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Introduction to the Abhyankar's conjecture for $\mathbb{P}^1 \setminus \{\infty\}$ for the case $G \neq G(S)$ after M. Raynaud

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1 Introduction

This note is a resume of my talk of the title "the Abhyankar's conjecture proved by Raynaud II" in a symposium entitled with "Rigid geometry and group actions", which was held at Kyoto in May 1998. In this note we make a brief introduction of Abhyankar's conjecture for $\mathbb{P}^1 \setminus \{\infty\}$ for the case $G \neq G(S)$ (See (1.2) below for the definition of $G(S)$), which was solved by Raynaud.

Let k be an algebraically closed field of finite characteristic $p > 0$ and let R be a complete discrete valuation ring of mixed characteristics with residue field k and fraction field K. Let $\mathcal{X}$ be a smooth proper curve over R and let $\phi \neq \mathcal{D} \subset \mathcal{X}$ be the relative normal crossing divisor over R. Put $\mathcal{U} := \mathcal{X} \setminus \mathcal{D}$. Let X (resp. U) be the special fiber of $\mathcal{X}$ and $\mathcal{U}$ and let $X_{\overline{K}}$ (resp. $U_{\overline{K}}$) be the geometric generic fiber of $\mathcal{X}$ and $\mathcal{U}$. Then it is shown in [SGA 1] that $\pi_1^{\mathrm{tame}}(U,*) \simeq \pi_1(U_{\overline{K}},*)$. Consequently $\pi_1^{\mathrm{tame}}(U,*)$ can be determined by the classical topological method since we may assume that $\overline{K}$ is the complex number field [SGA 1]. Here the superscription "tame" means the tame part of a profinite group. As a corollary we have the following: If U has an etale

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covering U' with finite Galois group G, then we have the surjection

$$\pi_1(U_{\overline{K}}, *) \longrightarrow G/p(G). \tag{1.1}$$

Here $p(G)$ is the quasi-p-part of G, that is, a group which is generated by the elements of G with p-power orders.

The Abhyankar's conjecture claims the converse: Let G be a finite group. If (1.1) is a surjection , G is realized as the Galois group of an etale covering U' over U. Raynaud has proved this conjecture for $\mathbb{P}^1_k \setminus \{\infty\}$ ([R2]) and Harbater has proved it for the case of the curves by using the result of Raynaud ([H]). Since $\pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus \{\infty\})$ is trivial, the Abhyankar conjecture for it says that, if $G = p(G)$, G is realized as a Galois etale covering of $\mathbb{P}^1_k \setminus \{\infty\}$. The proof by Raynaud for the conjecture is divided into two parts. Let S be a Sylow subgroup of G and let $G(S)$ be the following subgroup

$$G(S) := \langle G_i \subsetneq G | G_i : \text{quasi} - p - \text{group which has a Sylow subgroup contained in } S \}. \tag{1.2}$$

The proof in the case $G = G(S)$ is treated in [Su]. In this note we treat the case where $G \neq G(S)$ and G has non-trivial invariant p-subgroup of G. The induction on the order of G and a theorem which is stated in [Su] enables us to assume the latter condition.

The detailed proof is omitted. See [R2] for it. We try to explain the feeling and the meaning of statements.

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2 Plans of the proof

The proof of the conjecture in the case where $G \neq G(S)$ and G has non-trivial invariant p-subgroup of G is too complicated (M. Raynaud himself said to me that his success in the proof is a miracle). So we first give a rough idea of the proof. Let k be an algebraic closed field of finite characteristic $p > 0$ and let G be a quasi-p-group. The strategy for the proof of the conjecture is as follow:

1) We take a mixed characteristics complete discrete valuation ring R with residue field k and we consider a lifting of $\mathbb{P}^1_k$, that is, $\mathbb{P}^1_R$.
Let K be the fraction field of R.
2) We consider a ramified Galois covering Y_K of the generic fiber $\mathbb{P}^1_K$ with Galois group G such that the inertia groups generate G.
Let Y be the normalization of $\mathbb{P}^1_R$ in Y_K. In general the singularity of the special fiber Y_k is bad.
3) We take an extension of R (if necessary) and we have the semi-stable curve Y' by using the semi-stable reduction for curves.

