separates $[a, x_0]$ and $U(a)$ and,

(2) H_2 separates $[b, x_0]$ and $U(b)$.

Since $[a, x_0] \subset U(a)$ and $[b, x_0] \subset U(b)$, we can see that $[a, x_0] \subset U(a) \cap H_1$ and $[b, x_0] \subset U(b) \cap H_2$. By the condition of $\mathfrak{F}(P)$, these two faces are actually half lines. On the other hand, a, b, and x_0 cannot be in any single straight line because a and b are not comparable. Hence $[a, x_0]$ and $[b, x_0]$ are respectively included in two different lines, and in particular, both x_0 and y_0 belong to the intersection of those two lines. This means $x_0 = y_0$ and so $x_0 \in a \vee b$. Q.E.D.

Lemma 2. *Suppose that the positive cone P is algebraically closed and* $\operatorname{int} P \neq \emptyset$. *Then $\partial U(a) \cap \partial U(b) \neq \emptyset$ for every incomparable pair $a, b \in E$.*

proof. We can take an element $x \in U(a) \cap U(b)$. Indeed, $b - a$ can be written in the form $p - q$ with $p, q \in P$, and so $a + p = b + q \in U(a) \cap U(b)$. Since $a \notin U(b)$, and $U(b)$ is algebraically closed, there exists $\lambda_0 \in [0, 1)$ such that $\lambda_0 = \max\{\lambda > 0 \mid x + \lambda(a - x) \in U(b)\}$. Obviously, $z_0 \underset{\text{def}}{=} x + \lambda_0(a - x) \in U(a) \cap \partial U(b)$. Next we take $\lambda_1 = \max\{\lambda \mid z_0 + \lambda(b - z_0) \in U(a)\}$ similarly. Then $z_1 \underset{\text{def}}{=} z_0 + \lambda_1(b - z_0) \in \partial U(a)$. Moreover, since $b \leq z_1 \leq z_0 \in \partial U(b)$, it follows by Lemma 1 that $z_1 \in \partial U(b)$.

Applying Theorem 3 and Lemma 2, we can obtain the following.

Corollary 3. *Under the hypotheses in Theorem 3, $a \vee b \neq \emptyset$ holds for every $a, b \in E$. Moreover when a and b are not comparable, we have*

$$U(a, b) = (a \vee b) + P.$$

proof. The first statement of the theorem follows immediately from Theorem 3 and Lemma 2. To see the latter, it is sufficient to show $U(a, b) \subset (a \vee b) + P$. For an arbitrary element $x \in U(a, b)$, we can choose z_1 as in the proof of Lemma 2. Then $z_1 \leq x$ and $z_1 \in \partial U(a) \cap \partial U(b)$. Hence by Theorem 3, $z_1 \in a \vee b$, and this means that $x \in (a \vee b) + P$.

Theorem 4. *Under the hypotheses in Theorem 3,*

$$U(A) = (\operatorname{Sup} A) + P$$

holds for every subset $A \subset E$. In particular, the conclusions in Corollary 1 and Corollary 2 are valid.

Remark. The hypotheses of this theorem can be somewhat weakened. Moreover, using this theorem, we can simplify the proof of Lemma 2 and can obtain the second statement of Corollary 3 directly.

Lemma 3. *If $x \in \partial U(A)$ for a subset A of E, then $U(A)_x \subset \partial U(A)$ where $U(A)_x = \{y \in U(A) \mid y \leq x\}$.*

proof. Let y be an arbitrary point in $U(A)_x$. Since $x \in \partial U(A)$ there exists a point $z \in E$ such that $\{x + tz \mid t > 0\} \cap U(A) = \emptyset$. By the definition of $U(A)$, $U(A) + P = U(A)$, and this yields $\{y + tz \mid t > 0\} \cap U(A) = \emptyset$. This means that $y \in \partial U(A)$.

proof of Theorem 4. Let x_0 be an arbitrary point in $U(A)$. Since P is algebraically closed, P can not include any straight line. Indeed if $\{x+ty \mid t\in\mathbb{R}\}\subset P$ for some $y\neq 0$, then $\{ty \mid t\in\mathbb{R}\}\subset P\cup\partial P = P$ and this contradicts (P2). Hence for a positive element $x\neq 0$, there exists $t_1=\max\{t\geq 0 \mid x_0 - tx\in U(A)\}$. If we put $x_1 = x_0 - t_1 x$, then $x_1\in\partial U(A)$ and it follows from Lemma 3 that $U(A)_{x_1}\subset\partial U(A)$. Since $U(A)_{x_1}$ is a convex set and $\operatorname{int} U(A)\neq\emptyset$, we can apply the separation theorem and there exists a hyper plane H which separates $U(A)_{x_1}$ and $U(A)$. $U(A)_{x_1}\subset(x_1-P)\cap H$ and this is a straight half line by the assumption. Moreover, since $U(A)$ can not include the whole straight line, $U(A)_{x_1}$ is the form $\{\lambda x_1 + (1-\lambda)z \mid 0\leq\lambda\leq 1\}$ where $z\leq x_1$. Clearly, z is a minimal element of $U(A)$ and $z\leq x_0$, and this completes the proof. Q.E.D.

