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ON BRAID TYPE OF FIXED POINTS OF HOMEOMORPHISMS DEFINED ON THE TORUS

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1 Introduction

Huang and Jiang studied in [4] a method of estimating the number of periodic points of homeomorphisms f on the torus isotopic to the identity map. For any finite set of fixed points of f, the Jacobian matrix in Fox calculus can be defined from which one can obtain information about periodic points. They gave a method of calculating the Jacobian matrix. However, only one example was given there, and a systematic investigation was not done.

The purpose of the present paper is to study the Jacobian matrix. In our investigation, we assume that two fixed points have been found. Then, we can consider a homeomorphism on the torus with these two points deleted. It is known that the homomorphism on the fundamental group of the punctured torus induced by this homeomorphism can be identified with a braid on two strings. The braid group on two strings has two generators ρ and τ. Therefore the induced homomorphism is written as a product of ρ and τ. We study the case where the exponent sum of each of the two generators is zero. Moreover we only treat the case where the product has the simplest form. We compute the Jacobian matrix explicitly, and as an application of this computation, we show that the abelianization of the generalized Lefschetz number, which is an important invariant in fixed point theory, is a symmetric polynomial.

2 The Jacobian matrix and fixed points

We review some facts on the relation between the Jacobian matrix and fixed points obtained by Fadell and Husseini [4] and Huang and Jiang [5]. Let $x_1, x_2, \cdots, x_n$ be fixed points of a homeomorphism f on the 2-dimensional torus T^2 which is isotopic to the identity map id, and set $C = \{x_1, x_2, \cdots, x_n\}$, $M = T^2 - C$. Then we can consider $f : M \to M$. Pick a base point x_0 for M. The group $\pi_1(M, x_0)$ is a free group of rank $n+1$. Define elements $a_1, a_2, \cdots, a_n, b, c$ and $g_1, g_2, \cdots, g_n$ of $\pi_1(M, x_0)$ as shown in Figure 1.

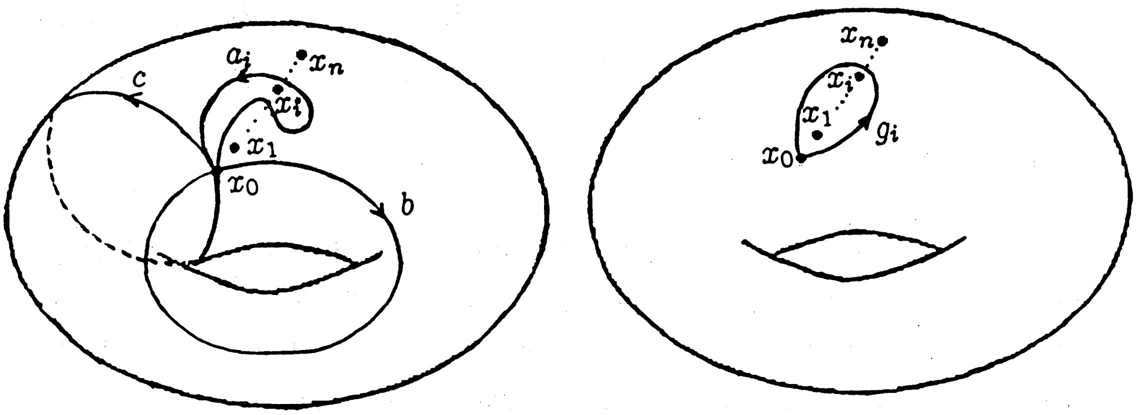

Figure 1

Obviously $g_i = a_1 a_2 \cdots a_i$ for $1 \le i \le n$. Now the 1-dimensional homology group $H_1(M)$ is an abelian group generated by $a_1, a_2, \cdots, a_n, b, c$ with a relation $a_1 a_2 \cdots a_n = 1$. Let Λ denote the group ring $\mathbf{Z}H_1(M)$. For $\varphi \in Aut\pi_1(M, x_0)$, let $\nu(\varphi)$ denote the homomorphisms on $H_1(M)$ and on Λ induced by φ.

