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RIGID GEOMETRY AND ÉTALE COHOMOLOGY OF SCHEMES

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§1. Introduction

In this article, we discuss basic properties of rigid geometry from the viewpoint of M. Raynaud [Ray 2], giving the formal flattening theorem and the comparison theorem of rigid-étale cohomology, as applications to algebraic geometry.

The estimate of cohomological dimension of Riemann space is included. We have also included conjectures on ramification of étale sheaves on schemes. In the appendix, a rigorous proof of the flattening theorem, which is valid over any valuation rings and noetherian formal schemes, is included. This appendix will be published separately.

There are two other approaches to the étale cohomology of rigid analytic spaces: V. Berkovich approach, R. Huber approach by adic spaces. We hope that the reader understands the freedom in the choice, and takes the shortest one according to the problems one has in the mind.

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Basic properties

To define the rigid analytic spaces, J. Tate regards rigid analytic spaces as an analogue of complex analytic spaces. On the other hand, M. Raynaud regards it as a formal schemes tensored with **Q**. The last approach is not only very beautiful, but much more convinient in the application to algebraic geometry. We take the Raynaud approach in the sequel.

By technical reasons, we consider coherent (= quasi-compact and quasi-separated) formal schemes which subject to one of the following conditions:

type n) X is a noetherian formal scheme.

type v) X is finitely generated over a complete valuation ring V with a-adic topology for some $a \in V$.

Those two assumptions ensure necessary Artin-Rees type theorems.

By $\mathcal{C}$ we denote the category of coherent (quasi-compact and quasi-separated) formal schemes, with coherent (quasi-compact and quasi-separated) morphisms.

We define the class of proper modification, called admissible blowing ups, as follows:

Let $\mathcal{I}$ be an ideal which contains an ideal of definition. When $X = \mathrm{Spf}\ A$ is affine, $\mathcal{I} = I \cdot \mathcal{O}_X$, the blowing up X' of X along $\mathcal{I}$ is just the formal completion of the blowing up of Spec A along I . In general X' is defined by patching. When X is the p-adic completion of some p-adic scheme Y, admissible blowing up means the (formal completion of) blowing up with a center whose support is concentrated in $p = 0$. So the following definition, due to Raynaud, will be suited for our purpose:

Definition (Raynaud [Ray 2]**).** *The category $\mathcal{R}$ of coherent rigid-analytic spaces is the quotient category of $\mathcal{C}$ by making all admissible blowing ups into isomorphisms, i.e.,*

$$\mathrm{Hom}_{\mathcal{R}}(X,Y) = \varinjlim_{X' \in \mathcal{B}_X} \mathrm{Hom}(X',\ Y).$$

For $X \in \mathcal{C}$, X viewed as an object of $\mathcal{R}$ is denoted by X^{rig} or X^{an}. X is called a formal model of X^{an}.

Note that we can fix a base if necessary. For example, in case of type v), it might be natural to work over the valuation ring V. Though the definition of rigid spaces seems to be a global one, i.e., there are no a priori patching properties, but it indeed does. The equivalence with the classical Tate rigid-spaces is shown in [BL].

Riemann space associated with a rigid space. *Let $\mathcal{X} = X^{\mathrm{an}}$ is a coherent rigid space. Then the projective limit*

$$< \mathcal{X} >= \varprojlim_{X' \in \mathcal{B}_X} X'$$

in the category of local ringed spaces exists. The topological space is quasi-compact. We call it the (Zariski-) Riemann space associated to $\mathcal{X}$. The projection $< \mathcal{X} > \to X$ is called the specialization map, and written as $\mathrm{sp} = \mathrm{sp}_X$. The structural sheaf $\tilde{\mathcal{O}}_{\mathcal{X}}$ yields

$$\mathcal{O}_{\mathcal{X}} = \varinjlim_{n} \mathrm{Hom}(\mathcal{I}^n \tilde{\mathcal{O}}_{\mathcal{X}},\ \tilde{\mathcal{O}}_{\mathcal{X}})$$

which is also (!) local ringed. This $\mathcal{O}_{\mathcal{X}}$ is the structural sheaf in rigid geometry ((classical) rigid geometry is a $\mathbf{Q}$*-theory, i.e., invert $\mathcal{I}$). $\tilde{\mathcal{O}}_{\mathcal{X}}$ is the (canonical) model of $\mathcal{O}_{\mathcal{X}}$.*

In the following we sometimes call the topology, or rather the Grothendieck topology associated to the topological space, admissible, to make it compatible with the classical terminology. The category $\mathcal{R}$, with the admissible topology, is called large admissible site.

Note that the model sheaf $\tilde{\mathcal{O}}_{\mathcal{X}}$ itself gives a local ringed space structure.

Why do we need such a topological space?

The typical example is $A = V\{\{X\}\}$, the ring of a-adic convergent power series ($V = \mathbf{Z}_p, a = p$ or $V = \mathbf{C}[[t]],\ a = t$), which should correspond to the unit disk over V. Put K= the fraction field of V, $\mathbf{C}_K$ = the completion of the algebraic closure of $\bar{K}$. $D(\mathbf{C}_K) = \{\beta \in \mathbf{C}\ ,\ |\beta| \leq 1\}$. In any a-adic analytic geometry, we expect $A_{\mathbf{C}_K}$ to be the ring of analytic functions to the closed unit disk $D(\mathbf{C}_K)$. Since the ring $A_{\mathbf{C}_K}$ is integral, the unit disk should be connected, but for the natural topology of $D(\mathbf{C}_K)$ this is false. The Riemann space of the unit disc is shown to be connected.

As in the Zariski case, each point x of $< \mathcal{X} >$ corresponds to a valuation ring V_x which is henselian along I = the inverse image of $\mathcal{I}$, i.e., x is considered as the image of the closed point of Spf $\hat{V}_x$. The local ring $A = \tilde{\mathcal{O}}_{\mathcal{X},x}$ has the following property: $B = \mathcal{O}_{\mathcal{X},x} = A[1/a]$ is a noetherian henselian local ring, whose residue field K_x is the quotient field of V_x (a is a generator of I) $A =$ the inverse image of V_x by the reduction map $B \to K_x$.

Conversely, any morphism Spf $V \to X$ from an adically complete valuation ring lifts uniquely to any admissible blowing ups by the valuative criterion, so the image of the closed point of V define a point x.

To define more general rigid spaces, which is inevitable if one treats the GAGA-functor, the following lemma is necessary:

Lemma. *For a coherent rigid space $\mathcal{X}$, the presheaf $\mathcal{Y} \to Hom_{\mathcal{R}}(\mathcal{Y}, \mathcal{X})$ on the large admissible site $\mathcal{R}$, is a sheaf.*

Definition. *A sheaf $\mathcal{F}$ on the big admissible site $\mathcal{R}$ is called a rigid space if the following conditions are satisfied:*

a) There is a morphism $\mathcal{Y} = \coprod_{i\in I} Y_i \to \mathcal{F}$ (Y_i are coherent representable sheaves) which is surjective.

b) Both projections $\mathrm{pr}_i : \mathcal{Y}\times_{\mathcal{F}}\mathcal{Y} \to \mathcal{Y}$ ($i = 1, 2$) are represented by open immersions.

c) $\mathcal{F}$ is quasi-compact if one can take quasi-compact $\mathcal{Y}$ in b).

d) $\mathcal{F}$ is quasi-separated if the diagonal $\mathcal{F} \to \mathcal{F} \times \mathcal{F}$ is quasi-compact.

We can show that if a rigid space in the above sense is quasi-compact and quasi-separated, then it is a representable sheaf, so the terminology " coherent rigid space " is compatible. Assume $\mathcal{F}$ is a quasi-separated rigid space. Then it is written as $\mathcal{F} = \varinjlim_{j\in J} \mathcal{X}_j$ where $\mathcal{X}_j$ is coherent , J is directed and all transition maps $\mathcal{X}_j \to \mathcal{X}_{j'}$ are open immersions. The definition has been used for a long time. For the construction of GAGA-functor for non-separated schemes quasi-separated spaces are not sufficient.

As an application of rigid-geometric idea, let me mention the following elementary example:

formal flattening theorem. *Let $\hat{f} : \hat{X} \to \hat{S}$ be a finitely presented morphism of formal schemes, with $\hat{S}$ coherent and of type v) or n). Assume $\hat{f}^{\mathrm{an}}$ is flat over $\hat{S}^{\mathrm{an}}$ (see the appendix for the definition of flatness). Then there is an admissible blow up $\hat{S}' \to \hat{S}$ such that the strict transform of $\hat{f}$ (kill torsions after taking the fiber product) is flat and finitely presented.*

The rigorous proof can be seen in the appendix. Another proof in case of noetherian formal schemes is found in [BL]. I explain the idea in case of flattening in the algebraic case [GR], i.e., when the morphism is obtained as the formal completion of a morohism of schemes $f : X \to S$. There is a principle to prove this kind of statement:

Principle. *Assume we have a canonical global procedure, an element of a cofinal subset $\mathcal{A}_S$ of all admissible blowing ups of S to achieve a property P. Assume the following properties are satisfied:*
a) P is of finite presentation.
b) The truth of $P(S')$ for $S' \in \mathcal{A}_S$ implies the truth of $P(S'')$ for all $S'' \in \mathcal{A}_S$ dominating S'.
c) P is satisfied at all stalks $\tilde{\mathcal{O}}_{\mathcal{X},x}$ of the model sheaf.
Then P is satisfied after some blowing up in $\mathcal{A}$.