Until here the proof is very natural. If the last step of the proof were as follows, the proof is able to understood for an ordinary man.
4) We find constructibly a suitable smooth component Z_k of Y'_k which is a covering with Galois group G of $\mathbb{P}^1_k$ and which is etale outside ∞_k.

The large part of this report is devoted to the non-real part 4).

3 Semi-stable reduction of curves

Let p be a prime and let G be a non-trivial quasi-p-group. We take a generator $\alpha_1, \dots, \alpha_m$ of G whose orders are p-powers. We take a large number m if necessary and we may assume $\alpha_1, \dots, \alpha_m$ has an relation $\alpha_1 \cdots \alpha_m = 1$. Let R be a mixed characteristics complete discrete valuation ring with residue field k. We put $P := \mathbb{P}^1_R$. We take different m-sections $h_1, \dots, h_m$ $(m \geq 1)$ $(\operatorname{supp} h_i \cap \operatorname{supp} h_j = \phi\ (i \neq j))$ of $P(R) = P_K(K)$ and we put $U := P \setminus \{h_1, \dots, h_m\}$. Then $\pi_1(U_{\overline{K}})$ has generators $\sigma_1, \dots, \sigma_m$ with one relation $\sigma_1 \cdots \sigma_m = 1$. Therefore we have a surjection

$$\pi_1(U_{\overline{K}}) \ni \sigma_i \longmapsto \alpha_i \in G.$$

Consequently we have an etale Galois covering V over $U_{\overline{K}}$ with Galois group G. Take the smooth projective model $Y_{\overline{K}}$ of V and make a normalization Y of P in $Y_{\overline{K}}$. In general the singularity of the special fiber Y_k is bad. Hence we need the semi-stable reduction theorem. Before giving the statement we recall the definition of the semi-stable curves.

Definition 3.1. Let g be a non-negative integer. Let S be a scheme and let X be an S-scheme. X/S is called an semi-stable curve of genus g if the following four conditions are satisfied:
1) X/S is proper and flat.
2) For all geometric points $\overline{s}$, $X_{\overline{s}}$ is reduced and connected and 1-dimensional.
3) For all geometric points $\overline{s}$, $X_{\overline{s}}$ has at most ordinary double points as singularities, that is, the completions of the structure sheaf of $X_{\overline{s}}$ at the closed points are isomorphic to $k(\overline{s})[[x]]$ or $k(\overline{s})[[x,y]]/(xy)$.
4) For all geometric points $\overline{s}$, $\dim_{k(\overline{s})} H^1(X_{\overline{s}}, \mathcal{O}) = g$.

Theorem 3.2 ([DM], [AW]). *Let A be a complete discrete valuation ring with fraction field F. Let X_F be a proper smooth geometrically connected curve over F. Then there exists a finite extension B of A and a semi-stable curve X' over B such that $X' \underset{B}{\otimes} \operatorname{Frac} B \simeq X_F \underset{F}{\otimes} \operatorname{Frac} B$. Moreover we can take a regular model X'.*