Define a map $B : Aut\pi_1(M, x_0) \to GL(n+1, \Lambda)$ by

$$B(\varphi) = \begin{pmatrix} \dfrac{\partial(g_i\varphi)}{\partial g_j} & \dfrac{\partial(g_i\varphi)}{\partial b} & \dfrac{\partial(g_i\varphi)}{\partial c} \\ \dfrac{\partial(b\varphi)}{\partial g_j} & \dfrac{\partial(b\varphi)}{\partial b} & \dfrac{\partial(b\varphi)}{\partial c} \\ \dfrac{\partial(c\varphi)}{\partial g_j} & \dfrac{\partial(c\varphi)}{\partial b} & \dfrac{\partial(c\varphi)}{\partial c} \end{pmatrix}^{Ab}_{1 \le i,j \le n-1},$$

for any $\varphi \in Aut\pi_1(M, x_0)$, where Ab denote the abelianization operator of the group ring $\mathbf{Z}\,\pi_1(M, x_0)$ and the partial derivatives here are the Fox derivatives. Note that the Fox derivatives are taken with respect to the basis $\{g_1, g_2, \cdots, g_{n-1}, b, c\}$, while the element of Λ is written in terms of the basis $\{a_2, a_3, \cdots, a_n, b, c\}$.

Now we choose an isotopy $\{f_t\}$, where $f_0 = id$, $f_1 = f$, then $\{f_t\}$ determines a subset $f_t(C) = \{f_t(x_1), \cdots, f_t(x_n)\}$ of T^2 with n points for each t. Let σ_C denote the braid represented by $f_t(C)$ [2], [7]. The braid σ_C is identified with an element of $Aut\pi_1(M, x_0)$. Then the homomorphism $f_* : H_1(M) \to H_1(M)$ coincides with the homomorphism $\nu(\sigma_C)$. We use the same notation f_* for the extension of f_* to Λ. Let $H = Coker(f_* - id)$. H is a quotient of $H_1(M)$ obtained by identifying each a_i with $a_i^{\nu_C}$, b with b^{ν_C} and c with c^{ν_C}, where $\nu_C = \nu(\sigma_C)$. Let μ_C stand for the projection $H_1(M) \to H$ as well as for its extention $\Lambda \to \mathbf{Z}H$.

We can derive some information about fixed points from the Jacobian matrix $B(\sigma_C)$. The generalized Lefschetz number $L(f)$ is a useful invariant to study fixed points. We shall be concerned with its abelianization $L(f)^{Ab}$, so we only review the definition of $L(f)^{Ab}$.

Definition 1. Denote $Fix(f)$ the set of fixed points of f. We shall classify $Fix(f)$ by the following equivalence relation:

$x, y \in Fix(f)$ are said to be abelianized Nielsen equivalent iff there exists a path ℓ from x to y such that $[(f \circ \ell)\ell^{-1}]$ is the zero element of $H_1(M)$.

Now, let $x \in Fix(f)$. We need to choose a path w from x_0 to $f(x_0)$, and a path c from x_0 to x. Then we can identify the abelianized Nielsen class $[x]$ with an element $[w(f \circ c)c^{-1}]$ of H naturally. This correspondence is evidently independent of the choice of c.

Definition 2. For $x \in Fix(f)$, let $H(x) = [w(f \circ c)c^{-1}] \in H$. For $\gamma \in H$, let $Fix_\gamma(f) = \{x \in Fix(f) \mid H(x) = \gamma\}$. Define $L(f)^{Ab}$ by

$$L(f)^{Ab} = \sum_{\gamma \in H} ind\big(f, Fix_\gamma(f)\big)\gamma \in \mathbf{Z}H,$$

where $ind\big(f, Fix_\gamma(f)\big)$ is the fixed point index of $Fix_\gamma(f)$ [3], [6].

From this definition, it is clear that $L(f)^{Ab}$ is a Laurant polynomial and the number of terms in $L(f)^{Ab}$ is a lower bound for the number of fixed points.