Let $S \backslash U = V(\mathcal{I})$ with $\mathcal{I}$ finitely generated. $\mathcal{A}_S$ is the set of $\mathcal{I}$-admissible blowing ups, for which the total transform of $\mathcal{I}$ is invertible. $P(S')$ is : The strict transform of $X \times_S S'$ is flat and finitely presented over S'.
a) follows from the finite presentation assumption of the strict transform. b) is clear. For c), take a point of the Zariski-Riemann space $< \mathcal{X} >$. Then the local ring $A = \tilde{\mathcal{O}}_{\mathcal{X},x}$ has the property mentioned before. To prove the flattening in this case, using the flatness of $X \times_S \operatorname{Spec} A$ over $A[1/a]$ $(I = (a))$, we are reduced to the valuation ring case. i.e., prove the claim restricted to "curves" passing $V(\mathcal{I})$. In the valuation ring case ("curve case") there is no need for blowing up, and the strict transform just means that killing torsions. But note that we need to check the finite presentation of the result, i.e.,

Lemma. *For a finitely generated ideal I of $V[X]$ V a valuation ring, the saturation $\tilde{I} = \{f \in V[X];\ af \in I \text{ for some } a \in V\backslash\{0\}\}$ is finitely generated.*

The proof of this lemma is not so easy, but I leave it as an exercise.

So the claim is true locally on $< \mathcal{X} >$, since we have the finite presentation property. The quasi-compactness of $< \mathcal{X} >$ implies the existence of a finite covering, which admit models with the desired flattening property. The patching is unnecessary, i.e., it is automatically satisfied since we have a canonical global procedure to achieve the flattening, and once the flattening is achieved, we have it for all admissible blow up in $\mathcal{B}_S$ dominating the model.

Sometimes we want to use just "usual curves " i.e., Spec of a discrete valuation ring rather than general valuations. Sometimes it is possible. This is plausible, since the general valuation rings does not have any good finiteness conditions. (The value group such as $\mathbf{Z}^n$ with the lexicographic order is good, but even these are not enough sometimes.)

Another "toy model " is given by Gabber's extension theorem of locally free sheaves, which played an important role in Vieweg's semipositivity of the direct image of the dualizing sheaves. The structure of locally free module with respect to $\tilde{\mathcal{O}}$ is used: it can be proved that such a module come from some formal model.

Separation: Relation with Berkovich space

Here we give the explanation of a notion which was unclear in the classical theory. Let $\mathcal{X}$ be a coherent rigid space. For a point $x \in< \mathcal{X} >$ with associated valuation ring V_x, the point of $\mathcal{X}$ which corresponds to the height one valuation of K_x is denoted by $y = \mathrm{sep}(x)$ and called the maximal generalization of x (y corresponds to the minimal prime ideal containing an ideal of definition). Let $[\mathcal{X}]$ be the subset of $< \mathcal{X} >$ consisting of height one points. Then we give $[\mathcal{X}]$ the quotient topology by surjection $\mathrm{sep} :< \mathcal{X} >\to [\mathcal{X}]$ (caution: the section corresponding to the natural inclusion $[\mathcal{X}] \to< \mathcal{X} >$ is not continuous). This space $[\mathcal{X}]$ has an advantage that it is much nearer to our topological intuition. For example

Proposition. *$[\mathcal{X}]$ is a compact Hausdorff space. Basis of closed sets is $\{\mathrm{sep}(\mathcal{U})\}$, $\mathcal{U}$ a quasi-compact open subset ($\mathrm{sep}^{-1}(\mathrm{sep}(\mathcal{U})) = \overline{\mathcal{U}}$, where $^-$ denotes the closure).*

holds. Especially there is ample supply of $\mathbf{R}$-valued functions on $[\mathcal{X}]$. Dually, a basis of open sets is obtained as follows : First we define the notion of tubes. For a model X' of $\mathcal{X}$ and a closed set C of X' $T_C = (\mathrm{sp}^{-1}(C))^{\mathrm{int}}$ (int denotes the interior), is called the tube of C. In fact, tube of C is defined as $\varinjlim_n \mathrm{sp}^{-1}(U_n)$, where U_n is the open set of the blowing up by $(\mathcal{I}_C)^n + \mathcal{I}$ where the inverse image of $\mathcal{I}$ generates the exceptional divisor. T_C is the complement of $\overline{\mathrm{sp}^{-1}(X' \setminus C)}$. For a tube $T = T_C$, $\mathrm{sep}^{-1}\,\mathrm{sep}(T) = T$ holds, and hence $\mathrm{sep}(T)$ is an open set of $[\mathcal{X}]$, which is not compact in general. Images of tubes form a basis of open sets in $[\mathcal{X}]$. For most cohomological questions both topological space give the same answer:

Proposition.
For a sheaf $\mathcal{F}$ on $< \mathcal{X} >$, $R^q\,\mathrm{sep}_*\,\mathcal{F} = 0$ if $q > 0$. For a sheaf $\mathcal{G}$ on $[\mathcal{X}]$, $\mathrm{sep}_*\,\mathrm{sep}^{-1}\,\mathcal{G} = \mathcal{G}$.

We check the claim fiberwise, and reduce to to the valuation ring case.

The proposition includes $H^q(\bar{\mathcal{U}}, \mathcal{F}) = H^q(\mathcal{U}, \mathcal{F}|_{\mathcal{U}})$ $(= H^q([\mathcal{U}], \mathcal{G})$) for a sheaf $\mathcal{F} = \mathrm{sep}^{-1}(\mathcal{G})$ on $\bar{\mathcal{U}}$. Note that this does not apply to coherent sheaves. This is quite important in the theory of overconvergent isocrystals of Berthelot.

The estimate of cohomological dimension

Here we give the estimate of cohomological dimension of the Riemann space of a coherent rigid space. The result can be applied to the estimate of the cohomological dimension of étale topos of a rigid space.

In the noetherian case or the height one case, the proof is rather easy, and follows from the limit argument in SGA4 [Fu]. We have treated rigid spaces over valuation rings which may not be of height one. The estimate of cohomological dimension in this case is not so evident, so the necessary tools are included.

Theorem. *Let $\mathcal{X}$ be a coherent rigid space over an a-adically complete valuation ring R, R' be the a-adically complete height one valuation ring associated to V. Let d be the relative dimension of $\mathcal{X}$ (which is equal to the dimension of $\mathcal{X} \times_R R'$). Then the cohomological dimension of the Riemann space $< \mathcal{X} >$ is at most d.*

Claim. *Assume V is a valuation ring with fraction field K, and X is a finitely presented scheme over V. Then $H^i(X_{ZR}, \mathcal{F}) = 0$ for $i > d$, where d is the dimension of $X \times_V K$, and X_{ZR} is the Zariski-Riemann space of X in the classical sense (as a scheme).*

Assuming the claim, one gets the estimate in the theorem: We write

$$\mathcal{X}_{ZR} = \varprojlim_{i \in I} X_i,$$

where X_i are flat model of $\mathcal{X}$. We put $V = R/\sqrt{a}$. Then

$$\mathcal{X}_{ZR} = \varprojlim_{i \in I} (X_i \times_R V)_{ZR}$$

holds. Since $X_i \times_R V$ is just a scheme over V, we can consider the Riemann space in the classical sense. Then we apply the claim.

First step: Reduction to finite height case

V is written as

$$V = \varinjlim_{i \in I} A_i,$$

where A_i is a subring of V which is finitely generated over $\mathbf{Z}$. Consider the Riemann space Z_i of $\operatorname{Spec} A_i$ in the classical sense. V determines a point x_i in Z_i, i.e., a valuation ring $V_i \subset V$ which dominate A_i. Since A_i is finitely generated over $\mathbf{Z}$, V_i has a finite height. Then we have

$$V = \varinjlim_{i \in I} V_i,$$

V_i is a valuation ring with finite height.

Since X is finitely presented over V, by the standard limit argument in EGA, there are $i_0 \in I$ and a finitely presented scheme X_0 over V_{i_0} such that $X = X_0 \times_{V_{i_0}} V$, and

$$X = \varprojlim_{i \geq i_0} X_i,$$

where $X_i = X_0 \times_{V_{i_0}} V_i$. By the usual argument, $X_{ZR} \simeq \varprojlim_{i \geq i_0} (X_i)_{ZR}$, and it suffices to prove the claim for each X_i. So we are reduced to the finite height case.

Second step: Reduction to valuation ring case

We may assume that the height of V is finite. We prove the claim by induction on the height. When the base is a field, you get the estimate as you do in your thesis. So the height 0 case is OK.

Since the height is finite, there is some $a \in V \setminus \{0\}$ such that $K = V[1/a]$. $\operatorname{Spec} K \to \operatorname{Spec} V$ is a finitely presented open immersion. Let $j : X \times_V K \to V$ be the induced open immersion. Take a sheaf $\mathcal{F}$ on X_{ZR}. Let $\mathcal{G}$ be the kernel of $\mathcal{F} \to j_* j^* \mathcal{F}$, and $\mathcal{H} = \mathcal{F}/\mathcal{G}$. $\mathcal{H} \hookrightarrow j_* j^* \mathcal{H}$.

Consider the exact sequence

$$.. \to H^i(X, \mathcal{G}) \to H^i(X, \mathcal{F}) \to H^i(X, \mathcal{H}) \to ..$$

The support of $\mathcal{G}$ is in $X_{ZR} \setminus (X \times_V K)_{ZR}$. We write $X_{ZR} = \varprojlim_{j \in J} X_j$, X_j is flat over V, and dominates X. Then

$$X_{ZR} \setminus (X \times_V K)_{ZR} = \varprojlim_{j \in J} (X_j \times_V V/\sqrt{a})_{ZR}$$

holds (check it). The height of $V/\sqrt{a}$ is strictly smaller than that of V. By our induction hypothesis $H^i(X_{ZR}, \mathcal{G}) = 0$ for $i > d$. So we are reduced to the case of $\mathcal{F}$, $\mathcal{F} \hookrightarrow j_* j^* \mathcal{F}$. Similarly, we can reduce to the case of $\mathcal{F} = j_* j^* \mathcal{F}$.