By this theorem, and by blowing ups and extending the base ring R, we may assume Y_k is simple normal crossing, i.e. all the irreducible components are smooth. Corresponding the vertexes (resp. edges) to the irreducible component of Y_k (resp. the double curves) we have the dual graph $\Gamma(Y_k)$ of Y_k. We can have the regular minimal model Y' of Y_K ([L]). Y' has semi-stable reduction. By the minimality of Y', the Galois group G acts on Y' and as a result it acts on $\Gamma(Y'_k)$. By [R1] Appendice, the quotient $P' = G \setminus Y'$ is a semi-stable curve. It is easy to check that the irreducible components of the special fiber P'_k does not have a node by (5.0) below and there is a component C which induces a finite morphism $C(\subset P') \longrightarrow P_k(\subset P)$. We take C as a origin of the dual graph of P'_k. These facts is essentially used only in the proof of (5.4) below. The horizontal sections $h_1, \ldots, h_m \in P(R) = P_K(K)$ define sections $h'_1, \ldots, h'_m \in P'(R)$. By blowing ups of P' and Y' if necessary we may assume the following:

Assumption $(*)$: $h'_1, \ldots, h'_m$ are on the smooth locus of P'.

This condition assures that we can take an etale neighborhood of a double point of a special fiber such that the morphism of the generic fiber $Y_K \longrightarrow P_K$ is etale. This condition is needed in the local calculation of dualizing sheaf Y'/R (See (5.2) below.).

4 Graphs

In this section we define an inertia group of a vertex of a finite graph. Let us consider the finite graph Γ on which there is an action of a finite group G. If $\Gamma = \Gamma(Y_k')$ and G is a quasi-group in §3, we call this situation the geometric case. In the geometric case, for a vertex v, let us denote by (v) the irreducible components corresponding to v. The decomposition group $D_{(v)}$ of (v) is recovered by the graph $\Gamma(Y_k')$: $D_{(v)}$ is equal to the stabilizer of v. However the inertia group of (v) is not. (The inertia group of a double point is recovered: the stabilizer of the edge (Note that the residue fields of the closed points are k, and hence it is algebraically closed.) We lose much information if we consider only the graph $\Gamma(Y_k')$. Therefore we need the definition of the inertia groups of the vertexes of a graph and we state the axioms of them: Let Γ be a finite graph and let G be a finite group which acts on Γ. Let h be the natural projection $h\colon \Gamma \longrightarrow \Gamma/G$.

Ax. 1: Γ is connected.

Ax. 2: $A' = \Gamma/G$ is a tree with origin o'. We fix the orientation of A' which diverges from o'.

Ax. 3: We take an oriented subtree A of Γ such that $h|_A\colon A \longrightarrow A'$ is an isomorphism of oriented trees and which satisfies the following **Ax. 8**. For a vertex s' of A', we denote by s in A the corresponding vertex to s'.

Ax. 4: (Notation) Let s be a vertex of A. Let A_s (resp. $A'_{s'}$) be the subtree of A (resp. A') which diverges from s (resp. s). Let us denote by Γ_s be the connected component of $h^{-1}(A'_{s'})$ which contains A_s. Let G_s (resp. D_s) be the stabilizer of Γ_s (resp. s).

Ax. 5: For a vertex $s \in \Gamma$ we are given an invariant subgroup I_s (, which is called an inertia group of s) of D_s such that $gI_sg^{-1} = I_{g(s)}$ $(\forall g \in G)$.

Ax. 6: a) For a vertex $s \in \Gamma$, I_s is a p-group. b) For a vertex $s \in \Gamma$, if s is not above a terminal point of A', then I_s is not trivial. For an edge γ of Γ let us denote by I_γ the stabilizer of oriented γ. As a result I_γ is a subgroup of $D_a \cap D_b$, where a and b are the edge points of γ.

Ax. 7: a) I_a is a subgroup of I_γ. (Therefore I_a is a invariant subgroup of I_γ by Ax. 5. By the second isomorphism theorem of group theory $\langle I_a, I_b\rangle$ is a p-subgroup.) b) The order of the $\underline{I}_\gamma := I_\gamma/\langle I_a, I_b\rangle$ is prime to p. For a vertex $s \in A$ we denote by $T(s)$ a subset of the vertexes of A which consists of the next points of s. Put $\underline{D}_s := D_s/p(D_s)$. We denote by $\gamma(t)$ is an edge which connects s and $t \in T(s)$. By Ax. 6 a) and Ax. 7 we have a natural morphism $\underline{I}_{\gamma(t)} \longrightarrow \underline{D}_s$.

