数理解析研究所講究録
1074 巻 1999 年 1-11



# Pseudo Dirichlet sets and a new cardinal invariant

Shizuo Kamo (Sakai) (加茂 静夫)

**Abstract**

Z. Bukovská [5] proved that $\mathbf{p} \leq \mathrm{non}(\mathcal{PD})$, where $\mathcal{PD}$ denotes the set of all pseudo-Dirichlet sets. In this paper, we shall show that $\mathbf{p}$ can be replaced by $\mathbf{h}$ in this inequality. It is known that $\mathbf{p} < \mathbf{h}$ is consisitent (see [1]). So, the equality $\mathbf{p} = \mathrm{non}(\mathcal{PD})$ can not be proved. This is a partial answer of problem 2 in [6]. Next, we shall introduce a certain cardinal invariant $\mathbf{f}$ and show that $\mathrm{add}(\mathcal{N}) \leq \mathbf{f} \leq \mathrm{non}(\mathcal{PD})$. Also, we shall construct two generic models such that one satisfies the inequality $\mathbf{b} < \mathbf{f}$ and another satisfies the inequality $\mathbf{f} < \mathrm{non}(\mathcal{PD})$.

## 1 Introduction

Throughout this paper, we shall use the standard terminologies for forcing of set theory and cardinal invariants on $\omega$ (see [3]). For each $a \in \mathbf{R}$, we denote by $\|a\|$ the distance of $a$ and the set of integers $\mathbf{Z}$. Let $A$ be a subset of the unit interval $[0,1]$. $A$ is called a *pseudo Dirichlet set*, if there exists an $X \in [\omega]^\omega$ such that

$$\forall a \in A\, \forall^\infty n \in X\ (\ \|na\| < \frac{1}{|X \cap n| + 1}\ ).$$

We denote the set of all pseudo Dirichlet sets by $\mathcal{PD}$. Z. Bukovská [5] showed that $\mathbf{p} \leq \mathrm{non}(\mathcal{PD})$. Let $\mathbf{h}$ be the least cardinal $\kappa$ such that the boolean algebra $\mathcal{P}(\omega)/\mathrm{fin}$ does not satisfy the $\kappa$-distributive law.

**Theorem 1.1** $\quad \mathbf{h} \leq \mathrm{non}(\mathcal{PD})$.

**Proof** For each $a \in [0,1)$, let $\|a\|^*$ denote the unique real number $r$ such that $0 \leq r < 1$ and $a = r \pmod{\mathbf{Z}}$. To show this theorem, let $A \subset [0,1]$ and $|A| < \mathbf{h}$.

For each $a \in A$, take a maximal almost disjoint set $W_a \subset [\omega]^\omega$ such that

$$\forall n \in X\, \forall m \in X \setminus n\ (\ | \|na\|^* - \|ma\|^* | < \frac{1}{|X \cap n| + 1}\ ), \text{ for all } X \in W_a.$$

Since $|A| < \mathbf{h}$, there exists a maximal almost disjoint set $W$ such that $W$ is an refinement of all $W_a$'s. Take a $Y \in W$. Choose some $Y' = \{ y_i \mid i < \omega \}_< \in [Y]^\omega$ such that

$$|Y \cap [y_i, y_{i+1})| \geq i \text{ and } y_{i+1} - y_i < y_{i+2} - y_{i+1}, \text{ for all } i < \omega.$$

Let $Z = \{ y_{i+1} - y_i \mid i < \omega \}$. We complete the proof by showing that

$$\forall^\infty n \in Z\ (\ \|na\| < \frac{1}{|Z \cap n| + 1}\ ), \text{ for all } a \in A.$$