Consider the higher direct image $R^s j_* j^* \mathcal{F}$. We calculate the fiber at $x \in X_{ZR}$. Since j induces quasi-compact and quasi-separated map on the Zariski-Riemann spaces (j is finitely presented), it is easy, and it is equal to $H^s(\operatorname{Spec} A[1/a], \mathcal{F}|_{\operatorname{Spec} A[1/a]})$. Here A is the valuation ring corresponding to x. Assume this vanishes for $s > 0$ at this moment. Then

$$H^i(X_{ZR}, j_* j^* \mathcal{F}) = H^i(X_{ZR}, R j_* j^* \mathcal{F}) = H^i((X \times_V K)_{ZR}, j^* \mathcal{F}).$$

We know the claim in the height 0 case. So we will finish the proof if we show the claim in the following case: $X = \operatorname{Spec} V$, V is a valuation ring with finite height.

Final step

Assume $X = \operatorname{Spec} V$. $H^i(X, \mathcal{F}) = 0$ for $i > 0$, since any open covering is refined by the total space X. The claim is proved.

§2. Comparison Theorems in rigid étale cohomology

Here fundamental theorems for rigid-étale cohomology are discussed. The origin for the study of rigid-étale theory is Drinfeld's work on p-adic upper half plane [D]. Most results here have applications in the study of modular varieties. The results, with many overlaps, are obtained by Berkovich for his analytic spaces (not rigid analytic one) over height one valuation fields. R. Huber has also obtained similar results for his adic spaces. The relation between these approaches will be discussed elsewhere.

We want to discuss étale cohomologies of rigid-analytic spaces. It is sometimes more convenient to use a variant of rigid-geometry, defined for henselian schemes instead of formal schemes.

In the affine case it is defined as follows. We take an affine henselian couple $(S, D) = (\text{Spec } A, \tilde{I})$: $D \subset S$ is a closed subscheme with $\pi_0(S' \times_S D) = \pi_0(S')$ for any finite S-scheme S' (hensel lemma). As an example, if S is $\mathcal{I}_D$-adically complete, $(S,\ D)$ is a henselian couple. Then to each open set $D \cap D(f) = \text{Spec } A[1/f]/I[1/f]$, $f \in A$, we attach the henselization of $A[1/f]$ with respect to $I[1/f]$. This defines a presheaf of rings on D. This is in fact a sheaf, and defines a local ringed space $\text{Sph}\, A$, called the henselian spectrum of A (as a topological space it is D, like a formal spectrum). General henselian schemes are defined by patching. See [Cox], [Gre], [KRP] for the details. We fix an affine henselian (or formal) couple (S, D). Put $U = S \setminus D$. We consider rigid geometry over S, i.e., rigid geometry over the henselian scheme attached to S. Of course we can work with formal schemes. Note on GAGA-functors: For a locally of finite type scheme X_U over U, there is a GAGA-functor which associates a general rigid space X_U^{rig} to X_U (X^{rig} is not necessarily quasi-compact, nor quasi-separated): Here are examples:

a) For X_U proper over U, $X_U^{\text{rig}} = (X^h)^{\text{rig}}$ (resp. $(\hat{X})^{\text{rig}}$). Here X is a relative compactification of X_U over S, the existence assured by Nagata. Especially the associated rigid space is quasi-compact (and separated) in this case.
b) In general X_U^{rig} is not quasi-compact, as in the complex analytic case. $(\mathbf{A}_U^1)^{\text{rig}}$ is an example. It is the complement of ∞_U^{rig} in $(\mathbf{P}_U^1)^{\text{rig}}$. This is associated with a locally of finite type formal (or henselian) scheme over S.
c) The GAGA-functor is generalized to the case of relative schemes of locally of finite presentation over a rigid space.

Rigid-étale topos

For simplicity I restrict to coherent spaces.

Definition.

a) A morphism $f : \mathcal{X} \to \mathcal{Y}$ is rigid-étale if it is flat (see the appendix for the definition of the flatness) and neat $(\Omega^1_{\mathcal{X}/\mathcal{Y}} = 0)$.

b) For a rigid space $\mathcal{X}$ we define the rigid étale site of $\mathcal{X}$ the category of étale spaces over $\mathcal{X}$, where covering is étale surjective. The associated topos is denoted by $\mathcal{X}_{\mathrm{et}}$.

For a coherent rigid space $\mathcal{X}$ the rigid-étale topos is coherent.

The reason for introducing the henselian version of the rigid analytic geometry in the study of étale topology lies in the following fact:

Categorical equivalence. *Let X be a henselian scheme which is good. Then consider the rigid henselian space $\mathcal{X} = X^{\mathrm{rig}}$. At the same time one can complete a henselian scheme, so we have a rigid-analytic space $\mathcal{X}^{\mathrm{an}} = (\hat{X})^{\mathrm{rig}}$. There is a natural geometric morphism*

$$\mathcal{X}^{\mathrm{an}}_{\mathrm{et}} \to \mathcal{X}_{\mathrm{et}}$$

since the completion of étale morphism is again étale, and surjections are preserved. Then the above geometric morphism gives a categorical equivalence.

The essential point here is the Artin Rees lemma, which assures the validity of Elkik's theorems on algebraization.

To prove the claim, we may restrict to coherent spaces. To show the fully-faithfulness one uses Elkik's approximation theorem [El] and some deformation theoretical argument to show morphisms are discrete. (The rigidity implies that an approximating morphism is actually the desired one.) For the essential surjectivity one can use Elkik's theorem in the affine case, since the patching the local pieces together is OK by the fully-faithfulness.

It is important to note the following consequence:

Corollary. *Let $(A_i, I_i)_{i\in I}$ be an inductive system of good rings, A_i I_i-adically complete. Then $\varprojlim_{i\in I} (\mathrm{Spf}\, A_i)^{\mathrm{an}}_{\mathrm{et}}$ is equivalent to $(\mathrm{Sph}\, A)^{\mathrm{rig}}_{\mathrm{et}}$, where $A = \varinjlim_{i\in I} A_i$, which is henselian along $I = \varinjlim I_i$. Here the projective limit is the 2-projective limit of toposes defined in SGA 4 .*

Since the above ring A is not I-adically complete in general (completion does not commute with inductive limit), the above equivalence gives the only way to calculate the limit of cohomology groups, especially calculation of fibers. This is the technical advantage of the introduction of henselian schemes. Moreover if we regard an affine formal scheme $X = \mathrm{Spf}\, A$ as a henselian scheme, i.e., $\tilde{X} = \mathrm{Sph}\, A$ with natural morphism $X \to \tilde{X}$ as ringed spaces, the induced geometric morphism $X^{\mathrm{rig}}_{\mathrm{et}} \to \tilde{X}^{\mathrm{rig}}_{\mathrm{et}}$ is a categorical equivalence so the "local" cohomological property of rigid analytic spaces is deduced from that of hensel schemes.

GAGA and comparison for cohomology

Let (S, D) be an affine henselian couple, X_U a finite type scheme over U. Then one has a geometric morphism

$$\epsilon : (X_U^{\mathrm{rig}})_{\mathrm{et}} \to X_{\mathrm{et}}$$

defined as follows: For an étale scheme Y over X_U, one associates Y^{rig}. Since GAGA-functor is left exact, and surjections are preserved, a morphism of sites is defined and gives ϵ. By the definition, $\epsilon^* F = F^{\ rig}$ for a representable sheaf F on X (we have used that F^{rig} is a sheaf on $(X_U^{\mathrm{rig}})_{\mathrm{et}}$). By abuse of notation we write $F^{\mathrm{rig}} = \epsilon^* F$ for a sheaf F on $(X_U)_{\mathrm{et}}$. Note that the morphism ϵ is not coherent, i.e., some quasi-compact object (such as an open set of X_U) is pulled back to a non-quasi compact object.

Theorem. *For a torsion abelian sheaf $\mathcal{F}$ on $(X_U)_{\mathrm{et}}$, the canonical map*

$$H^q_{\mathrm{et}}(X_U,\ \mathcal{F}) \simeq H^q_{\mathrm{et}}(X_U^{\mathrm{rig}},\ \mathcal{F}^{\mathrm{rig}})$$

is an isomorphism. The equivalence also holds in the non-abelian coefficient case, i.e., ind-finite stacks.

This especially includes Gabber's formal vs algebraic comparison theorem. The above theorem itself was claimed by Gabber in early 80's.

To deduce this form of comparison from the following form, Gabber's affine analogue of proper base change theorem [Ga] is used (if (S, D) is local, we do not have to use it). For the application to étale cohomology of schemes, see [Fu]. Especially the regular base change theorem, conjectured in SGA 4, is proved there (this is also a consequence of Popescu-Ogoma-Spivakovsky smoothing theorem).

Corollary (comparison theorem in proper case). *For $f : X \to Y$, proper morphism between finite type schemes over U, and a torsion abelian sheaf $\mathcal{F}$ on X, the comparison morphism*

$$(R^q f_* \mathcal{F})^{\mathrm{rig}} \to R^q f_*^{\mathrm{rig}} \mathcal{F}^{\mathrm{rig}}$$

is an isomorphism. Especially, for $\mathcal{F}$ constructible , $R^q f_*^{\mathrm{rig}} \mathcal{F}^{\mathrm{rig}}$ is again (algebraically) constructible (non-abelian version is also true, with a similar argument).

