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A Homogeneity Improvement Approach to the Heat Equation with Strong Absorption

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In this announcement we present results to appear concerning the regularity of the free boundary $\partial\{u>0\}$ of non-negative solutions of the heat equation with strong absorption

$$\partial_t u - \Delta u = -\frac{1+\gamma}{2} u^\gamma \, , \, \gamma \in (0,1) \, . \tag{1}$$

Equation (1) has been used in L. K. Martinson [15] and in Ph. Rosenau, S. Kamin [17] to describe the transport of thermal energy in plasma. Alternatively it has been derived as the asymptotic limit of a system proposed by C. Bandle and I. Stakgold in [3] as a simple model for a reaction diffusion process.

Concerning the case of one space dimension the solution's behaviour near *extinction points* has been extensively studied (see for example A. Friedman, M. A. Herrero [9], M. A. Herrero, J. J. L. Velazquez [12, 13]) and for initial data with compact support the number of extinction points has been estimated (see [9] and X.-Y. Chen, H. Matano, M. Mimura [5]). The paper [5] contains furthermore time-continuity of the set $\{u>0\} \subset (0,\infty)\times \mathbf{R}$

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with respect to Hausdorff distance and it tells us that the one-dimensional free boundary is a subset of a locally finite union of graphs of continuous functions.
For the *stationary* problem H. W. Alt and D. Phillips proved in [1] regularity of the free boundary in higher dimensions. Regarding the time-dependent problem in higher dimesions, regularity of the solution, estimates of the Hausdorff measure of the free boundary, the asymptotic behaviour near *horizontal free boundary points* (points in which the behaviour in time is dominant, see Proposition 2.2) and other results are contained in the paper [6] by H. J. Choe and the author.

Here we announce a regularity result for $\partial\{u > 0\}$ in higher dimensions: suppose that u is a solution of the Cauchy problem and that the initial data u^0 satisfy $0 \le u^0 \in C_0^{2,\sigma}(\mathbf{R}^n)$ and $(u^0)^{-\gamma}\Delta u^0 \in L^\infty(\mathbf{R}^n)$: then $\partial\{u > 0\}$ can be decomposed into a regular part $R = \{(t,x) \in ((0,\infty) \times \mathbf{R}^n) \cap \partial\{u > 0\}$: at least one blow-up limit of u at (t,x) is a half-plane solution $\}$ such that $\partial\{u > 0\}$ is locally in an open neighborhood of R a $\mathbf{C}^{\frac{1}{2},1+\mu}$-surface, and a singular part Σ which is ignored by spatial integration by parts in $\{u > 0\}$, i.e.

$$\int_{\{u(t)>0\}} \partial_i\zeta = \int_{\partial_{\mathrm{red}}\{u(t)>0\}} \zeta\,\nu_i\,\mathcal{H}^{n-1} = \int_{R\cap\{s=t\}} \zeta\,\nu_i\,\mathcal{H}^{n-1}$$

for a.e. $t \in (0,\infty)$ and every $\zeta \in C_0^{0,1}(\mathbf{R}^n)$ (Corollary 6.1); here the reduced boundary $\partial_{\mathrm{red}}\{u(t) > 0\}$ is the set of free boundary points at which the outer normal of H. Federer [7, 4.5.5] exists. Let us first remark that while Σ is in the just mentioned sense a set of less relevance, it is in general not a set of small measure: the steady-state solution $\left(\frac{1-\gamma}{2}|x_1|\right)^{\frac{2}{1-\gamma}}$ satisfies $R = \emptyset$, $\Sigma = \partial\{u > 0\}$ and $\mathcal{H}^{n-1}(\Sigma) > 0$. Even worse, when perturbing the stationary equation to $\Delta u = g\,u^\gamma$ where g is a strictly positive C^∞-function, we expect (in analogy to the counter-example by D. Schaeffer in [18, 2.9] for the case $\gamma = 0$) the appearance of free boundaries such that the *relative boundary* of R is a set of positive $n-1$-dimensional Hausdorff measure.