Let $a \in A$. Since $W$ is a refinement of $W_a$, there exists an $X \in W_a$ such that $Y \subset^* X$. Take an $i < \omega$ such that $Y \setminus y_i \subset X$. Then, for any $j \in [i+1, \omega)$, it holds that

$$\|(y_{j+1} - y_j)a\| \leq | \|y_{j+1}a\|^* - \|y_j a\|^* | \leq \frac{1}{|X \cap y_j| + 1} < \frac{1}{j+1}. \qquad \square$$

# 2 Combinatorial principle $\mathrm{wIn}_2$

T. Bartoszynski [2] introduced the notion of slalom and, using this, investigated systematically the relations between combinatorics and cardinal invariants which are associated by the null ideal $\mathcal{N}$ and the meager ideal $\mathcal{M}$. The following statement $\mathrm{In}_2$ and the theorem are some of them.

**Definition 2.1** *For $h \in {}^\omega\omega$ and $F \subset {}^\omega\omega$, define the statement $\mathrm{In}_2(F, h)$ by*

$$\mathrm{In}_2(F, h) \equiv \exists \varphi \in \prod_{n<\omega} [\omega]^{\leq h(n)}\ \forall f \in F\, \forall^\infty n < \omega\ (\ f(n) \in \varphi(n)\ ).$$

*The statement $\mathrm{In}_2(F, \mathrm{id}_\omega)$ is denoted by $\mathrm{In}_2(F)$, where $\mathrm{id}_\omega$ is the identity function on $\omega$.*

**Theorem 2.1** (Bartoszynski [2]) $\mathrm{add}(\mathcal{N}) = \min\{ |F| \mid F \subset {}^\omega\omega$ *and not* $\mathrm{In}_2(F) \}$.

In this section, we shall introduce the statement $\mathrm{wIn}_2$ which is some variant of $\mathrm{In}_2$. And we shall study relations between $\mathrm{wIn}_2$ and $\mathrm{non}(\mathcal{PD})$.

**Definition 2.2** *For $H$, $h \in {}^{\omega}\omega$ and $F \subset \prod_{n<\omega} H(n)$, define the statement* $\mathrm{wIn}_2(F, h, H)$ *by*

$$\mathrm{wIn}_2(F,h,H) \equiv \exists\, \varphi \in \prod_{n<\omega} [H(n)]^{\leq h(n)} \ \forall f \in F \ \forall^{\infty} n < \omega \ (\ f(n) \in \varphi(n)\ ).$$

$\mathrm{wIn}_2(F, H)$ *denotes the statement* $\mathrm{wIn}_2(F, \mathrm{id}_\omega, H)$. *Let*

$$\mathbf{f} = \min\{\, |F| \mid \exists\, H \in {}^{\omega}\omega\ (\ F \subset \prod_{n<\omega} H(n) \text{ and not } \mathrm{wIn}_2(F,H)\ )\,\}.$$

The following lemma can be easily proved by the result of Bartorszynski.

**Lemma 2.2** $\mathrm{add}(\mathcal{N}) = \min\{\, \mathbf{b}, \mathbf{f}\,\}$. □

The main result of this section is the following theorem.

**Theorem 2.3** $\mathbf{f} \leq \mathrm{non}(\mathcal{PD})$.

To show this theorem, we need some notations and lemmas.

A sequence $\langle\, I_n \mid n < \omega\,\rangle$ is called an *interval partition* of $\omega$, if there exists an increasing function $f \in {}^{\omega}\omega$ such that $f(0) = 0$ and, for all $n < \omega$, $I_n = \{\, k < \omega \mid f(n) \leq k < f(n+1)\,\}$.

The next lemma can be deduced from [6, Proposition 1]. But, for a convenience for the reader, we give a proof.

**Lemma 2.4** *Let $n < \omega$ and $0 < m,\ k < \omega$. Then, there exists some $p < \omega$ such that*

$$\forall\, a_0, \cdots, a_{m-1} \in [0,1]\ \exists\, s < \omega\ (\ n \leq s < p \text{ and } \forall\, i < m\ (\ \|sa_i\| < \frac{1}{k}\ )\ ).$$

**Proof** By induction on $1 \leq m < \omega$.

Case 1. $m = 1$.