In [4], Fadell and Husseini proved that the matrix $B(\sigma_C)$ is closely related to $L(f)$:

Theorem 1. ([4], abelianized version) *The polynomial* $1 - \mathrm{tr}\big(\mu_C B(\sigma_C)\big)$ *coincides with the abelianization* $L(f)^{Ab}$ *of the generalized Lefschetz number of* f.

We should note that B is not a homomorphism. However we have the product formula:

$$B(\varphi\psi) = B(\varphi)^{\nu(\psi)} B(\psi) \quad \textit{for } \varphi, \psi \in Aut\pi_1(M, x_0). \tag{1}$$

3 Statement of the result

We concider the case $C = \{x_1, x_2\}$. Let us first recall some facts about braids on the torus. The braids $\rho_i, \tau_i (i = 1, 2), \sigma_1$ used below are illustrated in Figure 2. We use the commutator notation $[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}$ in groups.

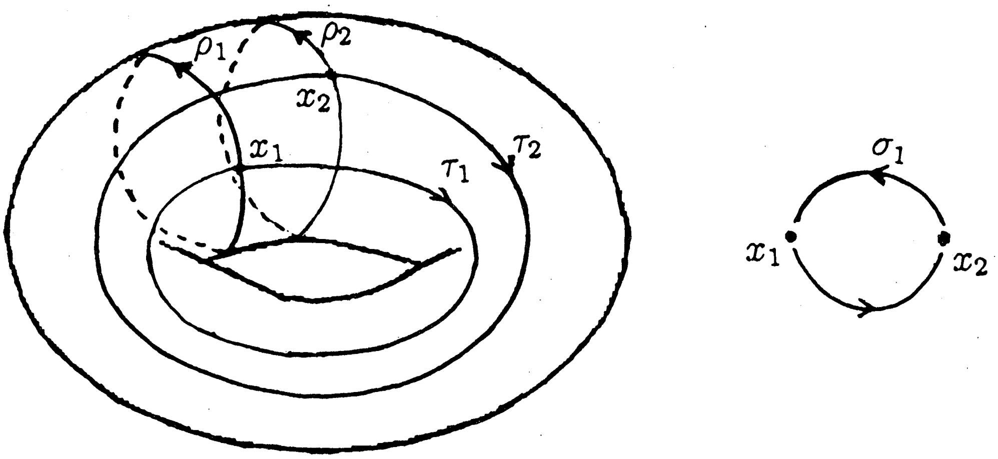

Figure 2

Proposition 1 .(Birman [1]) *The pure 2-braid group on* T^2 *admits the following presentation*:

Generators : $\rho_1, \rho_2, \tau_1, \tau_2$.

Relations : $[\rho_1, \rho_2] = [\tau_1, \tau_2] = 1$, $A_{12} = \tau_2^{-1}\rho_1\tau_2\rho_1^{-1}$, $A_{12}^{-1} = \rho_2^{-1}\tau_1\rho_2\tau_1^{-1}$,
$A_{12}^{-1} = (\tau_1\tau_2)A_{12}^{-1}(\tau_2^{-1}\tau_1^{-1})$, $A_{12} = (\rho_1\rho_2)A_{12}(\rho_2^{-1}\rho_1^{-1})$,
where $A_{12} = [\tau_1, \rho_1]$.

The full 2-braid group admits a presentation obtained from above by adding a generator σ_1 and relations :

$$\sigma_1^2 = A_{12},\ \rho_2 = \sigma_1\rho_1\sigma_1,\ \tau_2 = \sigma_1^{-1}\tau_1\sigma_1^{-1}.$$