This complicated behaviour of steady-state free boundaries distinguishes our equation from other equations like e.g. the porous medium equation. And we *have* to include the stationary behaviour fully into our considerations for two reasons: first, in the more physical context of [3] the reactant would be replenished at the boundary of some domain and thus be close to a non-trivial steady-state solution for large time. The second and more compelling reason is that by Proposition 2.2 the behaviour close to *each* non-horizontal free boundary point is that of a steady-state solution.
Let us furthermore point out that our situation is different from that of the one-phase Stefan problem (see A. Friedman, D. Kinderlehrer [10] and L. A. Caffarelli [4]) where it was physically justified to assume the temperature to be non-decreasing in time: in the setting of [3] we expect the formation of dead cores in finite time, so there have to be regions where the concentration u is decreasing. Regarding the propagation of a thermal pulse ([17]), the support of a smooth pulse with sufficiently steep slopes will first expand and later on shrink, so there are sign changes of $\partial_t u$.

We do not know whether it is possible to obtain our result by an extension of the sophisticated methods in [1]. We chose here a different approach which is related to the concept of Liapunov stability and to results concerning the Plateau problem.
In Section 1 we prove an "epiperimetric inequality" for the class of *half-plane solutions* $H = \{x \mapsto (\frac{1-\gamma}{2}\max(x\cdot\nu,0))^{\frac{2}{1-\gamma}} \;:\; \nu \in \partial B_1(0)\}$ and the *boundary-adjusted energy*

$$M(v) = \int_{B_1(0)} (|\nabla v|^2 + \max(v,0)^{1+\gamma}) - \frac{2}{1-\gamma}\int_{\partial B_1(0)} v^2 \, d\mathcal{H}^{n-1} \; :$$

if c is any non-negative homogeneous function of degree $\frac{2}{1-\gamma}$ which is close enough to the some $h \in H$, then there exists a function v with the same boundary values on $\partial B_1(0)$ but with a lower energy value

$$M(v) \;\leq\; (1-\kappa)M(c) + \kappa M(h) \;\; \text{(Theorem 1.1).} \tag{2}$$

In hommage to the inequality derived by E. R. Reifenberg for the perimeter, we call (2) by abuse of name "epiperimetric inequality." Our proof however owes nothing to the proof of the epiperimetric inequality in E. R. Reifenberg [16] or that in J. E. Taylor [19] as it works *completely* by indirect methods. The boundary-adjusted energy plays here the role of the Liapunov function, i.e. its scaled version satisfies a monotonicity formula: defining

$$\Phi_{(t_0,x_0)}(r) = r^{-n-\frac{2(1+\gamma)}{1-\gamma}} \int_{B_r(x_0)} \left(|\nabla u(t_0,\cdot)|^2 + \max(u(t_0,\cdot),0)^{1+\gamma}\right)$$
$$- \frac{2}{1-\gamma}\, r^{-n+1-\frac{4}{1-\gamma}} \int_{\partial B_r(x_0)} u(t_0,\cdot)^2 \, d\mathcal{H}^{n-1} ,$$

the function $r \mapsto \bar{C} r^\beta + \Phi_{(t_0,x_0)}(r)$ is non-decreasing for any $(t_0,x_0) \in \partial\{u > 0\}$ at which $|\partial_t(u^{1-\gamma})|$ is Hölder-continuous (Theorem 3.1).
The epiperimetric inequality (2) leads now to the differential inequality

$$\max(\Phi_{(t_0,x_0)}(r) - \Phi_{(t_0,x_0)}(0+), C_2 r^\beta)' \geq \Lambda \frac{1}{r} \max(\Phi_{(t_0,x_0)}(r) - \Phi_{(t_0,x_0)}(0+), C_2 r^\beta)$$

which in turn implies Hölder-continuity of $r \mapsto \Phi_{(t_0,x_0)}(r)$ and a convergence estimate for $\frac{u(t_0,x_0+r\cdot)}{r^{\frac{2}{1-\gamma}}}$ to the *unique blow-up limit* u_0 (Theorem 4.1).
This reminds very much of the use of Liapunov functions in the theory of linearized stability and of *Liapunov's direct approach* (compare e.g. to Theorem 18.7 and Remark 18.9 in H. Amann [2]). The convergence result itself on the other hand is reminiscent of a result by J. K. Hale and P. Massatt for differentiable gradient systems, by which one obtains single-point ω-limit sets in the case that the multiplicity of the eigenvalue 0 at critical points is 1 ([11, Theorem 4.3]). Let us however point out that our method also works for the obstacle problem where the second variation of the energy vanishes in more than one direction and that *our energy M is not of class C^2*, so a linearization regardless of the direction is not possible. This also means that we cannot apply the center manifold theorem and test the local center manifold for coincidence with the invariant manifold H in order to obtain our result.