We claim thta $p = nk + 1$ satisfies the condition. To show this, let $a \in [0,1]$. Define the mapping $\sigma : X = \{\, nj \mid j = 1, \cdots, k\,\} \to k$ by, for each $j < k$,

$$\sigma(nj) = \text{“ the unique } i \text{ such that } \frac{i}{k} \leq \|nja\|^* < \frac{i+1}{k} \text{ ”}.$$

If there exists some $nj$ such that $\sigma(nj) = 0$ or $k-1$, then $s = nj$ is a required one.

Otherwise, there exist $i < j \leq k$ such that $\sigma(ni) = \sigma(nj)$ and $s = n(j-i)$ is a required one.

Case 2. $m = m' + 1$.

By induction hypothesis, there exist $0 = p_0 < p_1 < \cdots < p_k$ such that

$$\forall a_0, \cdots, a_{m'-1} \in [0,1] \, \exists s < \omega \, ( \, p_j + n \leq s < p_{j+1} \text{ and } \forall i < m' \, ( \, \|sa_i\| < \frac{1}{2k} \, ) \, ),$$

for $j < k$.

We show that $p = p_k$ satisfies the condition. So, let $a_0, \cdots, a_{m'} \in [0,1]$. By the choise of $p_j$ (for $j < k$), there exist $s_0, \cdots, s_{k-1} < \omega$ such that

$$p_j + n \leq s_j < p_{j+1} \text{ and } \forall i < m' \, ( \, \|s_j a_i\| < \frac{1}{2k} \, ) \, ), \text{ for } j < k.$$

Then, it holds that

$$\|s_j a_{m'}\| < \frac{1}{k}, \text{ for some } j < k$$

or

$$\|s_j a_{m'} - s_{j'} a_{m'}\| < \frac{1}{k}, \text{ for some } j < j' < k.$$

In either cases, similar to case 1, we can take a required element $s$. □

**Corollary 2.5** *There is an interval partition $\langle I_n \mid n < \omega \rangle$ which satisfies*

$$(*) \begin{cases} \text{For any } n < \omega \text{ and } a_0, \cdots, a_{n-1} \in [0,1), \text{ there exists some } k \in I_n \text{ such that} \\ \qquad \|ka_i\| < 2^{-n}, \text{ for all } i < n. \end{cases}$$

□

**Proof of Thorem 2.3** Take an interval partition $\langle I_n \mid n < \omega \rangle$ which satisfies (*) in the previous corollary. Define $H \in {}^{\omega}\omega$ by

$$H(n) = 2^n \sum_{k \leq n} |I_k|, \text{ for all } n < \omega.$$

To show the theorem, let $A \subset [0,1]$ and $|A| < \mathbf{f}$. For each $a \in A$, define $f_a \in \prod_{n<\omega} H(n)$ by

$$\frac{f_a(n)}{H(n)} \leq a < \frac{f_a(n)+1}{H(n)}, \text{ for all } n < \omega.$$

Since $|A| < \mathbf{f}$, there exists a $\varphi \in \prod_{n<\omega} [H(n)]^n$ such that

$$\forall a \in A \, \forall^\infty n < \omega \, (\, f_a(n) \in \varphi(n) \,).$$

For each $n < \omega$, take $s_n \in I_n$ such that

$$\|s_n \frac{j}{H(n)}\| < 2^{-n}, \text{ for all } j \in \varphi(n).$$

We complete the proof by showing that

$$\forall a \in A \, \forall^\infty n < \omega \, (\, \|s_n a\| < 2^{-n+1} \,).$$