Ax. 8: The tree A satisfies the following equality for any vertex s of A.

$$\underline{D}_s = \langle \operatorname*{Im}_{t \in T(s)} (\underline{I}_{\gamma(t)} \longrightarrow \underline{D}_s)\rangle.$$

In §5 we explain that the axioms above are satisfied in the geometric case.

Remark 4.1. 1) In the geometric case it is shown that $\underline{I}_\gamma$ is a cyclic group of order prime to p ((5.2) 2) below).
2) Ax. 8 says that, if one ignores the quasi-p-part, the decomposition group of a smooth curve corresponding to s is generated by the inertia groups as in the characteristic 0. Indeed, Ax. 8 is shown by noting this observation.
3) G_s ($\forall s$) is a quasi-p-group by (4.3) 2) below. In particular, $G = G_o$ is so.

The following is a key theorem.

Theorem 4.2. *Let Γ and G be as above. Let S be a p-Sylow subgroup of G. If G does not have non-trivial invariant p-subgroup, then either of the following holds*:
1) $G(S) = G$
2) *There is a terminal point s of A such that $D_s = G$.*

For the proof of (4.2) we need the following lemma:

Lemma 4.3. 1) $G_s = \langle G_t | t \in T(s), D_s\rangle$.
2) $G_s = \langle G_t | t \in T(s), p(D_s)\rangle$.

The proof of 1) is purely graph theoretical (by using the connectedness of Γ). 2) follows immediately from 1) and Ax. 8.

Let us prove (4.2). (We see the graph from "far" points from the origin o.) Let us put $B = \{s \in A | G_s = G\}$. Since $o \in B$, B is not empty. Let t be the nearest point from s of the points between o and s. Then, by (4.3) 2), $G_t = \langle G_u | u \in T(t), p(D_t)\rangle$. Since the right hand side contains G_s, $G_t = G$. Therefore B is a subtree of A. Let s be a terminal point of B. We must consider the following two cases:
1) s is a terminal point of A: Then $D_s = G_s = G$.
2) s is not a terminal point of A: Note that $G_s = \langle G_t | t \in T(s), p(D_s)\rangle$. By Ax. 6 b), $I_s \neq 1$. Since I_s is an invariant p-subgroup of D_s, $I_s \subset p(D_s) \subset D_s \neq G$. Let $t \in T(s)$. Then $1 \neq I_s \subset I_{\gamma(t)} \subset D_t \subset G_t$. By the choice of t, $G_t \neq G$. Therefore, by (4.4) below, we see $G = G_s \subset G(S)$ (The following result is due to Hée.).

Proposition 4.4 ([R2] Appendice). *Let G be a finite group which does not have non-trivial invariant p-subgroup. Let P be a p-Sylow subgroup of G and let $L \underset{\neq}{\subset} G$ be a quasi-p-subgroup such that $L \cap P \neq 1$. Then $L \subset G(P)$.*

(4.4) says that P is not necessarily big, but $G(P)$ is so.
The following is the main theorem in this note.

Theorem 4.5. *Let G be a quasi-p-group such that $G(S) \neq G$ and G does not have non-trivial invariant p-subgroup. Then there is an etale covering of $\mathbb{P}^1_k \setminus \{\infty\}$ with Galois group G.*