In Sections 5 and 6 we derive – mainly by topological methods – the relative openness and $\mathbf{C}^{\frac{1}{2},1+\mu}$-regularity of the set R.
Let us conclude with the remark that it should be possible to obtain higher regularity of R.

1 THE EPIPERIMETRIC INEQUALITY

$H := \{x \mapsto (\frac{1-\gamma}{2}\max(x\cdot\nu,0))^{\frac{2}{1-\gamma}} \;:\; \nu \in \partial B_1(0)\}$ of all half-plane solutions as well as the *boundary-adjusted energy*

$$M(v) := \int_{B_1(0)} (|\nabla v|^2 + \max(v,0)^{1+\gamma}) - \frac{2}{1-\gamma}\int_{\partial B_1(0)} v^2 \, d\mathcal{H}^{n-1}$$

for $v \in H^{1,2}(B_1(0))$.
Then M takes for $h \in H$ the value $M(h) = \frac{1}{2}M((\frac{1-\gamma}{2}|x\cdot\nu|)^{\frac{2}{1-\gamma}})$

$$= \frac{1}{2}\int_{B_1(0)} \left(\frac{1-\gamma}{2}|x_1|\right)^{\frac{2}{1-\gamma}} \left(\frac{1-\gamma}{2}|x_1|\right)^{\frac{2\gamma}{1-\gamma}} \left(1-\frac{1+\gamma}{2}\right)$$

$$= \frac{1}{2}\frac{1-\gamma}{2}\int_{B_1(0)} \left(\frac{1-\gamma}{2}|x_1|\right)^{\frac{2(1+\gamma)}{1-\gamma}} =: \frac{\alpha_n}{2} > 0\,.$$

THEOREM 1.1 (the epiperimetric inequality) *There exist $\kappa \in (0,1)$ and $\delta \in (0,1)$ such that the following holds for each non-negative function c in $H^{1,2}(B_1(0))$ that is homogeneous of degree $\frac{2}{1-\gamma}$:*
if $\|c-h\|_{H^{1,2}(B_1(0))} \le \delta$ for some $h \in H$ then there exists a function $v \in H^{1,2}(B_1(0))$ such that $v = c$ on $\partial B_1(0)$ and $M(v) \le (1-\kappa)M(c) + \kappa\frac{\alpha_n}{2}$.

2 Introduction to the Problem and Properties of Solutions

Throughout this paper we assume that $\gamma \in (0,1)$, that $\Omega \subset \mathbf{R}^n$ is a bounded domain with $C^{2,\sigma}$ boundary for some $\sigma \in (0,1)$, that the boundary data g are non-negative and satisfy $g \in W^{1,2}_{\infty}((0,T) \times \Omega)$ for $T < \infty$ and that $u \in L^2((0,T); H^{1,2}(\Omega))$ for $T < \infty$ is a solution of

$$\begin{aligned} u = g \text{ on } (\{0\} \times \Omega) \cup ((0,\infty) \times \partial\Omega) \\ \text{and } \partial_t u - \Delta u = -\tfrac{1+\gamma}{2} \max(u,0)^\gamma \text{ in } (0,\infty) \times \Omega \end{aligned} \tag{3}$$

in the sense of distributions.

It is well known that (3) has a unique solution which is by the comparison principle bounded in $(0,T) \times \Omega$ for $T < \infty$. Taking $\min(u,0)$ as test function in the weak equation (3) we see immediately that u has to be non-negative. Standard energy estimates furthermore imply that $u \in H^{1,2}((0,T) \times \Omega)$ for $T < \infty$ and parabolic L^p-theory yields that $u \in W^{1,2}_p((0,T) \times \Omega)$ for $T < \infty$ and $1 \leq p < \infty$.

Let us also remark that $g(t,x) = 0$ for $t \geq T_0$ leads by comparison to $\max(0, (1-\gamma)\frac{1+\gamma}{2}(\frac{2(\sup u(T_0,\cdot))^{1-\gamma}}{(1+\gamma)(1-\gamma)} - t + T_0))^{\frac{1}{1-\gamma}}$ (in $\{t > T_0\} \times \Omega$) to the conclusion that $u(t,x) = 0$ for $t \geq T_0 + \frac{2(\sup u(T_0,\cdot))^{1-\gamma}}{(1+\gamma)(1-\gamma)}$.