So, let $a \in A$. Take an $m < \omega$ such that

$$\forall n \geq m \, (\, f_a(n) \in \varphi(n) \,).$$

Then, for any $n \geq m$, since $\frac{f_a(n)}{H(n)} \leq a < \frac{f_a(n)+1}{H(n)}$, it holds that

$$s_n \frac{f_a(n)}{H(n)} \leq s_n a < s_n \frac{f_a(n)+1}{H(n)}.$$

So,

$$\|s_n a\| \leq \|s_n \frac{f_a(n)}{H(n)}\| + \frac{s_n}{H(n)} < 2^{-n+1}. \qquad \square$$

Note that what we really proved is $\min\{\, |F| \mid \text{not } \mathrm{wIn}_2(F, H) \,\} \leq \mathrm{non}(\mathcal{PD})$, where $H$ is a function defined in the proof of Theorem 2.3.

# 3 The cardinal invariant f

In the previous section, we introduced the cardinal invariant $\mathbf{f}$ and showed the equality $\mathrm{add}(\mathcal{N}) = \min\{\, \mathbf{b}, \mathbf{f} \,\}$. Both of $\mathrm{add}(\mathcal{N})$ and $\mathbf{b}$ appear in the Cihoń's diagram. It seems to be an interesting problem to check the relations between $\mathbf{f}$ and other cardinals in the diagram. Since it is known that $\mathcal{PD} \subset \mathcal{N} \cap \mathcal{M}$, it holds that $\mathbf{f} \leq \min\{\, \mathrm{non}(\mathcal{N}), \mathrm{non}(\mathcal{M}) \,\}$. So, $\mathbf{f}$ seems to be not so large. If the inequality $\mathbf{f} \leq \mathbf{b}$ always holds, then $\mathbf{f}$ is equal to $\mathrm{add}(\mathcal{N})$ and $\mathbf{f}$ does not become a new cardinal invariant. In this section, we shall show that there exists a generic model which satisfies the inequality $\mathbf{b} < \mathbf{f}$.

**Definition 3.1** *For each $H \in {}^\omega\omega$, define the forcing notion $Q(H)$ by*

$$Q(H) = \{\, p \in \prod_{n<\omega} [H(n)]^{\leq n} \mid \exists k < \omega \, \forall n < \omega \, (\, |p(n)| \leq k \,) \,\},$$

$$q \leq p \quad \textit{iff} \quad \forall n < \omega \ ( \ p(n) \subset q(n) \ ).$$

*Define* $\tau_H : Q(H) \to \omega$ *by*

$$\tau_H(p) = \min\{ \ k < \omega \mid \forall n < \omega \ ( \ |p(n)| \leq k \ ) \ \}.$$

Using the density argument, the following lemma can be proved easily.

**Lemma 3.1** *Let* $H \in {}^{\omega}\omega$ *and* $\mathcal{G}$ *be* $V$*-generic on* $Q(H)$*. In* $V[\mathcal{G}]$*, define* $\varphi \in \prod_{n<\omega} \mathcal{P}(H(n))$ *by*

$$\varphi(n) = \bigcup \{ \ p(n) \mid p \in \mathcal{G} \ \}.$$

*Then, it holds that*

(1) $|\varphi(n)| \leq n$, *for all* $n < \omega$,

(2) $\forall g \in (\prod_{n<\omega} H(n))^V \ \forall^{\infty} n < \omega \ ( \ g(n) \in \varphi(n) \ ).$ □

**Lemma 3.2** $Q(H)$ *satisfies the* $\omega_1$*-chain condition.*

**Proof** Let $W \subset Q(H)$ and $|W| = \omega_1$. Replace $W$ by a certain subset of $W$, if necessary, we can assume that, for some $k < \omega$,

$$\tau_H(p) = k \text{ and } p \restriction 2k = p' \restriction 2k, \text{ for all } p, \ p' \in W.$$

Then, every elements of $W$ are mutually compatible. □

**Lemma 3.3** *Every unbounded family of functions in* ${}^{\omega}\omega \cap V$ *is still unbounded in* $V^{Q(H)}$.

Bartszinski and Judah [3, Theorem 6.4.13] proved that any finite support iteration by forcing notions which preserved the unboundedness in ${}^{\omega}\omega$ does not add a dominating function. So, starting a ground model which satisfies CH, by choosing appropreate $H$'s, we can construct an $\omega_2$-stage finite support iteration $P$ such that $V^P$ satisfies $\mathbf{b} = \omega_1$ and $\mathbf{f} = \omega_2$.

