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PRODUCTS OF k-SPACES, AND k-NETWORKS

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We give some affirmations and negations to the following *Hypothesis*. We assume that spaces are regular and T_1.

Hypothesis: Let X and Y be k-spaces with point-countable k-networks. Then, $X \times Y$ is a k-space if and only if one of the following holds.

(K_1) X and Y have point-countable bases.

(K_2) X or Y is locally compact.

(K_3) X and Y are locally k_ω-spaces.

Note that the " if " part of *Hypothesis* holds for any k-spaces X and Y. Lasnev spaces; quotient s-images of metric spaces; or CW-complexes are k-spaces with a point-countable k-network.

We recall some definitions used in this note. Let X be a space, and let $\mathcal{C}$ be a cover of X. Then,

$\mathcal{C}$ is *point-countable* (resp. *compact-countable; star-countable)* if any point $x \in X$ (resp. compact $K \subset X$; element $C \in \mathcal{C}$) meets at most countably many $D \in \mathcal{C}$.

X is determined by $\mathcal{C}$, if $F \subset X$ is closed in X iff $F \cap C$ is closed in C for every $C \in \mathcal{C}$. Here, we can replace " closed " by " open ".

X *is dominated by* $\mathcal{C}$, if for any subcollection $\mathcal{C}^*$ of $\mathcal{C}$, $\cup \mathcal{C}^*$ is a closed subset determined by $\mathcal{C}^*$.

A space is a *k-space* if it is determined by a cover of compact subsets.

Every space is determined by any open cover. Every space is dominated by any HCP (=hereditarily closure-preserving) closed cover. Every CW-complex is a k-space dominated by a cover of compact metric subsets.

Let α be an infinite cardinal. Then, a space is a *k_α-space* if it is determined by a cover $\mathbf{C}$ of compact subsets with $|\mathbf{C}|\leqq\alpha$. A space X is *locally $<k_\alpha$* if each $x\in X$ has a nbd whose closure is a $k_{\alpha(x)}$-space, here $\alpha(x)<\alpha$. A space X is *locally k_ω* if X is locally $<k_\alpha$, $\alpha=\omega_1$.

Let X be a space, and let $\mathcal{P}$ be a cover of X. Then,
$\mathcal{P}$ is a *k-network* if, for any nbd U of a compact set K in X, there is a finite $\mathcal{P}^*\subset\mathcal{P}$ such that $K\subset\cup\mathcal{P}^*\subset U$.
$\mathcal{P}$ is a *cs*-network* (resp. *cs-network*) if, for any open nbd U of $x\in X$, and for any sequence L converging to x, there is $P\in\mathcal{P}$ with $x\in P\subset U$, and P contains L frequently (resp. eventually).
A k-network $\mathcal{P}$ is *closed* if elements of $\mathcal{P}$ are closed.
Obviously, cs-networks; or *closed* k-networks are cs*-networks.

Spaces with a countable k-network (resp. σ-locally finite k-network) are *$\aleph_0$-spaces* (resp. *$\aleph$-spaces*).
Spaces with a countable network are *cosmic spaces*.

Let ${}^\omega\omega$ be the set of all functions from ω to ω. For f, $g\in{}^\omega\omega$, $f\geqq g$ if $\{n\in\omega: f(n)<g(n)\}$ is finite. Let $\mathbf{b}=\min\{|A|: {}^\exists\text{unbounded } A\subset{}^\omega\omega\}$.
Let $BF(\omega_2)$ mean "$\mathbf{b}\geqq\omega_2$", and $\neg BF(\omega_2)$ mean "$\mathbf{b}=\omega_1$". Then, (CH) $\Rightarrow$ $\neg BF(\omega_2)$, and (MA+ $\neg$CH) $\Rightarrow$ $BF(\omega_2)$.

Let X and Y be $\aleph$-spaces; or closed s-images of metric spaces. Then, Hypothesis holds ([T_1]).

Let X and Y be Lašnev spaces. Then, (CH) $\Rightarrow$ <u>*Hypothesis*</u> holds ([T_2]); and, $\neg BF(\omega_2)$ $\Leftrightarrow$ <u>*Hypothesis*</u> holds ([G]).

Let X and Y be CW-complexes (or closed images of CW-complexes). Then, $\neg BF(\omega_2)$ $\Leftrightarrow$ <u>*Hypothesis*</u> holds ([T_3]; [TZ]).

Now, let us consider following properties.

(C) Closed σ-compact, cosmic subsets are $\aleph_0$-spaces.
(A_1) Space with a σ-locally countable cs*-network.
(A_2) Fréchet space with a point-countable cs*-network.
(A_3) Space with a star-countable cs*-network.
(A_4) Space with a point-countable cs-network.
(A_5) Space with a compact-countable cs*-network.
(B_1) Fréchet space with a point-countable k-network.
(B_2) Space with a σ-HCP k-network.
(B_3) Space with a star-countable k-network.
(B_4) Space with a σ-compact-finite k-network.
(B_5) Space with a compact-countable k-network.