There is another (more primitive) version which includes nearby cycles. We will state the claim, with a brief indication of the proof. X a scheme, $i : Y \hookrightarrow X$ a closed subscheme with $U = X \setminus Y$. $j : U \hookrightarrow X$. Let $T_{Y/X} = \mathcal{X}_{\mathrm{et}}$, $\mathcal{X} = (X^h|_Y)^{\mathrm{rig}}$. (It is the analogue of (deleted) tubular neighborhood of Y in X). For any étale sheaf $\mathcal{F}$ on U one associates, by a patching argument, an object of $T_{Y/X}$ which we write as $\mathcal{F}^{\mathrm{rig}}$ ("restriction of $\mathcal{F}$ to the tubular neighborhood"). Note that there is a geometric morphism $\alpha_X : T_{Y/X} \to Y_{\mathrm{et}}$ ("fibration over Y").

Theorem. *For a torsion abelian sheaf $\mathcal{F}$ on U, there is an isomorphism*

$$i^* R j_* \mathcal{F} \simeq R(\alpha_X)_* \mathcal{F}^{\mathrm{rig}}.$$

If we apply this claim to a finite type scheme over a trait (or the integral closure of it in a geometric generic point), one knows that rigid-étale cohomology in the quasi compact case is just the hypercohomology of the nearby cycles:

Corollary. *Let V be a height one valuation ring, with separably closed quotient field $K = V[1/a]$. Let X be a finitely presented scheme over V, or $X = \mathrm{Spf}\, A$, A a good ring of type v) which is finitely presented over V. Let $\mathcal{F}$ be a torsion sheaf on X_K, or a torsion sheaf on* Spec $A[1/a]$. *Then*

$$R\Gamma((\hat{X})^{\mathrm{rig}},\ \mathcal{F}^{\mathrm{rig}}) = R\Gamma(X_s,\ i^* Rj_* \mathcal{F})$$

holds. Here $i : X_s = X \times_V \mathrm{Spec}\ V/\sqrt{a} \to X$ (or $i : \mathrm{Spec}\ A \times_V \mathrm{Spec}\ V/\sqrt{a} \to \mathrm{Spec}\ A$ in the affine formal case) and $j : X_K \to X$ (or $j : \mathrm{Spec}\ A_K \to \mathrm{Spec}\, A$ in the affine formal case).

The above mentioned comparison theorem follows from this theorem, using the Gabber's affine analogue of proper base change theorem. Let me give a brief outline of the proof. The underlying idea is quite topological. Put

$$Z = \varprojlim_{X' \in \mathcal{B}_X} X'_{\mathrm{et}}$$

($\mathcal{B}_X$ is the set of admissible blowing ups (in the scheme sense),

$$T^{\mathrm{unr}}_{Y/X} = \varprojlim_{X' \in \mathcal{B}_X} (X' \times_X Y)_{\mathrm{et}}$$

($T^{\mathrm{unr}}_{Y/X}$ is the analogue of tubular neighborhood of Y). The limit is taken as toposes. Then $U_{\mathrm{et}} \overset{j^{\mathrm{unr}}}{\to} Z \overset{i^{\mathrm{unr}}}{\leftarrow} T^{\mathrm{unr}}_{Y/X}$ is a localization diagram (U is an "open set" and T^{unr} is a "closed set" of Z.) Using the proper base change for usual schemes (here the assumption that $\mathcal{F}$ is torsion is used), one shows that

$$R\beta_*(i^{\mathrm{unr}\,*} Rj^{\mathrm{unr}}_* \mathcal{F}) = i^* Rj_* \mathcal{F}$$

($\beta : T^{\mathrm{unr}} \to Y_{\mathrm{et}}$). So we want to do a comparison on $T^{\mathrm{unr}}_{Y/X}$.

In fact, there is a morphism $\pi : T_{Y/X} \to T^{\mathrm{unr}}_{Y/X}$ ("inclusion of deleted tubular neighborhood") such that $R\pi_* \mathcal{F}^{\mathrm{rig}} = i^{\mathrm{unr}\,*} Rj^{\mathrm{unr}}_* \mathcal{F}$ (this formula is valid for any sheaf!). The construction is canonical. To calculate the fibers, one needs to treat a limit argument, so we take here an advantage of henselian version, not formal one.

In the non-proper case, i.e., f is of finite type but not assumed proper, the comparison is not true unless we restrict to constructible coefficients, torsion prime to residual characteristic of S. (Since the analytic topos involved is not coherent in this case, one can not use limit argument to deduce general torsion coefficient case. This is the same as **C**-case.) Though the author thinks that comparison is always true for finite type morphism between quasi-excellent schemes, the only known result, which is free from resolution of singularities, is the following height one case (a corresponding result for Berkovich type analytic spaces is obtained earlier in [Be]).

Theorem (comparison theorem in the non-proper case). *Let V be a height one valuation ring, with separably closed quotient field K. $f : X \to Y$ morphism between finite type schemes over K. Then*

$$(R^q f_* \mathcal{F})^{\mathrm{rig}} \to R^q f_*^{\mathrm{rig}} \mathcal{F}^{\mathrm{rig}}$$

is an isomorphism for $\mathcal{F}$ constructible sheaf, torsion prime to residual characteristics of V.

This is proved in [Fu] by a new variant of Deligne's technique in [De], without establishing the Poincaré duality. This geometric argument, more direct, reduces the claim for open immersions (evidently the most difficult case) to a special case, i.e., to o an open immersion of relative smooth curves over a smooth base. Moreover one can impose good conditions, such as smoothness and tameness of $\mathcal{F}$. In this case one can make an explicit calculation. Of course the comparison in the proper case, which is already stated, is used.

Using the comparison theorems, it is easy to see the comparison for $\otimes^{\mathbf{L}}$, $RHom$, f^*, f_*, $f_!$. The claim for $f^!$ follows from the smooth case. For the Poincaré duality in this case, using all the results I mentioned already, there are no serious difficulties except various compatibility of trace maps. Berkovich and Huber have announced such results already for their analytic spaces.

§4. Geometric ramification conjecture

In the following we discuss a geometric version of the upper numbering filtration on the absolute Galois group of a complete discrete valuation field.

Grothendieck has conjectured the following: $X = \mathrm{Spec}\ R$, R a strictly hensel regular local ring, $D = V(f) \subset X$ a regular divisor. Then for n invertible on X

$$H^i_{\mathrm{et}}(X \backslash D,\ \Lambda) = 0$$

if $i > 1$, $\Lambda = \mathbf{Z}/n\mathbf{Z}$. (For $i = 0,\ 1$ the group is easy to calculate.) Note that the conjecture is quite essential in the construction of cycle classes on general regular schemes. Moreover this conjecture implies the following: Assume the dimension of X is greater than 1. Then $\mathrm{Br}(X \setminus \{s\})_\ell = 0$. Here s denotes the closed point, Br means the Brauer group (we can take cohomological Brauer group) and ℓ is a prime invertible on X.

Gabber has announced that he can prove the absolute purity conjecture [Ga 4] (there is a note by the author of Gabber's lecture).

We try to explain how this conjecture is related to the birational geometry of X. In fact, our approach is similar to Hironaka's proof of "non-singular implies rational " in the continuous coefficient case. In his proof a stronger form of resolution of singularities was used, and we will try to do the same thing in the discrete coefficient case. But it turns out that the spectral sequence involved are bit complicated in the naive approach, so we will use log-structures of Fontaine-Illusie-Kato to avoid the difficulty.

The form of embedded resolution we want to use is the following:
For a pair $(X,\ Y)$, where X is a quasi-excellent regular scheme and Y is a reduced

normal crossing divisor, we define a good blowing up $(X',\ Y')$ by X' is the blowing up of X along D, where D is a regular closed subscheme of X which cross normally with Y. (The last condition implies that étale locally we can find a regular parameter system $\{f_j\}$, $1 \leq j \leq n$ such that Y is defined by $\prod_{i=1}^{m} f_i = 0$ and D is defined by $\{f_j = 0,\ j \in J\}$ for a subset J of $\{1,,...n\}$.) $Y' =$ total transform of Y_{red}.

We say $\pi : (X',\ Y') \to (X,\ Y)$ is a good modification if π is a composition of good blowing ups. The point is we can control normal crossing divisors.

Conjecture (Theorem of Hironaka in characteristic 0 [H] **).**
Let $\mathcal{C}_{X,Y}$ be the category of all good modifications of (X,Y), and $\mathcal{B}_{X,Y}$ the category of proper modifications of X which becomes isomorphic outside Y.
Then $\mathcal{C}_{X,Y}$ is cofinal in $\mathcal{B}_{X,Y}$.

Note that it is even not clear that $\mathcal{C}_{X,Y}$ is directed. Since any element in $\mathcal{B}_{X,Y}$ is dominated by admissible blowing ups, this conjecture is equivalent to the existence of a good modification which makes a given admissible ideal invertible.

So the conjecture is a strong form of simplification of coherent ideals, which is shown by Hironaka in characteristic zero. It is easy to see the validity of conjecture in dimension 2, but I do not know if it is true in dimension 3.

The implication of the conjecture in rigid geometry is the following: We define the tame part $T^{\text{tame}}_{Y/X}$ of $T_{Y/X} = \mathcal{X}_{rig-et}$ by

$$T^{\text{tame}}_{Y/X} = \varprojlim_{(X',\ Y') \in \mathcal{B}_{X,\ Y}} Y'_{log-et}$$

Here we give X' the direct image log-structure from $X' \backslash Y'$, and Y' the pullback log-structure. The limit is taken in the category of toposes. Since Y' is normal crossing, the behavior is very good. By the conjecture, we can determine the points of this tame tubular neighborhood (note that the topos has enough points by Deligne's theorem on coherent toposes in [SGA 4]) .