In order to prove Lemma 3.3, we need a result of Brendle and Judah [4]. Let $P$ be a forcing notion which satisfies the $\omega_1$-chain condition and $\tau : P \to \omega$ be a homomorphism. Following Brendle and Judah [4], we say that $(P, \tau)$ is *nice*, if it satisfies

$$\left\{ \begin{array}{l} \text{For any predence set } \{ p_i \mid i < \omega \} \subset P \text{, it holds that} \\ \qquad \forall m < \omega \, \exists n < \omega \, \forall q \in P \, ( \text{ if } \tau(q) \leq m \text{, then } \exists i < n \, ( q \uparrow p_i ) ). \end{array} \right.$$

**Theorem 3.4** (Brendle and Judah [4]) *Let $(P, \tau)$ be a nice forcing notion. Then, every unbounded family of functions in ${}^\omega\omega \cap V$ is still unbounded in $V^{Q(H)}$.* □

**Proof of Lemma 3.3** It suffices to show that $(Q(H), \tau_H)$ is nice. So, let $\{ p_i \mid i < \omega \}$ be a predence subset of $Q(H)$ and $m < \omega$. To get a contradiction, assume that, for each $n < \omega$, there exists a condition $q_n \in Q(H)$ such that

$$\tau_H(q_n) \leq m \text{ and } \forall i < n \, ( q_n \perp p_i ).$$

Since $\{ q_n {\restriction} k \mid n < \omega \}$ is a finite set for every $k < \omega$, we can choose $X_k \in [\omega]^\omega$ by induction on $k < \omega$ such that

$$X_{k+1} \subset X_k \text{ and } \forall n, n' \in X_k \, ( q_n {\restriction} (k+1) = q_{n'} {\restriction} (k+1) ).$$

Define $r \in Q(H)$ by

$$r(k) = q_n(k), \text{ for some/all } n \in X_k.$$

Note that $\tau_H(r) \leq m$. Since $\{ p_i \mid i < \omega \}$ is predence, there exists $i < \omega$ such that $r$ is compatible with $p_i$. Let $k = \tau_H(p_i) + m$. Take $n \in X_k$ such that $i < n$. Since $i < n$, it holds that $p_i$ and $q_n$ are incompatible. Since $\tau_H(p_i) + \tau_H(q_n) \leq k$, it holds that $\exists j < k \, ( |p_i(j) \cup q_n(j)| > j )$. By this and the fact that $r {\restriction} k = q_n {\restriction} k$, $r$ is incompatible with $p_i$. This is a contradiction. □

# 4 Consistency of $\mathbf{f} < \mathrm{non}(\mathcal{PD})$

Concerning about the cardinal invariant associated by $\mathrm{In}_2$, T. Bartoszynski [2] pointed out implicitly that, if two functions $h_0$, $h_1 \in {}^\omega\omega$ satisfies that

$$\lim_{n<\omega} h_i(n) = \infty, \text{ for } i = 0, 1,$$

then $\min\{ |F| \mid \text{ not } \mathrm{In}_2(F, h_0) \} = \min\{ |F| \mid \text{ not } \mathrm{In}_2(F, h_1) \}$.