By Proposition 1, the full 2-braid group is generated by σ_1, ρ_1, τ_1. Let $M = T^2 - \{x_1, x_2\}$. We can choose an isotopy $\{f_t\}$ suitably to satisfy $f_t(x_2) \equiv x_2$ $(0 \le t \le 1)$. Then the braid σ_C is written as a product of $\rho_1^{\pm 1}$ and $\tau_1^{\pm 1}$. For brevity, we shall write $a = a_2$, $g = g_1, \rho = \rho_1$ and $\tau = \tau_1$. Obviously $g_2 = [b, c]$ and $g = g_2 a^{-1} = [b, c]a^{-1}$. $\pi_1(M, x_0)$ has two useful bases $\{a, b, c\}$ and $\{g, b, c\}$. Define the automorphisms $\rho, \tau : \pi_1(M, x_0) \to \pi_1(M, x_0)$ to be those determined by the corresponding geometric braids ρ, τ. By geometric inspection, we can write down the automorphisms $\rho^{\pm 1}, \tau^{\pm 1}$ in terms of the basis $\{g, b, c\}$ as follows:

$$\rho : \begin{cases} g \mapsto cgc^{-1} \\ b \mapsto g^{-1}b \\ c \mapsto c \end{cases}, \qquad \rho^{-1} : \begin{cases} g \mapsto c^{-1}gc \\ b \mapsto c^{-1}gcb, \\ c \mapsto c \end{cases}$$

$$\tau : \begin{cases} g \mapsto bgb^{-1} \\ b \mapsto b \\ c \mapsto gc \end{cases}, \qquad \tau^{-1} : \begin{cases} g \mapsto b^{-1}gb \\ b \mapsto b \\ c \mapsto b^{-1}g^{-1}bc \end{cases}.$$

The Jacobian matrix B for the automorphisms $\rho^{\pm 1}, \tau^{\pm 1} : \pi_1(M, x_0) \to \pi_1(M, x_0)$ become:

$$B(\rho) = \begin{pmatrix} c & 0 & a^{-1}(a-1) \\ -a & a & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B(\rho^{-1}) = \begin{pmatrix} c^{-1} & 0 & a^{-1}c^{-1}(1-a) \\ c^{-1} & a^{-1} & a^{-1}c^{-1}(1-a) \\ 0 & 0 & 1 \end{pmatrix},$$

$$B(\tau) = \begin{pmatrix} b & a^{-1}(a-1) & 0 \\ 0 & 1 & 0 \\ 1 & 0 & a^{-1} \end{pmatrix}, \quad B(\tau^{-1}) = \begin{pmatrix} b^{-1} & a^{-1}b^{-1}(1-a) & 0 \\ 0 & 1 & 0 \\ -ab^{-1} & b^{-1}(a-1) & a \end{pmatrix}.$$

Thier actions on Λ are given by:

$$(2) \qquad \nu(\rho) : \begin{cases} a \mapsto a \\ b \mapsto ab, \\ c \mapsto c \end{cases} \qquad \nu(\rho^{-1}) : \begin{cases} a \mapsto a \\ b \mapsto a^{-1}b, \\ c \mapsto c \end{cases}$$

$$\nu(\tau):\begin{cases} a\mapsto a\\ b\mapsto b\\ c\mapsto a^{-1}c\end{cases},\qquad \nu(\tau^{-1}):\begin{cases} a\mapsto a\\ b\mapsto b\\ c\mapsto ac\end{cases}.$$

These expressions and the product formula (1) enable one to calculate $B(\sigma_C)$ for $\sigma_C \in Aut\pi_1(M,x_0)$ that is written as a product of ρ,τ and their inverses.

For $m\geq 2,\ n\geq 2$ we have:

$$(3)\qquad B(\tau^m)=\begin{pmatrix} b^m & (1-a^{-1})\sum\limits_{k=0}^{m-1} b^k & 0\\ 0 & 1 & 0\\ a^{1-m}\sum\limits_{k=0}^{m-1}(ab)^k & \sum\limits_{k=0}^{m-2}(1-a^{k+1-m})b^k & a^{-m}\end{pmatrix},$$

$$(4)\qquad B(\rho^n)=\begin{pmatrix} c^n & 0 & (1-a^{-1})\sum\limits_{k=0}^{n-1} c^k\\ -a^n\sum\limits_{k=0}^{n-1}(a^{-1}c)^k & a^n & \sum\limits_{k=0}^{n-2}(1-a^{-k-1+n})c^k\\ 0 & 0 & 1\end{pmatrix}.$$