Lemma. *Let $\epsilon : T^{\text{tame}}_{Y/X} \to T^{\text{unr}}_{Y/X}$ be the canonical projection (defined assuming the conjecture). Then for a point x of $T^{\text{unr}}_{Y/X}$, which corresponds to strictly hensel valuation ring $V = V_x^{sh}$, the fiber product $T^{\text{tame}}_{Y/X} \times_{T^{\text{unr}}_{Y/X}} (\operatorname{Sph} V)^{\text{unr}}$ is equivalent to $(\operatorname{Sph} V)^{\text{tame}}$.*

So the points above x is unique up to non-canonical isomorphisms, which corresponds to the integral closure of V in the maximal tame extension of the fraction field of V. Using this structure of points we have

Proposition. *For any torsion abelian sheaf $\mathcal{F}$ on $T^{\text{tame}}_{Y/X}$ order prime to residual characteristics, we have*

$$R\alpha_* \alpha^* \mathcal{F} = \mathcal{F}.$$

Here α denotes the projection from $T_{Y/X}$.

This is just the fiberwise calculation (α is cohomologically proper), using that the Galois cohomology of henselian valuation fields without any non-trivial Kummer extension. (This part is completely the same as one dimensional cases.) Then our theorem is the following:

Theorem. *The conjecture implies Grothendieck's absolute purity conjecture.*

To see this, we use comparison theorem first.

$$R\Gamma(X\backslash Y,\ \Lambda) = R\Gamma(T_{Y/X},\ \Lambda)$$

By the proposition, this is equal to $R\Gamma(T^{\mathrm{tame}}_{Y/X},\ \Lambda)$. So we want to calculate this cohomology. Since the topos T^{tame} is defined as a 2-projective limit, we have

$$H^q(T^{\mathrm{tame}}_{Y/X},\ \Lambda) = \varinjlim_{(X',\ Y')\in\mathcal{C}_{X,Y}} H^q(Y'_{log-et},\ \Lambda)$$

So we conclude by the following lemma:

Lemma. *For a good modification* $\pi : (X',\ Y') \to (\tilde{X},\ \tilde{Y})$

$$R\pi'_*\Lambda = \Lambda,$$

where $\pi' : Y'_{log-et} \to \tilde{Y}_{log-et}$.

In fact, this is a consequence of the absolute purity conjecture. To prove the lemma, we may assume that π is a good blowing up. In this case we use proper base change theorem in log-etale theory, and reduce the claim to equicharacteristic cases. Especially to the relative purity theorem over a prime field.

Geometric Ramification Conjecture: Wild case

We end with a heuristic discussion on ramifications in the wild case, with the hope that the rigid-geometric method might be effective in dealing with the problem.

The ringed topos $T^{\mathrm{tame}}_{Y/X}$ should be the tame part of the full tubular neighbourhood $T_{Y/X}$, with the canonical projection $T_{Y/X} \to T^{\mathrm{tame}}_{Y/X}$. Even in the general case, we expect to have a filtration which generalizes the upper numbering filtration of the absolute Galois group of a complete discrete valuation field: $T_{Y/X}$ has a (enormously huge) log-structure with the following monoid:

$$M_{Y/X} = \tilde{\mathcal{O}}_{\mathcal{X}} \cap \mathcal{O}^{\times}_{\mathcal{X}}.$$

Here $\mathcal{X} = (\hat{X}|_Y)^{\mathrm{rig}}$ is the associated rigid space, and $\tilde{\mathcal{O}}_{\mathcal{X}}$ is the integral model of the structure sheaf $\mathcal{O}_{\mathcal{X}}$. $M^{\mathrm{gr}} = M/\tilde{\mathcal{O}}^{\times}_{\mathcal{X}}$ is the associated sheaf of groups. The stalk of M^{gr} at a point x is $K^{\times}_x/V^{\times}_x$.

Let $\mathcal{I}$ be the defining sheaf of ideals of Y in X. This choice of $\mathcal{I}$ determines a real valued map $\mathrm{ord}_{\mathcal{I}} : M^{\mathrm{gr}} \to \prod_{x\in\mathcal{X}} \mathbf{R}$, sending the local section m to $(\mathrm{ord}_{\tilde{x}}\, m)_{x\in\mathcal{X}}$. Here $\tilde{x}$ is the maximal generalization of x, i.e., the point corresponding to the height one valuation associated to x, and $\mathrm{ord}_{\tilde{x}}$ is the $\mathbf{R}$-valued additive valuation normalized as $\mathrm{ord}_{\tilde{x}}(a) = 1,\ (a) = \mathcal{I}_x$.

The kernel Ker $\mathrm{ord}_{\mathcal{I}}$ is independent of any choice of $\mathcal{I}$ (or formal models), and we denote it by $[M^{\mathrm{gr}}]$.

We put $N_{\mathbf{R}} = [M^{\mathrm{gr}}] \otimes_{\mathbf{Z}} \mathbf{R}$.

Then $N_{\mathbf{R}}$ has the following filtration $\{N^{\geq s}_{\mathbf{R}}\}_{s\in\mathbf{R}_{\geq 0}}$ indexed by $\mathbf{R}_{\geq 0}$:

$$N^{\geq s}_{\mathbf{R}} = \mathrm{ord}^{-1}_{\mathcal{I}}(\mathbf{R}_{\geq s}).$$

Here we embed $\mathbf{R}$ diagonally in $\prod_{x\in\mathcal{X}} \mathbf{R}$.

We can define $N^{>s}_{\mathbf{R}}$ in the same way.

Problem. *For each submonoid $\mathcal{N}$ of $N_{\mathbf{R}}$ containing $N_{\mathbf{R}}^{\geq s}$ for some $s > 0$, find a topos $T^{\mathcal{N}}$ with a projection $p_{\mathcal{N}} : T_{Y/X} \to T^{\mathcal{N}}$ with the following properties:*

1. $T^{N_{\mathbf{R}}^{\geq 0}} = T_{Y/X}^{\mathrm{tame}}$. *$p_{\mathcal{N}}^* : T^{\mathcal{N}} \to T_{Y/X}$ is fully-faithful. Moreover the filtration is exhaustive, i.e.,* $\cup_s p_{\mathcal{N}}^*(T^{N_{\mathbf{R}}^{\geq s}}) = T_{Y/X}$. $\mathbf{R}p_{\mathcal{N}*}p_{\mathcal{N}}^*\mathcal{F} \simeq \mathcal{F}$ *for a torsion "over-convergent" sheaf (in the sense of P. Schneider, the notion equivalent to étale sheaves in the sense of Berkovich) $\mathcal{F} \in T^{\mathcal{N}}$ with order prime to residual characteristics.*
2. *Assume X is regular, and Y is a normal crossing divisor. The "étale homotopy type" of $T^{\mathcal{N}}$ depends only on logarithmic scheme $X_n = \mathrm{Spec}\ \mathcal{O}_X/\mathcal{I}^n$ with the induced log-structure if $N_{\mathbf{R}}^{\geq n} \subset T^{\mathcal{N}}$. i.e., for two (Y, X) and (Y', X') with $X_n \simeq X'_n$ with log-structure, there is a correspondence between finite étale coverings $\mathcal{F}$, $\mathcal{F}'$ in $T_{Y/X}^{N_{\mathbf{R}}^{\geq n}}$, $T_{Y'/X'}^{N_{\mathbf{R}}^{\geq n}}$ with order prime to residual characteristics, and*

$$R\Gamma(T_{Y/X}^{N_{\mathbf{R}}^{\geq n}}, \mathcal{F}) \simeq R\Gamma(T_{Y'/X'}^{N_{\mathbf{R}}^{\geq n}}, \mathcal{F}')$$

   holds.

For classical complete discrete valuation rings (with perfect residue fields) the invariance in Problem 2 was found by Krasner in the naive form (the precise version is found in [De 2]). Except this case, the problem of defining the upper numbering filtration is quite non-trivial (the imperfectness of the residue field causes a difficulty). There is a very precise conjecture by T. Saito on the upper numbering filtration in this case. There is an attempt using the notion of "s-étaleness" which generalizes logarithmic étaleness, though the full detail will not be available so soon.

The more appropriate candidate than $N_{\mathbf{R}}$, including non-overconvergent sheaves, seems to be $M^{\mathrm{gr}} \otimes_{\mathbf{Z}} \mathbf{Q}$, i.e., "before $\mathbf{R}$", and expect filtration indexed by $\mathbf{Q}_{\geq 0}$.

Appendix: A Proof of Flattening Theorem in the Formal Case

§0. Introduction

In the following a proof of the flattening theorem in the formal case is given. The flattening theorem in the algebraic case was proved by L. Gruson and M. Raynaud [GR]. The corresponding theorem in the formal case is proved by M. Raynaud [R] for formal schemes over discrete valuation rings. F. Mehlmann [M] has given a detailed proof for formal schemes over height one valuation rings. S. Bosch and W. Lütkebohmert [BL2] treated both noetherian formal schemes and formal schemes over height one valuation rings. The proofs of [R], [M] and [BL2] are similar to the algebraic case in [GR].

We treat noetherian formal schemes and formal schemes over a valuation ring of arbitrary height. Our approach here is different from [R], [M], and [BL2], and analogous to O. Zariski's proof of resolution of singularities of algebraic surfaces. First we prove the theorem locally on the Zariski-Riemann space associated to the rigid space defined by the formal scheme. Using the quasi-compactness of the Zariski-Riemann space, we get the claim globally. The principle is quite general, and many problems can be treated in this way.

§1.Rigid Geometry

In this paper we consider adic rings which are good, i.e., a couple (A, I) which is either of the following:

type n) A is noetherian and I is arbitrary.

type v) A is topologically finitely generated over an a-adically complete valuation ring V and $I = (a)$.