In this section, we shall show that, for any $H \in {}^\omega\omega$, $\mathbf{f}$ may not be equal to $\min\{ |F| \mid \text{ not } \mathrm{wIn}_2(F, H) \}$. Using this, we shall prove the consistency of $\mathbf{f} <$

non($\mathcal{PD}$). Henceforce, $H \in {}^\omega\omega$ is an arbitrary, but fixed function on $\omega$. For each $k < \omega$, let

$$T_k\ (= T_k^H) = \{\, q \in Q(H) \mid \tau_H(q) \le k \,\}.$$

Define $H_0,\ H_1 : \omega \times \omega \to \omega$ by

$$H_0(k,m) = \min\left\{ l < \omega \;\middle|\; \begin{array}{c} \forall\, \delta : l \to [\omega_2]^{\le k}\ \exists\, S \in [\,l\,]^m \exists\, v \in [\omega_2]^{\le k} \\ \forall\, i,\, j \in S\ (\text{ if } i \ne j, \text{ then } \delta(i) \cap \delta(j) = v\ ) \end{array} \right\},$$

$$H_1(k,m) = \min\left\{ l < \omega \;\middle|\; \begin{array}{c} \forall\, \delta : l \to T_k\ \exists\, S \in [\,l\,]^m \exists\, q \in Q(H) \\ \forall\, i \in S\ (\ q \le \delta(i)\ ) \end{array} \right\}.$$

Note that $H_0$ is a recursive function. And, $H_1$ is an $H$-recursive function, since it holds that

$$\exists\, q' \in Q(H)\ \forall\, q \in S\ (\ q' \le q\ ) \quad \text{iff} \quad \forall\, i < mk\ (\ |\bigcup_{q \in S} q(i)| \le i\ )\ , \text{ for any}$$

$S \in [T_k]^m$.

Define $H_2,\ H^* : \omega \to \omega$ by

$$H_2(k) = \underbrace{H_1(k, H_1(k, H_1(\cdots, H_1}_{k \text{ times}}(k, k+1)\cdots))),$$

$$H^*(k) = H_0(k, H_2(k)).$$

Define an $\omega_2$-stage finite support iteration $P_\alpha$ (for $\alpha \le \omega_2$) associated with $\dot{Q}_\alpha$ (for $\alpha < \omega_2$) by

$$\Vdash_\alpha \dot{Q}_\alpha = Q(H)\ , \quad \text{for all } \alpha < \omega_2.$$

Let $P(H) = P_{\omega_2}$. It holds that

$$V^{P(H)} \models \forall\, F \subset \prod_{n<\omega} H(n)\ (\text{ if } |F| \le \omega_1, \text{ then } \mathrm{wIn}_2(F, H)\ ).$$

The purpose of this section is to show

**Theorem 4.1** $\quad V^{P(H)} \models$ *not* $\mathrm{wIn}_2((\prod_{n<\omega} H^*(n))^V, H^*)$.

**Corollary 4.2** *Suppose that* $V \models$ CH. *Let* $H \in {}^\omega\omega$ *be the function which is defined in the proof of Theorem 2.3. Then, it holds that*

$$V^{P(H)} \models\ \mathbf{f} = \omega_1 \text{ and } \mathrm{non}(\mathcal{PD}) = \omega_2.$$

□

To show Theorem 4.1, we need some definitions and lemmas. Let

$$D = \{ p \in P(H) \mid \forall \alpha \in \mathrm{supp}(p) \ ( \ p{\restriction}\alpha \text{ decides } \tau_H(p(\alpha)) \ ) \}.$$

The following lemma can be proved easily.

**Lemma 4.3** *$D$ is dense in $P(H)$.* □

Define $\rho : D \to \omega$ by

$$\rho(p) = \min \left\{ k < \omega \;\middle|\; \begin{array}{l} |\mathrm{supp}(p)| \leq k \\ \text{and} \\ \forall \alpha \in \mathrm{supp}(p) \ ( \ p{\restriction}\alpha \Vdash_\alpha \tau_H(p(\alpha)) \leq k \ ) \end{array} \right\}.$$

For each 各 $k < \omega$, let

$$D_k = \{ p \in D \mid \rho(p) \leq k \}.$$

**Lemma 4.4** *Let $k < \omega$ and $\delta : H^*(k) \to D_k$. Then, there exist $p^+ \in P(H)$ and $P(H)$-name $\dot{S}$ which satisfy* (1), (2).