For $m\geq 1,\ n\geq 1$ we have:

$$(5)\qquad B(\tau^{-m})=\begin{pmatrix} b^{-m} & a^{-1}b^{-m}(1-a)\sum\limits_{k=0}^{m-1} b^k & 0\\ 0 & 1 & 0\\ -ab^{-m}\sum\limits_{k=0}^{m-1}(ab)^k & b^{-m}\sum\limits_{k=0}^{m-1}(a^{k+1}-1)b^k & a^m\end{pmatrix},$$

$$(6)\qquad B(\rho^{-n})=\begin{pmatrix} c^{-n} & 0 & c^{-n}(a^{-1}-1)\sum\limits_{k=0}^{n-1} c^k\\ c^{-n}\sum\limits_{k=0}^{n-1}(a^{-1}c)^k & a^{-n} & a^{-n}(1-a)\sum\limits_{k=0}^{n-1}\left(\sum\limits_{i=1}^{k+1}c^{-i}\right)a^k\\ 0 & 0 & 1\end{pmatrix}.$$

From (1) ,(2) ,(3) ,(4) ,(5) ,(6) , we can calculate the Jacobian matrix $B(\tau^m\rho^n) = (x_{ij})_{1\le i,j\le 3}$ and $B(\tau^{-m}\rho^{-n}) = (y_{ij})_{1\le i,j\le 3}$:

Proposition 2. *For $m, n \in \mathbf{N}$*

$$
\begin{aligned}
x_{11} &= c^n(a^nb)^m - a^n(1-a^{-1})\Big\{\sum_{k=0}^{m-1}(a^nb)^k\Big\}\sum_{k=0}^{n-1}(a^{-1}c)^k,\\
x_{12} &= a^n(1-a^{-1})\sum_{k=0}^{m-1}(a^nb)^k,\\
x_{13} &= c^{n-1}(1-a^{-1})\sum_{k=0}^{n-1}\Big\{\sum_{j=0}^{m}(a^nb)^j - \sum_{j=0}^{m-1}(a^nb)^ja^k\Big\}c^{-k},\\
x_{21} &= -a^n\sum_{k=0}^{n-1}(a^{-1}c)^k,\\
x_{22} &= a^n,\\
x_{23} &= \begin{cases} 0 & (n=1),\\ \displaystyle\sum_{k=0}^{n-2}(1-a^{n-1-k})c^k & (n\ge 2),\end{cases}\\
x_{31} &= a^{mn+1}b^m\sum_{k=1}^{m}\Big\{a^{n-1}\big(\sum_{j=0}^{n-1}(a^{-1}c)^j\big)(a-a^k)+c^n\Big\}(a^{n+1}b)^{-k},\\
x_{32} &= \begin{cases} 0 & (m=1),\\ \displaystyle a^n(a-1)\sum_{k=1}^{m-1}\big(\sum_{i=0}^{k-1}(a^{n+1}b)^i\big)a^{-k} & (m\ge 2),\end{cases}\\
x_{33} &= \begin{cases} \dfrac{a^{-m}\{a^{2m}b^m(a-1)+a(ab-1)\}}{a^2b-1} & (n=1),\\ \displaystyle a^{-m}\Bigg[(a-1)\sum_{k=0}^{m-1}\Big\{\sum_{j=0}^{n-1}c^j+\big(\sum_{j=0}^{n-2}\sum_{i=0}^{j}(a^{-1}c)^ia^j\big) & \\ \qquad\qquad\times(a-a^{m-k})\Big\}(a^{n+1}b)^k+1\Bigg] & (n\ge 2).\end{cases}
\end{aligned}
$$

$$
\begin{aligned}
y_{11} &= a^{mn}b^{-m}c^{-n}\Big\{(a^{-1}-1)\big(\sum_{k=0}^{m-1}(a^{-n}b)^k\big)\sum_{k=0}^{n-1}(a^{-1}c)^k+1\Big\},\\
y_{12} &= a^{n(m-1)}b^{-m}(a^{-1}-1)\sum_{k=0}^{m-1}(a^{-n}b)^k,
\end{aligned}
$$