Then we know that for any finitely presented algebra B over A and a finitely generated B-module M the Artin-Rees lemma is valid, $\hat{M} = M \otimes_B \hat{B}$, and $\hat{B}$ is flat over B (see [Fu] in case of type v)).

We say a coherent (quasi-compact and quasi-separated) formal scheme S is good of type n) (resp. type v)) if it is noetherian (resp. it is finitely generated over V).

This is compatible with the above definition for adic rings. When S is good of some type, we just say S is good.

Later we need some ideas from rigid geometry, so we review it here briefly. Let S be a coherent formal scheme with the ideal of definition $\mathcal{I}$. Then we define a local ringed space $<\mathcal{S}>$, the Zariski-Riemann space of $\mathcal{S}$, by

$$<\mathcal{S}>= \varprojlim_{S' \to S} S',$$

where S' runs over all admissible blowing ups [Fu, 4.1.3]. The structural sheaf obtained as the limit is denoted by $\tilde{\mathcal{O}}_S$. Call the canonical projection $<\mathcal{S}>\to S$ the specialization map, and denote it by $\mathrm{sp}_{\mathcal{S}}$. This map is surjective if some ideal of definition is invertible. It is easy to see that $<\mathcal{S}>$ is quasi-compact as a topological space.

We denote by $<\mathcal{S}>^{\mathrm{cl}}$ the points of $<\mathcal{S}>$ which define locally closed analytic subspaces of $\mathcal{S}$, and call an element a classical point of $\mathcal{S}$.

When $\mathcal{S}$ is defined by a good formal scheme we say $\mathcal{S}$ is good. In this case we have the following:

a) The rigid-analytic structural sheaf $\mathcal{O}_{\mathcal{S}} = \varinjlim_n \mathrm{Hom}(\mathcal{I}^n, \tilde{\mathcal{O}}_S)$ is coherent.

b) For an affine formal scheme $S = \mathrm{Spf}\, A$ with an ideal of definition $\mathcal{I} = (a)$, the coherent $\mathcal{O}_{\mathcal{S}}$-module $\mathcal{F}$ associated with an $A[1/a]$-module M satisfies $\Gamma(\mathcal{S}, \mathcal{F}) = M$. ($\mathcal{S} = S^{\mathrm{rig}}$.)

In case of type v), let V' be the height one valuation ring V localized at $\sqrt{a}$. V' is aV'-adically complete.

A coherent rigid space $\mathcal{X} = X^{\mathrm{rig}}$ over V defines $\mathcal{X}_{V'} = \mathcal{X} \times_V V'$ by base change. Let $j : \mathcal{X}_{V'} \to \mathcal{X}$ be the canoncal morphism. Then $<\mathcal{X}_{V'}>$ is a subspace of $<\mathcal{X}>$, and $\mathcal{O}_{\mathcal{X}} = j_* \mathcal{O}_{\mathcal{X}_{V'}}$ holds. The properties a) and b) are reduced to the height one case, where the claim is well known.

For $s \in \mathcal{S}$ the local ring $A = \tilde{\mathcal{O}}_{S,s}$ at s with $I = (\mathcal{I}\tilde{\mathcal{O}}_S)_s$ has the following property:

I is finitely generated and any finitely generated ideal containing a power of I is invertible.

We call such rings I-valuative [Fu, §3].

The notion of flatness in rigid geometry is defined as follows:

Definition 1.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a finite type morphism of rigid spaces and $\mathcal{F}$ a finite type $\mathcal{O}_{\mathcal{X}}$-module. $\mathcal{F}$ is called (rigid-analytically) f-flat iff all fibers $\mathcal{F}_x$ are flat $\mathcal{O}_{\mathcal{Y},f(x)}$-modules for all $x \in <\mathcal{X}>$.*

Proposition 1.2. *Assume $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{F}$ are defined by good adic rings B, A, a finitely generated B-module M, I is generated by a regular element a, and fibers of $\mathcal{F}$ are flat at all classical points. Then $M[1/a]$ is a flat $A[1/a]$-module.*

proof. Take a finitely generated $A[1/a]$-module N and take a resolution

$$\cdots \to L_1 \to L_0 \to N \to 0$$

with L_i finite free. Then the induced

$$\cdots \to \mathcal{L}_1 \to \mathcal{L}_0 \to \mathcal{G} \to 0$$

is exact. Then consider

$$\cdots \to \mathcal{F} \otimes \mathcal{L}_i \to \mathcal{F} \otimes \mathcal{L}_{i-1} \to .. \to \mathcal{F} \otimes \mathcal{L}_0 \to \mathcal{F} \otimes \mathcal{G} \to 0. \tag{1.3}$$

We see that the sequence 1.3 is exact since the coherence of the cohomology sheaves implies that they are zero iff their fibers at all classical points are zero. By the assumption that $\mathcal{F}_x$ is flat for all classical point x, cohomology sheaves vanish and hence the exactness follows.

Applying the global section functor Γ to 1.3 and using Γ is exact on coherent sheaves defined by $B[1/a]$-modules we know that

$$\to M \otimes L_i \to M \otimes L_{i-1} \to .. \to M \otimes N \to 0$$

is exact, i.e., $M[1/a]$ is flat.

Corollary 1.4. *Assume $\mathcal{Y}$ is good. Then $\mathcal{F}$ is f-flat iff $\mathcal{F}_x$ are flat $\mathcal{O}_{\mathcal{Y},f(x)}$-modules for all $x \in <\mathcal{X}>^{\mathrm{cl}}$.*

§2 Fibers

Let $\mathcal{S} = S^{\mathrm{rig}}$ be a coherent rigid space, $s \in \mathcal{S}$. Then we say a formal scheme T is a formal neighborhood of s if and only if T is an open subformal scheme of some admissible blowing up S' of S with $s \in T^{\mathrm{rig}}$.

Theorem 2.1. *Let B be a topologically finitely generated algebra over A, A a good I-adic ring and assume I is generated by a regular element a. Take $y \in \mathcal{Y}$ and put $V = V_y = \varinjlim_{A'} A'/J_{A'}$, $D = D_y = \varinjlim_{A'} B\hat{\otimes}_A A'/J_{A'}$, where $\operatorname{Spf} A'$ runs over formal neighborhoods of y, and $J_{A'}$ is the defining ideal of the closure of $\eta' \in \operatorname{Spec} A'$ (η' is the image of the generic point of V_y). Then for a finitely generated D-module P the strict transform of P is finitely presented.*

proof. We may assume $B = A\{\{X\}\} = A\{\{X_1, \dots, X_n\}\}$. Take a surjection from D^m to the strict transform of P. Let N be the kernel. N is a-saturated, i.e., $\{x \in D^m, a^s x \in N \text{ for some } s \in \mathbf{N}\} = N$. We prove any a-saturated submodule N of D^m is finitely generated.
First we prove the case $m = 1$, i.e., $N = I$ is an ideal of D.

We prove 2.1 using a formal version of the Groebner basis. Since our situation is different from the known cases, we establish a division lemma of Hironaka-Weierstrass type.

Put $L = \mathbf{N}^n$, with the standard monoid structure and the following total order (homogeneous lexicographic order):
For $\mu = (m_1, \dots, m_n)$ and $\mu' = (m'_1, \dots, m'_n)$,

$$\mu > \mu' \Leftrightarrow (\sum_{i=1}^{n} m_i, m_1, \dots, m_n) \text{ is bigger than } (\sum_{i=1}^{n} m'_i, m'_1, \dots, m'_n)$$

in the lexicographic order.

We say a submonoid $E \subset L$ is an ideal of L iff $E + L = E$. Then Dickson's lemma claims that any ideal of L are finitely generated, i.e., there exists a finite subset J of E such that

$$E = \cup_{j \in J}(j + L).$$

(Consider the sub $\mathbf{Z}[X]$-module of $\mathbf{Z}[X]$ generated by $X^e, e \in E$, and use the noetherian property.)

We define the notion of coefficients for an element in D. $V = \varinjlim_{A'} A'/J_{A'}$ dominates $A'/J_{A'}$ with $A'/J_{A'}$ integral. By this assumption transition maps $A'/J_{A'} \to A''/J_{A''}$, $B\hat{\otimes}_A A'/J_{A'} \to B\hat{\otimes}_A A''/J_{A''}$ are injective. We take a model A' where f is represented by $F \in A'\{\{X\}\}$.Then for $\mu \in L$, the μ-coefficient in V of expansion

$$F = \sum_{\nu \in L} a_\nu X^\nu$$

of F is independent of a choice of A'. We call this element in V μ-coefficient of f.

Next we claim the ideal

$$C_F = (a_\mu,\ \mu \in L,\ a_\mu \text{ is the } \mu\text{-coefficient of } F)$$

of V generated by the coefficients of f is finitely generated, and hence generated by one element since V is a valuation ring.

Take A' such that f comes from an element $F \in B\hat{\otimes}_A A'$. Since F is a-adically convergent series, there exists some s such that some coefficient is not in $I^s + J_{A'}$. The ideal I' generated by coefficients of F and I^s is a finitely generated admissible ideal. I' gives C_F.

If we denote a generator of C_F by $\mathrm{cont}(f)$,

$$\mathrm{cont}(fg) = \mathrm{cont}(f)\,\mathrm{cont}(g)$$

holds modulo units (Gauss's lemma).