(1) $\Vdash \dot{S} \subset H^*(k)$ *and* $|\dot{S}| \geq k+1$.

(2) $\forall i < H^*(k) \, \forall p' \leq p^+$ ( *if* $p' \Vdash i \in \dot{S}$, *then* $p' \leq \delta(i)$ ).

**Proof** Let $k < \omega$. Define $l_m$ (for $m \leq k$) by

$$\begin{cases} l_0 & = \ k+1 \\ l_{m+1} & = \ H_1(k, l_m) \end{cases}.$$

Note that $H^*(k) = H_0(k, l_k)$. Assume that $\delta : H^*(k) \to D_k$. Since $\langle \mathrm{supp}(\delta(i)) \mid i < H^*(k) \rangle : H^*(k) \to [\omega_2]^{\leq k}$, by the choise of $H_0$, there exist $S_0 \in [H^*(k)]^{l_k}$ and $v \in [\omega_2]^{\leq k}$ such that

$$\forall i, j \in S_0 \ ( \text{ if } i \neq j, \text{ then } \mathrm{supp}(\delta(i)) \cap \mathrm{supp}(\delta(j)) = v \ ).$$

Define $p \in P(H)$ by

$$\mathrm{supp}(p) = \bigcup \{ \mathrm{supp}(\delta(i)) \mid i \in S_0 \} \setminus v,$$

$$p(\alpha) = \delta(i)(\alpha), \text{ if } \alpha \in \mathrm{supp}(\delta(i)) \text{ and } i \in S_0.$$

Let $n = |v|$ and $v = \{ \alpha_1, \cdots, \alpha_n \}_<$. Note that $n \leq k$. By induction on $1 \leq m \leq n$, choose $P_{\alpha_m}$-names $\dot{S}_m$, $\dot{q}_m$ such that

(3) $\Vdash_{\alpha_m} \dot{S}_m \in [\dot{S}_{m-1}]^{l_{k-m}}$ and $\dot{q}_m \in \dot{Q}_{\alpha_m}$,

(4) $p{\restriction}\alpha_m \cup \langle \dot{q}_j \mid 1 \leq j < m \rangle \Vdash_{\alpha_m} \dot{q}_m \leq \delta(i)(\alpha_m)$, for all $i \in \dot{S}_m$.

We must show that these can be chosen. Assume that $m \le n$ and $\dot{S}_j$, $\dot{q}_j$ were chosen, for $j < m$. Since $H_1$ is abusolute and $H_1(k, l_{k-m}) = l_{k-m+1}$, it holds that

$p{\restriction}\alpha_m \cup \langle \dot{q}_j \mid 1 \le j < m \rangle \Vdash_{\alpha_m} \exists q \in Q(H)\ \exists S \in [\dot{S}_{m-1}]^{l_{k-m}}\ \forall i \in S\ (\ q \le \delta(i)(\alpha_m)\ )$.

Using this, it can be possible to choose $\dot{S}_m$, and $\dot{q}_m$.

Let $p^+ = p \cup \langle \dot{q}_m \mid 1 \le m \le n \rangle$, $\dot{S} = \dot{S}_n$. It is clear that this $p^+$ and $\dot{S}$ satisfy (1) in the lemma. In order to show that these satisfy (2), assume that

$$i < H^*(k) \text{ and } p' \le p^+ \text{ and } p' \Vdash i \in \dot{S}.$$

Since $\Vdash_P \dot{S} = \dot{S}_n \subset \dot{S}_{n-1} \subset \cdots \subset S_0$, $i \in S_0$. For each $m = 1, \cdots, n$, since $\dot{S}_m$ is a $P_{\alpha_m}$-name, it holds that $p'{\restriction}\alpha_m \Vdash_{\alpha_m} i \in \dot{S}_m$. By this, since $p' \le p^+$, we have that

$$p'{\restriction}\alpha_m \Vdash_{\alpha_m} \dot{q}_m \le \delta(i)(\alpha_m)\ , \text{ for all } m = 1, \cdots, n.$$