Then, (A_i) or (B_i) $\Rightarrow$ (C); (A_2) $\Rightarrow$ (B_1); (B_2) or (B_3) $\Rightarrow$ (B_4), etc. We note that first countable spaces with (B_4) are metric, etc.

Let X be a space, and let $\{L_\gamma : \gamma < \omega_1\}$ be a collection of disjoint sequences in X with $L_\gamma \to x_\gamma \notin L_\gamma$. Let $S = \cup\{L_\gamma : \gamma < \omega_1\} \cup L$, here $L = \{x_\gamma : \gamma < \omega_1\}$. For a compact metric subset K of X with $K \cap S = L$, let $K^* = S \cup K$ be a subspace of X with $K^*/K = (\Sigma L_\gamma^*)/L$, where $L_\gamma^* = L_\gamma \cup \{x_\gamma\}$. Here, K^*/K is a quotient space obtained from K^* by identifying all points of K to a single point, and ΣL_γ^* is the topological sum of $\{L_\gamma^* : \gamma < \omega_1\}$.

By <u>the</u> X and Y, we mean the spaces X and Y in *Hypothesis*; that is, the X and Y are k-spaces with point-countable k-networks.

In the following theorem, (C) is essential by Theorem 3(3) below.

Theorem 1. Let the X and Y satisfy (C). Then,

(1) If neither X nor Y contains a closed copy of any K*, *Hypothesis* is valid.

(2) *Hypothesis* is valid iff $\neg BF(\omega_2)$.

(3) If X=Y, *Hypothesis* is valid.

Corollary. Let the X satisfy one of (A_i), and let the Y do so. Then, *Hypothesis* is valid.

【The result for (A_1); (A_2); and (A_4) is due to [S]; [S or LLi]; and [LiL] resp. The result for (A_5) is due to [LLi], where spaces have compact-countable *closed* k-networks】

Corollary. Let the X satisfy one of (B_i), and and let the Y do so. Then, *Hypothesis* is valid iff $\neg BF(\omega_2)$. If X=Y, *Hypothesis* is valid.

【The result for (B_2); (B_3); (B_5) is due to [L]; [LT_1]; [LT_2] resp.】

Corollary. Let X and Y be k-spaces with point-countable *closed* k-networks such that $so(X) \leqq 2$, and $so(Y) \leqq 2$, here so(X) is the sequential order of X (see [AF]). Then, *Hypothesis* is valid iff $\neg BF(\omega_2)$.

If X=Y, *Hypothesis* is valid.

【The result is due to [S] under (CH)】

Theorem 2. (MA). Let the X satisfy one of (B_i), and let the Y satisfy one of (A_i) or (B_i). Then, *Hypothesis* is valid, but replace (K_3) by $(K_3{}^*)$: One of X and Y is locally k_ω, and another is locally $<k_c$, $c=2^\omega$.

Let us say that an operation is CD if it is the decomposition of operators " closed maps " and " dominations ". Let CD(metric) be the class of all spaces obtained from metric spaces under CD. Then, for example, spaces dominated by Lašnev spaces belong to CD(metric).

Note that spaces in CD(metric) have σ-compact-finite k-networks, and spaces in CD(separable metric) have star-countable k-networks.

Corollary. Let X and Y be spaces in the class CD(metric). Then,

(1) Under $\neg BF(\omega_2)$, $X \times Y$ is a k-space iff one of the following holds.

(k_1) X and Y are metric.

(k_2) X or Y is locally compact metric.

(k_3) X is dominated by a countable cover of locally compact metric spaces, and so is Y. If X=Y, the result holds without $\neg BF(\omega_2)$.

(2) Under (MA), the result in (1) holds, but replace (k_3) by

(k_3*) One of X and Y is dominated by a countable cover of locally compact metric spaces, and another is the topological sum of k_α-spaces, here $\alpha < 2^\omega$.

(3) X^ω is a k-space iff X is metric.

Theorem 3. (1) Under $BF(\omega_2)$, *Hypothesis* is not valid for Lašnev, CW-complexes X and Y with star-countable k-networks.

(2) Under $BF(\omega_2)$, *Hypothesis* is not valid for k-spaces X and Y with point-countable *closed* k-networks (indeed, X and Y are quotient, finite-to-one images of locally compact metric spaces) ([LiL]).

(3) Under (CH), *Hypothesis* is not valid for σ-compact, k-spaces X and Y, where X is a Lašnev, $\aleph_0$-space which is neither locally compact nor metric, and Y has a point-countable *closed* k-network with so(Y)=3, but Y is not locally k_ω (X and Y are quotient s-images of locally compact metric spaces) ([S]).

Question: Let X and Y be k-spaces with point-countable k-networks.

(1) If X^2 is a k-space, then does X satisfy (K_1), or (K_3) ?

(2) As a characterization for $X \times Y$ to be a k-space, are there different types of properties on X, Y from (K_1), (K_2), (K_3), and (K_3^*) ?

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