Define $\nu(f) \in L$ by

$$\nu(f) = \{\sup \nu, a_\nu/\,\mathrm{cont}(f) \text{ is a unit} \quad (a_\nu \text{ is the } \nu\text{-coefficient of } f)\}.$$

It is easy to see ν satisfies

$$\nu(fg) = \nu(f) + \nu(g)$$

for $f,\ g \in D \setminus \{0\}$, and ν defines a valuation on D. The initial term of f is defined as

$$\mathrm{in}(f) = a_{\nu(f)} X^{\nu(f)}.$$

For any ideal $I \subset D$,

$$\nu(I) = \text{the ideal of } L \text{ generated by } \{\nu(f),\ f \in I\}.$$

Lemma 2.5 (division lemma). *For $f,\ g \in D$, assume $\mathrm{cont}(g) = 1$. Then there is unique $\beta \in D$ such that $f - \beta \cdot g$ has no exponents in $\nu(g) + L$.*

proof. Take a polynomial b in $V[X]$, and $C \in \sqrt{a}$ such that all ν-coefficients of $f - bg$, g, $\nu > \nu(g)$, and $\nu(g)$-coefficient of $f - bg$ are divisible by C.

Take a formal neighborhood A' of y such that $f - bg$, g are represented by $F, G, \in A'/J'_A\{\{X\}\}$, b, C by $\tilde{b}$, $\tilde{C} \in IA'/J_{A'}$.

Then, by induction on ℓ, we prove the existence of polynomials $\beta_\ell \in A'/J_{A'}[X]$ such that ν-coefficient of $G_\ell = F - \beta_\ell G$ for $\nu \in \nu(g) + L$, and $\beta_{\ell+1} - \beta_\ell$ are divisible by $\tilde{C}^\ell$ in $A'/J_{A'}$.

For $\ell = 0$ this is true with $\beta_0 = 0$. For an element H of $A'/J_{A'}\{\{X\}\}$, let $\mu_\ell(H)$ be the maximal exponent of H in $\nu(G) + L$ whose coefficient $A_{\mu_\ell(H)}$ is not divisible by $\tilde{C}^{\ell+1}$ in $A'/J_{A'}$.

Put $H' = H - A_{\mu_\ell(H)} X^{\mu_\ell(H) - \nu(G)} G$. $\mu_\ell(H')$ is strictly less than $\mu_\ell(H)$. Continuing this process finite times from $H = G_\ell$, we have such $\beta_{\ell+1}$.

The sequence $\{\tilde{b}+\beta_\ell\}$ is $\tilde{C}$-adically convergent in $A'/J_{A'}\{\{X\}\}$, and the existence of β is proved. Uniqueness is clear from the additivity of ν.

Sublemma 2.6. *There is a unique upper-triangular matrix Γ with entries in D and the diagonal component 1 such that $(\tilde{n}_i) = \Gamma \cdot (n_i)$ satisfy the following: $\tilde{n}_i$ has no exponent in F_{i+1}.*

We prove this by a descending induction on j starting from $j = \ell$. Assume we have constructed $\tilde{n}_j$, $j > \alpha$. Applying division lemma to $f = n_t$ and $g = n_\ell = \tilde{n}_\ell$, there is α_ℓ such that $n_t^1 = n_t - \alpha_\ell \cdot n_\ell$ has no exponent in F_ℓ. Since n_t^1 and $\tilde{n}_{\ell-1}$ have no exponents in F_ℓ, any D-linear combination of them has no exponent in $F_\ell \backslash F_{\ell-1}$. By division lemma again for $f = n_t^1$ and $g = \tilde{n}_{\ell-1}$, we can find $\alpha_{\ell-1}$ such that $n_t^2 = n_t^1 - \alpha_{\ell-1} \cdot \tilde{n}_{\ell-1}$ has no exponents in $\nu_{\ell-1} + L$ and hence $F_{\ell-1}$. Continuing this process, we have $\tilde{n}_t$ with the desired property.

Put $E(N) =$ the ideal of L generated by $\{\nu(f); f \in N\}$.

Take generators $\nu_i (i \le i \le \ell)$ of $E(N)$ such that $F_j =$ the ideal generated by ν_s, $s \ge j$, satisfies $F_\ell \not\subset F_{\ell-1} \not\subset \ldots \not\subset F_1$.
For each ν_i we take $n_i \in N$ satisfying $\nu(n_i) = \nu_i$ and $\mathrm{cont}(n_i) = 1$ using the saturation hypothesis.

To prove claim, it suffices to show the following:

Claim 2.4. *$\{n_i\}_{1 \le i \le \ell}$ generates N.*

For $n \in N$ there is unique $\tilde{\beta}_\ell \in D$ such that $m_\ell = n - \tilde{\beta}_\ell \cdot \tilde{n}_\ell$ has no exponent in F_ℓ by division lemma applied to $f = n$ and $g = \tilde{n}_\ell$. Continuing this for m_ℓ and $\tilde{n}_{\ell-1}$,.., we have $\tilde{\beta}_i$ such that $n = \sum_{i=1}^{\ell} \tilde{\beta}_i \cdot \tilde{n}_i$, and the existence of β_i follows from sublemma. This $\{\tilde{\beta}_i\}$ has the property that $\sum_{i \le j} \tilde{\beta}_i \cdot \tilde{n}_i$ has no exponents in F_{j+1}. For the uniqueness, if we have a presentation $0 = \sum \tilde{\beta}_i \cdot \tilde{n}_i$ we may assume $\tilde{\beta}_j = 1$ and $\tilde{\beta}_i = 0$ for some $i > j$. Then the exponent ν_j should appear in $\sum_{1 \le i \le j-1} \tilde{\beta}_i \cdot \tilde{n}_i$ which is contradiction.

Now we prove the general case by Nagata's trick, assuming $B = A\{\{X\}\}$. Take an a-saturated submodule N of D^m. From the ideal case we have just proved, $D[1/a]$ is a noetherian ring since any ideal I of $D[1/a]$ admits an a-saturated extension $\tilde{I}$ to D, which is finitely generated. Then we can find a finitely generated submodule $\tilde{N}$ of N such that $\tilde{N}[1/a]$ generates $N[1/a]$. This $\tilde{N}$ comes from a finitely generated submodule N' of $A'\{\{X\}\}$ by a standard limit argument, where $\mathrm{Spf}\, A'$ is a formal open neighborhood of y. We may assume $A = A'$ by replacing A, i.e., there is a finitely generated B-submodule $\mathcal{N}$ of B^m such that $\mathcal{N}$ gives $\tilde{N}$. Since $N/\tilde{N}$ is a-torsion it suffices to prove a-torsions in $M \otimes_B D$ is finitely generated as a D-module, where $M = B^m/\mathcal{N}$.

For M, we put $B_*M = B \oplus M$, the split algebra extension of B by M. So the multiplication rule is $(b_1, m_1) \cdot (b_2, m_2) = (b_1 \cdot b_2,\ b_1 m_2 + b_2 m_1)$. Since M is a finitely presented B-module, $\hat{M} = M \otimes_B \hat{B} = M$, and hence B_*M is a-adically complete. Moreover $D_{B_*M} = D \oplus M \otimes_B D$ holds. Applying the ideal case to D_{B_*M}, we get that the a-torsions in $M \otimes_B D$ form a finitely generated D-module.

Remark. We have a canonical way to choose β_i in $n = \sum \beta_i \cdot n_i$.

Corollary 2.7. *$\hat{D}$ is faithfully flat over D.*

§3 Flattening

Continuity lemma 3.1 (cf. EGA chap IV lemme 11.2.5). *Assume we are given a projective system $\{A_j\}_{j\in J}$ of good adic rings. Assume there is a minimal element $j_0 \in J$, and I is an ideal of definition of $A_0 = A_{j_0}$. B_0 is a topologically finitely presented A_0-algebra, and M_0 is a finitely presented B_0-module. Put $B_j = B\hat{\otimes}_A A_j$, $M_j = M \otimes_{B_0} B_i$, $A = \varinjlim_{j\in J} A_i$, $B = \varinjlim_{j\in J} B_j$ and $M = \varinjlim_{i\in J} M_j = M_0 \otimes_{B_0} B$. Assume M/IM is a flat A/I-module and $\mathrm{Tor}_1^A(M, A/I) = 0$. Then $M_{j'}$ is $A_{j'}$-flat for some $j' \geq j_0 \in J$.*

proof. Since $M/IM = \varinjlim_{j\in J} M_j/IM_j$ is flat over A/I, by [GD, corollaire 11.2.6.1] there is $j_1 \geq j_0 \in J$ such that M_j/IM_j is flat over A_j/IA_j for $j \geq j_1$. Since $\mathrm{Tor}_1^{A_{j_1}}(M_{j_1}, A_{j_1}/IA_{j_1})$ is a finitely generated B_{j_1}-module, the vanishing of $\mathrm{Tor}_1^A(M, A/IA)$ means that there exists $j' \in J$, $j' \geq j_1$ such that the image of $\mathrm{Tor}_1^{A_{j_1}}(M_{j_1}, A_{j_1}/IA_{j_1})$ in $\mathrm{Tor}_1^{A_{j'}}(M_{j'}, A_{j'}/IA_{j'})$ is zero. We apply the following lemma.

Sublemma 3.2. *Let A be a good I-adic ring, A' a good IA'-adic A algebra, B a topologically finitely presented algebra over A, and M a finitely presented B-module. $B' = B\hat{\otimes}_A A'$, $M' = M \otimes_B B'$. Then the canonical map*

$$\mathrm{Tor}_1^A(M, A/IA) \otimes_A A' \to \mathrm{Tor}_1^{A'}(M', A'/IA')$$

is surjective if M/IM is a flat A/I-module.

proof. We may assume $B = A\{\{X\}\}$. We take $0 \to N \to L \to M \to 0$, L: a finite free B-module, N: a finitely generated B-module. Since A' is a good adic ring, we have the exactness of $0 \to N' \to L_0\hat{\otimes}_B B' \to M' \to 0$, $N' = N\hat{\otimes}_B B'$. Since $L\hat{\otimes}_B B'$ are A'-flat,

$$0 \to \mathrm{Tor}_1^{A'}(M', W) \to N' \otimes_{A'} W \to L\hat{\otimes}_B B' \otimes_{A'} W \to M' \otimes_{A'} W \to 0$$

for any A'-module W. Especially, $\mathrm{Tor}_1^{A'}(M', A'/IA') = \mathrm{Tor}_1^{A'}(M \otimes_A A', A'/IA')$. Then the claim follows from [GD, lemme 11.2.4].