Proof. By (4.2) there is a terminal point s such that $D_s = G$. Since D_s has no non-trivial invariant p-subgroup, $I_s = 1$ by Ax. 6 a). Let $C(\subset Y'_k) \longrightarrow \mathbb{P}^1_k(\subset P'_k)$ be the corresponding components to s and s'. Then either of the following two cases arises:
Case I; P'_k is irreducible:
Then $Y'_k \longrightarrow P'_k = \mathbb{P}^1_k$ is etale by (5.1) and (5.2) 1) b) below because $I_s = 1$, and consequently $G = 1$.
Case II; P'_k is not irreducible:
Since s is a terminal point, $C(\subset Y'_k) \longrightarrow \mathbb{P}^1_k$ is etale outside a unique double point on $\mathbb{P}^1_k$ by (5.2) 1) b) below. Therefore (4.5) is shown. □

5 The geometric case

In this section we explain that the geometric case satisfies axioms of inertia groups Ax. 2, Ax. 6 a), b) and Ax. 7 b) and some facts which are needed in the proof of (4.5). The other axioms are easy to prove.

Ax. 2: The fact that the graph $\Gamma(P'_k)$ is a tree follows from the following weight spectral sequence [M] (3.15) and (3.22) and the vanishing of $H^1_{\text{log-dR}}(P'_k/k')$:

$$E_1^{-r,i+r} = \bigoplus_{\substack{j\geq 0 \\ j\geq -r}} H^{i-2j-r}_{\text{dR}}(P'^{(2j+r+1)}_k/\kappa) \Longrightarrow H^i_{\text{log-dR}}(P'_k/k'). \tag{5.0}$$

Here $P'^{(j)}_k$ is the disjoint union of all j-fold intersections of the different irreducible components of P'_k ($j \in \mathbb{Z}_{\geq 1}$). The fact that $\Gamma(P'_k) = \Gamma(Y'_k)/G$ follows from the following lemma:

Lemma 5.1. *The image of any double point of Y'_k is also a double point of P'_k.*

The proof of (5.1) is easy: Indeed let y be a double point of Y'_k. We may assume that $G = I_y$ by considering an etale neighborhood of y and that the morphism between generic fibers is etale by $(*)$ in §3. We claim that G fixes the two components which pass y. Assume that it is not. Let I' be the stabilizer of them. Then Y'/I' be a semi-stable curve [R1] and $Y'/I' \longrightarrow P'$ is etale outside points of codimension ≥ 2. Hence $Y'/I' \longrightarrow P'$ is etale by a theorem of Zariski [SGA2] Exp. X, Th. (3.4). This is a contradiction.

Ax. 6 a), Ax. 7 b): We need the following.

Lemma 5.2. *Let $f\colon Y' \longrightarrow P' = G \setminus Y'$ be the Galois covering which are considered in §3.*
1) Let C be an irreducible component of the special fiber Y'_k and let η be the generic point of C. Then the following hold:
a) The inertia group I_η is a p-group.
b) The inertia groups of closed points of C which are not double points are all the same and are equal to I_η.
2) Let y be a double point which is the intersection of irreducible components (a) and (b). Let I_a (resp. I_b) be the inertia group of the generic point of (a) (resp. (b)). Then the following holds:
$I_y/\langle I_a, I_b\rangle \simeq \mathbb{Z}/m$, where m is a natural number prime to p. In particular $\langle I_a, I_b\rangle$ is a p-Sylow subgroup of I_y.

The sketch of the proof of (5.2) is as follows:
1) a): Since C is generically unramified over the image of C in P'_k, I_η is a p-group by [Se] Chap. I Prop. 21.
1) b): b) follows from the following lemma ($n = 1$ in the notation of (5.2)):

Lemma 5.3. *Let $\mathcal{Y}$ be a smooth curve over* $\operatorname{Spec} R$ *on which a finite group G acts on $\mathcal{Y}$. Put $\mathcal{X} := G \setminus \mathcal{Y}$. Let $y \in \mathcal{Y}_K(K)$ be a closed point with specialization point $o \in \mathcal{Y}_k(k)$. Let $x \in \mathcal{X}_K(K)$ be the image of y and let η be the generic point such that $o \in \overline{\{\eta\}}$. Let np^a ($(n,p) = 1$) be the ramification index of y/x. Then $I_o/I_\eta \simeq \mathbb{Z}/n$.*

This lemma follows from a variation of a lemma of Abhyankar (See [R2] (6.3.2).).
2): We may assume $G = I_y$ by considering an etale neighborhood of y and $I_a = I_b = 1$ by considering a quotient covering. Here we used [R1] Appendice.