§3 Examples

Let E be the space of all symmetric matirces of $M_2(\mathbb{R})$, and let P be the set of all positive semi definite matrices in E. Then (E,P) is m.o.c., but it is not a lattice. E and P can be identified with $\mathbb{R}^3$ and

$$P=\{(x,y,z)\in\mathbb{R}^3 \mid z^2\leq xy,\ 0\leq x,\ 0\leq y\}$$

respectively. It is easy to see that every exposed face of the positive cone P is 1-dimensional except the trivial face $\{0\}$, and P satisfies the condition in Theorem 3. Hence, by some simple culculations, we can determine the set $a\vee b$ for incomparable pair $a,b\in E$.

Next we investigate the relation between the condition of Theorem 2 and that of Theorem 3. For a partially ordered linear space (E,P), we say that the positive cone P satisfies condition $(\mathfrak{F})$ when $\dim C\leq 1$ for every $C\in\mathfrak{F}(P)$. In finite dimensional cases, P does not satisfy the condition $(\mathfrak{F})$ when P is a closed convex cone generated by a finite set. On the other hand, such a positive cone satisfies monotone order completeness. This means that monotone order completeness does not imply the condition $(\mathfrak{F})$. Now we show an example in order to see the converse implication is also not true.

Let E be the linear space consisting of all sequences $x=(x_1,x_2,\cdots)$ $(x_i\in\mathbb{R})$ such that $x_i=0$ except for finite number of $i=1,2,\cdots$. We define

$$P=\{\ x=(x_1,x_2,\cdots) \mid x_1\geq\left(\sum_{i=2}^{\infty}x_i^2\right)^{\frac{1}{2}}\ \}.$$

Then it is easy to see that P is algebraically closed and $\operatorname{int} P\neq\emptyset$. Indeed $(1,0,0,\cdots)\in\operatorname{int} P$. Let $C\in\mathfrak{F}(P)$ and let $x=(x_1,x_2\cdots), y=(y_1,y_2,\cdots)$ be two points in $C\setminus\{0\}$. Since $x,y\in\partial P$, $x_1^2=\sum_{i=2}^{\infty}x_i^2$, *and* $y_1^2=\sum_{i=2}^{\infty}y_i^2$. By the convexity of C, we also have $\frac{1}{2}(x+y)\in\partial P$, and hence $(x_1+y_1)^2=\sum_{i=2}^{\infty}(x_i+y_i)^2$. By simple calculation, we obtain $x=\lambda y$ for some $\lambda>0$. This means that $\dim C=1$, and that P satisfies the condition $(\mathfrak{F})$. Thus Theorem 3 and Theorem 4 are applicable in this case.

We will show that (E,P) is not m.o.c. We define a sequence $\{a_n\}\subset E$ by

$$a_n=(\frac{1}{2^n},\frac{1}{2},\frac{1}{4},\frac{1}{8},\cdots,\frac{1}{2^n},0,0,\cdots)\qquad(n=1,2,\cdots).$$

Then we have $a_1\geq a_2\geq a_3\geq\cdots$. Moreover, since $(\frac{1}{2})^2+(\frac{1}{4})^2+(\frac{1}{8})^2+\cdots=\frac{1}{3}$, we can see that $(-\sqrt{\frac{1}{3}},0,0,\cdots)$ is a lower bound of $\{a_n\}$. Let $b=(b_1,b_2,\cdots,b_i,0,0,\cdots)$

be an arbitrary lower bound of $\{a_n\}$. Then an element of the form $c = (b_1 + \lambda, b_2, b_3, \cdots, b_i, \mu, 0, 0, \cdots)$ always satisfies $b \not\geq c$ when $\lambda > 0$. It is easy to see that we can choose λ and μ such that c is also a lower bound of $\{a_n\}$. This means that the greatest lower bound of $\{a_n\}$ does not exist, and (E, P) is not m.o.c..

References

[1] I. Amemiya, *A generalization of Riesz -Fisher's theorem*, J.Math.Soc.Japan **5** (1953), 353-354.

[2] T. Ando, *On fundamental properties of a Banach space with cone*, Pacific J. Math. **12** (1962), 1163–1169.

[3] R. B. Holmes, *Geometric Functional Analysis and its Applications*, Springer-Verlag (1975).

[4] D. T. Luc, *Theory of vector optimization*, Springer-Verlag (1989).

[5] S.Koshi, *A generalization of the Amemiya's theorem*, to appear in Czech Math. J.

[6] ———, *Lattice structure of partially ordered linear space*, Memoirs of the Hokkaido Institute of Technology **25** (1997), 1–7.

[7] R.T.Rockafellar, *Convex Analysis*, Princeton University Press (1970).

[8] A. C. Zaanen, *Riesz space II*, North Holland Math. Libr. **30** (1983).

N.Komuro
Hokkaido University of Education at Asahikawa
Hokumoncho 9 chome Asahikawa
070 Japan
e-mail: komuro@ atson.asa.hokkyodai.ac.jp