$$
\begin{aligned}
y_{13} &= a^{mn}b^{-m}c^{-n}(1-a^{-1})\sum_{k=0}^{n-1}\Big\{(1-a^{-(k+1)})\sum_{j=0}^{m-1}(a^{-n}b)^j - 1\Big\}c^k,\\
y_{21} &= c^{-n}\sum_{k=0}^{n-1}(a^{-1}c)^k,\\
y_{22} &= a^{-n},\\
y_{23} &= c^{-n}\sum_{k=0}^{n-1}\big(a^{-(k+1)}-1\big)c^k,\\
y_{31} &= a^n b^{-1}c^{-n}\sum_{k=0}^{m-1}\Big\{\big(\sum_{j=0}^{n-1}(a^{-1}c)^j\big)(a^m-a^k)-a^m\Big\}(a^{n-1}b^{-1})^k,\\
y_{32} &= (1-a^{-1})b^{-1}\sum_{k=1}^{m}\Big\{\sum_{i=0}^{k-1}(a^{n-1}b^{-1})^i\Big\}a^k,\\
y_{33} &= a^m\Big[a^{n-m-1}b^{-1}c^{-n}(a-1)\sum_{k=0}^{m-1}\Big\{a^m\sum_{j=0}^{n-1}c^j+\big(\sum_{j=0}^{n-1}\sum_{i=0}^{j}(a^{-i}c^j)\big)\\
&\qquad\qquad\times(a^k-a^m)\Big\}(a^{n-1}b^{-1})^k+1\Big].
\end{aligned}
$$

Since $H_1(M)$ is generated by a, b, c, we have:

$$H = \mathbf{Z}a \oplus \mathbf{Z}b \oplus \mathbf{Z}c / Im(f_* - id). \tag{7}$$

When $\sigma_C = \tau^{m_1}\rho^{n_1}\tau^{m_2}\rho^{n_2} \in Aut\pi_1(M, x_0)$, we have by (2):

$$Im(f_* - id) = \mathbf{Z}(m_1+m_2)a + \mathbf{Z}(n_1+n_2)a. \tag{8}$$

We consider the case of $m_1+m_2 = n_1+n_2 = 0$. From (7),(8) we have that $Im(f_* - id) = 0$ and $\Lambda = \mathbf{Z}[a,b,c]$, the ring of polynomials on a, b, c. Therefore, $L(f)^{Ab}$ is a polynomial on a, b, c. For $x \in Fix(f)$, let $I(x)$ be the coefficient of a in the monomial $H(x)$. We call $I(x)$ the intersection number of x. This number coincides with the usual intersection number of the loop $w(f\circ c)c^{-1}$ with the segment connecting x_1 to x_2.

Let $B'(\sigma_C)$ denote the simplified matrix of $B(\sigma_C)$ obtained by substituting 1 for b and c. Then we have:

$$1 - \operatorname{tr}\big(\mu_C B'(\sigma_C)\big) = \sum_i ind(f, Fix_i(f))a^i,$$

where $Fix_i(f) = \{x \in Fix(f) \mid I(x) = i\}$. The following theorem asserts that $L(f)^{Ab}$ is a symmetric polynomial:

THEOREM 2. *Let* $\sigma_C = \tau^m \rho^n \tau^{-m} \rho^{-n}$. *Then we have the following equality:*

$$ind\big(f, Fix_i(f)\big) = ind\big(f, Fix_{2mn-i}(f)\big) \quad \textit{for any } i.$$

4 Proof of Theorem 2

Theorem 2 follows easily from the following Lemma:

LEMMA. *Let* $B'(\sigma_C) = \big(z_{ij}(a)\big)_{1 \le i,j \le 3}$, *where* $\sigma_C = \tau^m \rho^n \tau^{-m} \rho^{-n}$. *Then the following equalities hold:*

(i) $z_{11}(a) = 1$,

(ii) $z_{ii}(a) = a^{2mn} z_{ii}(a^{-1}) \quad (i = 2, 3)$.