The following Theorem 2.1, Proposition 2.1, Corollary 2.1, Lemma 2.1, Proposition 2.2 and Corollary 2.2 have been stated and proved as Theorem 3.1, Proposition 4.1, Corollary 4.2, Lemma 5.1, Proposition 5.2 and Corollary 5.3 in [6]. Note however that the equation considered in [6] does not contain the factor $\frac{1+\gamma}{2}$, so several of the statements differ from those in [6] by a factor.

THEOREM 2.1 (regularity) *There exists a constant $\bar{C} < \infty$ depending only on n, γ, T, M and $\delta \in (0,1)$ such that for each solution u of (3) with respect to n, γ and $g \in \mathbf{C}^{\frac{2+\sigma}{2}, 2+\sigma}([0,T] \times \bar{\Omega})$ satisfying*

$$1 + \sup_{(0,T)\times\Omega} g + \sup_{\{0\}\times\Omega} |g^{-\gamma}\Delta g| + \sup_{(0,T)\times\partial\Omega} |g^{-\gamma}\partial_t g| \leq M < \infty$$

the estimates

$$\|\nabla(u^{\frac{1-\gamma}{2}})\|_{L^\infty((\delta,T-\delta)\times\Omega_\delta)} + \|\partial_t(u^{1-\gamma})\|_{L^\infty((0,T)\times\Omega)} \le \bar{C}$$
$$\text{and } \|u^{\frac{1-\gamma}{2}}\|_{\mathbf{C}^{\frac{1}{2},1}((\delta,T-\delta)\times\Omega_\delta)} \le \bar{C}$$

hold (here $\Omega_\delta := \Omega - B_\delta(\partial\Omega)$*).*

PROPOSITION 2.1 (non-degeneracy) *There exists a constant* $c > 0$ *depending only on* n *and* γ *such that the solution* u *of (3) satisfies for every* $(t_0, x_0) \in \overline{\{u > 0\}}$ *and every* $Q_r(t_0, x_0) \subset (0, \infty) \times \Omega$ *the estimate*

$$\sup_{Q_r^-(t_0,x_0)} u \ge c\, r^{\frac{2}{1-\gamma}} .$$

COROLLARY 2.1 (finite propagation speed of $\{u > 0\}$) *There exists a constant* $1 \le S < \infty$ *depending only on* n, γ *and* M *such that for each solution* u *of (3) with respect to* n, γ *and* $g \in \mathbf{C}^{\frac{2+\sigma}{2}, 2+\sigma}([0,\infty) \times \bar{\Omega})$ *satisfying*

$$1 + \sup_{\{0\}\times\Omega} |g^{-\gamma}\Delta g| + \sup_{(0,\infty)\times\partial\Omega} |g^{-\gamma}\partial_t g| \le M < \infty$$

and for every $Q_r^+(t_0, x_0) \subset (0, \infty) \times \Omega$ *the implication*

$$u(t_0, \cdot) = 0 \text{ in } B_r(x_0) \Rightarrow u(t_0 + s^2, \cdot) = 0 \text{ in } B_{\max(0, r - Ss)}(x_0)$$

holds.

LEMMA 2.1 (subsolution property) *The function* $w := |\partial_t(u^{1-\gamma})|^{\frac{1}{1-\gamma}}$ *is subcaloric in* $(0, \infty) \times \Omega$ *in the sense that any caloric function* $v \in L^\infty(Q_r(t_0, x_0)) \cap C^0(\overline{Q_r(t_0, x_0)} \cap \{u > 0\})$ *satisfying* $Q_r(t_0, x_0) \subset (0, \infty) \times \Omega$, $0 \le v$ *a.e. in* $Q_r(t_0, x_0)$ *and* $(1-\gamma)|\partial_t u| \le u^\gamma v^{1-\gamma}$ *in* $(\{t_0 - r^2\} \times B_r(x_0)) \cup ((t_0 - r^2, t_0 + r^2) \times \partial B_r(x_0))$ *must be* $\ge w$ *a.e. in* $Q_r(t_0, x_0)$.