By the sublemma $\mathrm{Tor}_1^{A_j}(M_j, A_j/IA_j)$ vanishes. Note that local criterion for the flatness is true for good adic rings using the Artin-Rees lemma, so M_j is flat over A_j.

Theorem 3.3 (Flattening theorem). *Let Y be a good coherent formal scheme, $f : X \to Y$ a finitely generated morphism and $\mathcal{F}$ a finitely generated module which is rigid-analytically f-flat. Then there exists an admissible blowing up $Y' \to Y$ such that the strict transform of $\mathcal{F}$ is flat and finitely presented.*

proof. We may assume that $X = \mathrm{Spf}\ B$, $Y = \mathrm{Spf}\ A$ are affine, the defining ideal I of A is generated by a regular element a, and $\mathcal{F}$ is defined by a B-module M.

Take $y \in \mathcal{Y}$ and put $\tilde{A} = \tilde{\mathcal{O}}_{\mathcal{Y},y}$, $\tilde{B} = \varinjlim B\hat{\otimes}_A A'$, where Spf A' runs over affine formal neighborhoods of y. Then $\tilde{A}$ is $a\tilde{A}$-valuative, $\tilde{A}[1/a]$ is a local ring with residue field K, $J = \cap_n a^n\tilde{A}$ is the maximal ideal of $\tilde{A}[1/a]$, and $\tilde{A}/J$ is a valuation ring separated for $a\tilde{A}/J$-adic topology.

First we prove the claim locally on the Zariski-Riemann space of Y^{rig}.

Lemma 3.4. *The strict transform of $M \otimes_B \tilde{B}$ is a finitely presented $\tilde{B}$-module.*

By our assumption of rigid-flatness $M \otimes_{\tilde{B}} \tilde{B}[1/a]$ is flat (proposition 1.3). Then $\mathrm{Tor}_1^{\tilde{A}}(M \otimes \tilde{B}[1/a], K) = 0$. From

$$0 \to J \to \tilde{A} \to \tilde{A}/J\tilde{A} \to 0$$

we have

$$0 \to \mathrm{Tor}_1^{\tilde{A}}(M, \tilde{A}/J\tilde{A}) \to J \otimes_{\tilde{A}} M \to M \to M/JM \to 0$$

Since $J \otimes_{\tilde{A}} M$ is a-torsion free, $\mathrm{Tor}_1^{\tilde{A}}(M, \tilde{A}/J) = 0$ by our assumption that this module is a-torsion and $JM = J \otimes_{\tilde{A}} M$ is a-torsion free. We show that $(M \otimes_B \tilde{B})_{a-\mathrm{tors}} \to (M/JM)_{a-\mathrm{tors}}$ is bijective. The injectivity is clear since JM has no non-zero a-torsions. For the surjectivity, take an element $m \in M$ which is mapped to an a-torsion element in M/JM. $am \in JM$, thus there exists $n \in JM$ such that $an = am$, and $m - n$ have the same image as m in M/JM.

$(M \otimes_B \tilde{B})_{a-\mathrm{tors}} = (M/JM)_{a-\mathrm{tors}}$ is finitely generated as a $\tilde{B}$-module by 2.1.

We globalize the local result using the quasi-compactness of the Zariski-Riemann space.

We take $\mathcal{B}$ as the category of the admissible blowing up Y' for which $\mathcal{I}\mathcal{O}_{Y'}$ is invertible. For an admissible blowing up $Y' \in \mathcal{B}$, let $C_{Y'}$ be the subset of Y' defined as

$$C_{Y'} = \{y' \in Y', \text{ the strict transform of } \mathcal{F}_{Y'} \text{ is not flat over } \mathcal{O}_{Y',y'}\}.$$

Then $C_{Y'}$ is closed. This follows from the continuity lemma 3.1. When there is a morphism $Y'' \to Y'$ in $\mathcal{B}$, $C_{Y''}$ is mapped to $C_{Y'}$. We prove $C_{Y'} = \emptyset$ for some $Y' \in \mathcal{B}$. If not, $\mathrm{sp}_{Y'}^{-1} C_{Y'} \subset < \mathcal{Y} >$ are non-empty, using the surjectivity of specialization map. For a finite set I and $Y_i \in \mathcal{B}$, $i \in I$ there is $\tilde{Y} \in \mathcal{B}$ dominating all Y_i, $i \in I$. This means that $\{\mathrm{sp}_{Y'}^{-1} C_{Y'}\}_{Y' \in \mathcal{B}}$ has finite intersection property. Since $< \mathcal{Y} >$ is quasi-compact, the intersection is non-empty, and we have a point y of $< \mathcal{Y} >$ which is mapped to $C_{Y'}$ by the specialization map for any $Y' \in \mathcal{B}$.

Over the local ring $\tilde{A} = \tilde{\mathcal{O}}_{\mathcal{Y},y}$ the strict transform of $M \otimes_B \tilde{B}$ is finitely presented. Writing $\tilde{A}$ as the limit of affine formal neighborhoods $\mathrm{Spf}\, A'$ of y, by the standard limit argument, there exists an affine formal neighborhood $\mathrm{Spf}\, A'$ of y such that there exists a finitely presented $B\hat{\otimes}_A A'$ module N with $M\hat{\otimes}_A A' \xrightarrow{\lambda} N$ which gives the strict transform of $M \otimes_B \tilde{B}$ over $\tilde{A}$. Both $\mathrm{Ker}\,\lambda$ and $\mathrm{Coker}\,\lambda$ are finitely generated since N is finitely presented. Since the limit of $N\hat{\otimes}_{A'} A''$, where $\mathrm{Spf}\, A''$ runs over formal neighborhoods of y dominating $\mathrm{Spf}\, A'$, is a flat $\tilde{A}$-module by our construction, by continuity lemma 3.1 $N\hat{\otimes}_{A'} A''$ is flat for some A''. Replacing A'' further, we may assume $\mathrm{Ker}\,\lambda_{A''}$ is an a-torsion module and $\mathrm{Coker}\,\lambda_{A''}$ is zero using the finite generation. So there exists some affine open neighborhood U_y of y in $Y'' \in \mathcal{B}$, on which $\mathcal{F}_{U_y}/\mathcal{F}_{U_y\ a-\mathrm{tors}}$ is f_{U_y}-flat and finitely presented. This is a contradiction, and the strict transform of $\mathcal{F}_{Y'}$ is $f_{Y'}$-flat for some $Y' \in \mathcal{B}$, and hence for all $Y'' \in \mathcal{B}$ dominating Y'.

For the finite presentation, note that we have find an affine open neighborhood $U_y \subset W_y$, $W_y \in \mathcal{B}$ for any $y \in < \mathcal{Y} >$ where the strict transform is finitely presented.

We take a finite subset $I \subset < \mathcal{Y} >$ such that U_i^{rig}, $i \in I$ covers $< \mathcal{Y} >$ by the quasi-compactness. By blowing up Y' further, we may assume Y' dominates all W_i, $i \in I$. Let $\pi_i : Y' \to W_i$ be the projection. Then $\pi_i^{-1}(U_i)$ covers Y' by the surjectivity $\mathrm{sp}_{Y'}$, and the strict transform of $\mathcal{F}_{Y'}$ is finitely presented.

Corollary 3.5.
Assumptions are as in 3.3, and assume moreover that $\operatorname{Supp} \mathcal{F} = \mathcal{X}$. *Then f is an open map.*

§4 Theorem of Gerrizen-Grauert

The following theorem is used to give the relation between rigid spaces in the classical sense and the definition of Raynaud.

Theorem.
Let $f : X \to Y$ is a separated morphism of finite type between good formal schemes, and let f^{rig} be the morphism between the asociated rigid spaces $f^{\mathrm{rig}} : \mathcal{X} = X^{\mathrm{rig}} \to \mathcal{Y} = Y^{\mathrm{rig}}$. Assume f^{rig} satisfies $\mathcal{O}_{\mathcal{Y}, f^{\mathrm{rig}}(x)} \to \mathcal{O}_{\mathcal{X},x}$ is an isomorphism for $x \in < \mathcal{X} >^{\mathrm{cl}}$. Then there is an admissible blowing up $Y' \to Y$ such that the strict transform of $f_{Y'}$ is an open immersion.

This immediately follows from the formal flattening theorem. Here we give a simpler proof based on Elkik's approximation theorem.

We may assume $\mathcal{Y}$ is affine, and the ideal of definition is generated by one element a. f^{rig} is a rigid etale morphism in the sense of [Fu, 5.1.2]. By [Fu, 5.1.3], there is a henselian scheme $\tilde{f} : \tilde{X} \to \tilde{Y}$ such that $\hat{\tilde{X}} = X$. Here $\tilde{Y} = \operatorname{Sph} \Gamma(Y, \mathcal{O}_Y)$. By [Fu, lemma 3.2.2], there is an admissible blowing up $\tilde{Y}'$ of $\tilde{Y}$ such that the strict transform of $\tilde{f}_{\tilde{Y}'}$ is flat. (Note that we have used the flattening theorem in the algebraic case only.) By passing to the completion, the strict transform of $f_{Y'}$ is flat for $Y' = \hat{\tilde{Y}}'$ (in case of type v), we use Gabber's theorem [Fu, 1.2.3]). The rest of the argument is the same.

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