PROOF. From Proposition 2, in the case at least one of m or n is 1, we have:

$$z_{11}(a) = 1, \; z_{22}(a) = z_{33}(a) = \begin{cases} a^n & (m = 1), \\ a^m & (n = 1). \end{cases}$$

We have the conclusion of this Lemma.

Consider the case $m \ge 2$, $n \ge 2$. Let $\mathbf{v}_i$ denote the i-th row vector of $B(\tau^m \rho^n)^{\nu(\tau^{-m}\rho^{-n})}$, where we put $b = c = 1$. Let $\mathbf{w}_j$ denote the j-th column vector of $B(\tau^{-m}\rho^{-n})$, where we put $b = c = 1$. We use abbreviation as follows:

$$A_n^m = \sum_{k=1}^{n} a^{km}.$$

From Proposition 2, we have:

$$\begin{aligned}
\mathbf{v}_1 &= \Big[a^{mn} - ma^{n-m+1}(1 - a^{-1})A_n^{m-1}, \; ma^n(1 - a^{-1}), \\
&\qquad\qquad a^{mn}(1 - a^{-1})\Big\{(m+1)A_n^{-m} - ma^{-1}A_n^{1-m}\Big\}\Big], \\
\mathbf{v}_2 &= \Big[-a^{n-m+1}A_n^{m-1}, \; a^n, \; a^{-m}(A_{n-1}^m - a^n A_{n-1}^{m-1})\Big], \\
\mathbf{v}_3 &= \Big[a^{n-m+1}(aA_m^{-1} - m)A_n^{m-1} + a^{mn+1}A_m^{-1}, \; a^{n-1}(a-1)\sum_{k=1}^{m-1} A_k^1 a^{-k},
\end{aligned}$$

$$a^{-m}\Big\{(1-a^{-1})a^{-m}\big(A^1_m A^m_n + a\sum_{k=1}^{m}(\sum_{j=1}^{n-1} A^{m-1}_j a^j)a^k$$

$$-ma^{m+1}\sum_{j=1}^{n-1} A^{m-1}_j a^j\big) + 1\Big\}\Big],$$

$$\mathbf{w}_1 = \Big[a^{mn}\Big\{a^n(1-a)A^{-n}_m A^{-1}_n + 1\Big\},\ aA^{-1}_n,$$

$$a(a^m A^{n-1}_m A^{-1}_{n-1} - A^n_m A^{-1}_n)\Big],$$

$$\mathbf{w}_2 = \Big[a^{mn}(a^{-1}-1)A^{-n}_m,\ a^{-n},\ a^{-n}(a-1)\sum_{k=1}^{m} A^{n-1}_k a^k\Big],$$

$$\mathbf{w}_3 = \Big[a^{mn}(1-a^{-1})\Big\{a^n(n-A^{-1}_n)A^{-n}_m - n\Big\},\ A^{-1}_n - n,$$

$$(a-1)\big\{na^m A^{n-1}_m + (A^n_m - a^{m+1}A^{n-1}_m)\sum_{j=1}^{n} A^{-1}_j\big\} + a^m\Big].$$

Since $z_{ij}(a) = \mathbf{v}_i \cdot \mathbf{w}_j$, where $\cdot$ is the inner product, we have $z_{11}(a) = 1$. This completes the proof of (i).

Similarly we have:

$$z_{22}(a) = \frac{A^m_{n-1}A^n_m - a^n A^{m-1}_{n-1}A^n_{m-1}}{A^1_{n-1}},$$

$$z_{33}(a) = a^{mn} - \frac{(m-1)a^n A^{m-1}_{n-1}(A^1_{mn-1} - nA^n_{m-1})}{A^1_{n-1}}$$

$$+\frac{mA^m_{n-1}(A^1_{(m+1)n-1} - nA^n_m)}{A^1_{n-1}} - \frac{a^{2mn}\sum_{k=1}^{n-1} A^1_k}{A^1_n} - \frac{\sum_{k=1}^{n-1} A^{-1}_k}{A^{-1}_n}.$$

From above, we can easily prove the equalities (ii).

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