We define u to be a solution of the *Cauchy problem* if

$$\begin{gathered} u^0 \in C_0^{2,\sigma}(\mathbf{R}^n)\,,\ (u^0)^{-\gamma}\Delta u^0 \in L^\infty(\mathbf{R}^n)\,, \\ u = u^0 \text{ on } \{0\} \times \mathbf{R}^n\,, \\ u \ge 0 \text{ and } \partial_t u - \Delta u = -\tfrac{1+\gamma}{2} \max(u, 0)^\gamma \text{ in } (0, \infty) \times \mathbf{R}^n \end{gathered} \tag{4}$$

in the sense of distributions.
Note that this u coincides by Corollary 2.1 in $(0,\infty)\times B_Z(0)$ with the solution of (3) with respect to $g(t,x) = u^0(x)$ and $\Omega = B_Z(0)$ provided that Z has been chosen large enough in terms of n, γ and u^0.
For a solution u of the Cauchy problem (4) we know in addition the following from [6]:

PROPOSITION 2.2 (horizontal and non-horizontal points) *The following dichotomy holds at each free boundary point $(t_0, x_0) \in ((0,\infty)\times\Omega)\cap \partial\{u>0\}$:*
either $\limsup_{\{u>0\}\ni(t,x)\to(t_0,x_0)} |\partial_t(u^{1-\gamma})| = (1-\gamma)\frac{1+\gamma}{2}$ *in which case (t_0,x_0) is called a* horizontal point *and* $\max(0,(1-\gamma)\frac{1+\gamma}{2}(-t))^{\frac{1}{1-\gamma}}$ *is the unique* blow-up limit *with respect to every* blow-up sequence $u_m(t,x) = {\rho_m}^{-\frac{2}{1-\gamma}}\, u(t_0 + {\rho_m}^2\, t, x_0 + \rho_m\, x)$ *(where $\rho_m \to 0$ as $m\to\infty$),*
or $\limsup_{\{u>0\}\ni(t,x)\to(t_0,x_0)} |\partial_t(u^{1-\gamma})| = 0$ *in which case (t_0,x_0) is called a* non-horizontal point *and every blow-up limit is a non-trivial steady-state solution of (3) which is homogeneous of degree $\frac{2}{1-\gamma}$.*

Throughout this paper we will often use the term *homogeneous solution of degree* $\frac{2}{1-\gamma}$. This will always denote a non-negative homogeneous solution of degree $\frac{2}{1-\gamma}$ of class C^2 of the equation $\Delta v = \frac{1+\gamma}{2} v^\gamma$. Moreover, let us denote by Hor the *set of horizontal points.*

COROLLARY 2.2 (relative openness of non-horizontal points) *The set of non-horizontal free boundary points $\partial\{u>0\}$ –* Hor *is open relative to $((0,\infty)\times \mathbf{R}^n)\cap\partial\{u>0\}$.*

3 THE MONOTONICITY FORMULA

A powerful tool is now given by the monotonicity formula introduced by the author in an elliptic version ([20, Theorem 3.1]) and in a parabolic version ([21, Theorem 3.1]). As we are going to need both versions in the subsequent sections, we state here the monotonicity formula for the perturbed elliptic case, Theorem 3.1, and that for the parabolic case, Theorem 3.2. Since Theorem 3.1 has the advantage of being *local* we shall prefer it to Theorem 3.2 in the remaining part of the paper. This way yields (by very minor modifications of the following proofs) also an independent proof of regularity for the stationary boundary value problem.

THEOREM 3.1 (the monotonicity formula) *Suppose that* $\beta \in (0,1)$, *that* $C < \infty$ *and that* $K \subset\subset ((0,\infty)\times\Omega)\cap\partial\{u>0\}$ *satisfies for* $(t_0,x_0)\in K$ *and* $\delta := \frac{1}{2}\inf_K \operatorname{dist}(\cdot,\partial\Omega)$ *the estimate* $|\partial_t(u^{1-\gamma})(t_0,x)| \le C|x-x_0|^\beta$ *for* $x \in B_\delta(x_0)$. *Then there exists* $\bar{C} < \infty$ *such that for all* $0<\rho<\sigma<\delta$ *and every* $(t_0,x_0)\in K$ *the function*

$$\Phi_{(t_0,x_0)}(r) := r^{-n-\frac{2(1+\gamma)}{1-\gamma}} \int_{B_r(x_0)} \left(|\nabla u(t_0,\cdot)|^2 + \max(u(t_0,\cdot),0)^{1+\gamma}\right)$$