So, $p' \le \delta(i)$. □

**Lemma 4.5** *Let $k < \omega$. Assume that a $P(H)$-name $\dot{a}$ satisfies*

$$\Vdash \dot{a} \in [H^*(k)]^{\le k}.$$

*Then, there exists some $j < H^*(k)$ such that*

$$\forall p \in D_k\ (\ \textit{not}\ p \Vdash j \in \dot{a}\ ).$$

**Proof** Suppose not. Take $\delta : H^*(k) \to D_k$ such that

$$\delta(j) \Vdash j \in \dot{a}, \text{ for all } j < H^*(k).$$

By the previous lemma, there exist $p^+ \in P(H)$ and $P(H)$-name $\dot{S}$ such that

$$\Vdash \dot{S} \subset H^*(k) \text{ and } |\dot{S}| \ge k+1,$$

$$\forall i < H^*(k)\ \forall p' \le p^+\ (\text{ if } p' \Vdash i \in \dot{S}, \text{ then } p' \le \delta(i)\ ).$$

Then, it holds that $p^+ \Vdash \dot{S} \subset \dot{a}$. This contradicts that $p^+ \Vdash |\dot{S}| \ge k+1$ and $|\dot{a}| \le k$.

□

**Proof of Theorem 4.1** Assume that $\Vdash_{P(H)} \dot{\varphi} \in \prod_{k<\omega} [H^*(k)]^k$. Using the previous lemma, for each $k < \omega$, take a $j_k < H^*(k)$ such that

$$\forall p \in D_k\ (\text{ not } p \Vdash j_k \in \dot{\varphi}(k)\ ).$$

We claim that $\Vdash \exists^\infty k < \omega$ ( $j_k \notin \dot{\varphi}(k)$ ). Suppose not. Then, there exist $p \in D$ and $n < \omega$ such that

$$p \Vdash \forall k > n \, ( j_k \in \dot{\varphi}(k) ).$$

Take $k > n$ such that $p \in D_k$. Then, it holds that $p \Vdash j_k \in \dot{\varphi}(k)$. But, this contradicts the choise of $j_k$. □

**Added in proof:**

After the completion of this paper, Dr. Kada [7] have proved that $\mathbf{d} < \mathrm{non}(\mathcal{PD})$ is consistent with ZFC.

# References


[1] B. Balcar, J. Pelant and P. Simon, *The space of ultrafilters on* $\mathbf{N}$ *covered by nowhere dense sets*, Fund. Math., 110 (1980) pp. 11-24.

[2] T. Bartoszyński, *Combinatorial aspects of measure and category*, Fund. Math., 127 (1987) pp. 225-239.

[3] T. Bartoszyński and H. Judah, *Set Theory*, A K peters, Wellesley, Massachusetts, 1995

[4] J. Brendle and H. Judah, *Perfect sets of random reals*, Israel J. of Math., 83 (1993) pp. 153-176.

[5] Z. Bukovská, *Thin sets in trigonometrical series and quasinormal convergence*, Math. Slovaca, 40 (1990) pp. 53-62.

[6] Z. Bukovská and L. Bukovský, *Adding small sets to an N-set*, Proc. AMS., 123 (1995) pp. 3867-3873.

[7] M. Kada, *Slalom, prediction, and permitted trigonometric thin sets*, manuscript



Department of Mathematics

University of Osaka Prefecture

Gakuen-chou, Sakai, Japan

email:kamo@center.osakafu-u.ac.jp