Let $\operatorname{Spec} O'$ (resp. $\operatorname{Spec} O$) be an affine etale neighborhood of y (resp. the image of y) such that the horizontal ramification points does not belong to $\operatorname{Spec} O'$ ((*) in §3). Let ω' (resp. ω) be the dualizing sheaf of $\operatorname{Spec} O'/\operatorname{Spec} R$ (resp. $\operatorname{Spec} O/\operatorname{Spec} R$). Then $f^*(\omega) \simeq \omega'$ by [EGA IV-2] (5.10.6) because $f^*(\omega) \longrightarrow \omega'$ is an isomorphism at the points of codimension ≤ 1. Therefore the covering is tame and hence cyclic.

Ax. 6 b): Ax. 6 b) is equivalent to the following in the geometric case.

Lemma 5.4. *Let $C \subset Y_k$ be the irreducible component of Y_k with generic point η. Let $D \subset P_k'$ be the image of C. If $I_\eta = 1$, then D is a terminal point of the dual graph $\Gamma(P_k')$.*

(5.4) follows from the minimality of Y: Indeed, we can contract a subtree of $\Gamma(P_k')$ which diverges from D by [BLR] Prop. 4 p. 169. Let us consider the corresponding inverse image and the covering $Y^* \longrightarrow P^*$. By the minimality $Y^* = Y$. Here we have used that the ramification points in the generic fiber are specialized to different sections.

References

[AW] M. Artin and G. Winters. *Degenerate fibers and stable reduction of curves.* Topology 10, (1971) 373-383.

[BLR] S. Bosch, W. Lütkebohmert and M. Raynaud. *Néron models.* Ergebnisse der Math. und ihrer Grenz. 3. Folge, Band 21, Springer- Verlag (1990).

[DM] P. Deligne and D. Mumford. *The irreducibility of the space of curves of given genus.* I. H. E. S. Publ. Math. 36 (1969), 75-109.

[EGA IV-2] A. Grothendieck and J. Dieudonné. *Éléments de géométrie algébrique* IV-2. I. H. E. S. Publ. Math. (1965).

[H] D. Harbater. *Abhyankar's conjecture on Galois groups over curves.* Inv. Math. 117, (1994), 1-25.

[L] S. Lichtenbaum. *Curves over discrete valuation rings.* Amer. J. Math. 25, (1968), 380-405.

[M] A. Mokrane. *La suites spectrale des poids en cohomologie de Hyodo-Kato.* Duke Math. Jour. 72, (1993), 301-336.

[R1] M. Raynaud. *p-groupes et réduction semi-stable des courbes.* In: The Grothendieck Festschrift, P. Cartier et al. (eds.), Vol. 3, Birkhäuser (1990), 179-197.

[R2] M. Raynaud. *Revêtements de la droite affine en caractéristique* $p > 0$ *et conjecture d'Abhyankar.* Inv. Math. 116, (1994), 425-462.

[Se] J.-P. Serre. *Corps Locaux.* Hermann (1990), 179-197.

[SGA 1] A. Grothendieck et M. Raynaud. *Revêtements étales et Groupe Fondamentals.* Lec. Notes in Math. 224, Springer-Verlag, (1971).

[SGA 2] A. Grothendieck. *Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetsz locaux et globaux.* Séminaire de géométrie algébrique du Bois-Marie 1962 Adv. Stu. Pure Math. vol. 1240 North-Holland and Paris, Masson (1968).

[Su] N. Suwa. *Conjecture d'Abhyankar proved by M. Raynaud I.* In this volume.