$$- \frac{2}{1-\gamma}\, r^{-n+1-\frac{4}{1-\gamma}} \int_{\partial B_r(x_0)} u(t_0,\cdot)^2 \, d\mathcal{H}^{n-1} ,$$

defined in $(0,\delta)$, *satisfies* $\Big|\Phi_{(t_0,x_0)}(\sigma) - \Phi_{(t_0,x_0)}(\rho)$

$$- \int_\rho^\sigma r^{-n-\frac{2(1+\gamma)}{1-\gamma}} \int_{\partial B_r(x_0)} 2\left(\nabla u(t_0,\cdot)\cdot\nu - \frac{2}{1-\gamma}\,\frac{u(t_0,\cdot)}{r}\right)^2 d\mathcal{H}^{n-1}\, dr\Big|$$

$\le \bar{C}\,(\sigma^\beta - \rho^\beta)$.

THEOREM 3.2 *Assume that* u *is a solution of the Cauchy problem (4), that* $(t_0,x_0)\in(0,\infty)\times\mathbf{R}^n$, *that* $T_r^-(t_0) = (t_0-4r^2, t_0-r^2)\times\mathbf{R}^n$, *that* $0<\rho<\sigma<\frac{\sqrt{t_0}}{2}$ *and that*

$$G_{(t_0,x_0)}(t,x) = 4\pi(t_0-t)\,|4\pi(t_0-t)|^{-\frac{n}{2}-1} \exp\left(-\frac{|x-x_0|^2}{4(t_0-t)}\right)$$

is the backwards heat kernel. Then

$$\Psi^-_{(t_0,x_0)}(r) = r^{-2-\frac{2(1+\gamma)}{1-\gamma}} \int_{T_r^-(t_0)} \left(|\nabla u|^2 + \max(u,0)^{1+\gamma}\right) G_{(t_0,x_0)} +$$

$$- \frac{2}{1-\gamma}\frac{1}{2} r^{-\frac{4}{1-\gamma}} \int_{T_r^-(t_0)} \frac{1}{t_0 - t} u^2\, G_{(t_0,x_0)}$$

satisfies the monotonicity formula

$$\Psi^-_{(t_0,x_0)}(\sigma) - \Psi^-_{(t_0,x_0)}(\rho) = \int_\rho^\sigma r^{-1-\frac{4}{1-\gamma}} \int_{T_r^-(t_0)} \frac{1}{t_0-t} \Big(\nabla u \cdot (x - x_0)$$

$$- 2(t_0 - t)\partial_t u - \frac{2}{1-\gamma} u \Big)^2 G_{(t_0,x_0)}\, dr \geq 0 .$$

Let $\Psi^-_{(t_0,x_0)}(0+)$ *denote the right limit* $\lim_{r\to 0} \Psi^-_{(t_0,x_0)}(r) \in [-\infty, \infty)$. *Then* $(t_0, x_0) \mapsto \Psi^-_{(t_0,x_0)}(0+)$ *is an upper semicontinuous function in* $(0,\infty) \times \mathbf{R}^n$.

4 An energy decay estimate and uniqueness of blow-up limits

In this section we state that an epiperimetric inequality always implies an energy decay estimate and uniqueness of blow-up limits. More precisely:

THEOREM 4.1 (energy decay, uniqueness of blow-up limits) *Suppose that* $\beta \in (0,1)$, *that* $C < \infty$ *and that* $K \subset\subset ((0,\infty)\times\Omega)\cap\partial\{u > 0\}$ *satisfies for* $(t_0, x_0) \in K$ *and* $\delta := \frac{1}{2}\inf_K \operatorname{dist}(\cdot, \partial\Omega)$ *the estimate* $|\partial_t(u^{1-\gamma})(t_0, x)| \leq C|x - x_0|^\beta$ *for* $x \in B_\delta(x_0)$. *Assume furthermore that the epiperimetric inequality holds with* $\kappa \in (0,1)$ *for each* $c_r(x) := \left(\frac{|x|}{r}\right)^{\frac{2}{1-\gamma}} u(t_0, x_0 + r\frac{x}{|x|})$ *such that* $r \leq r_0 < 1$, *and let* u_0 *denote an arbitrary blow-up limit of* u *at the point* (t_0, x_0) .
Then for each $\Lambda \in \left(0, \min\left(\beta, \left(n + \frac{2(1+\gamma)}{1-\gamma}\right)\frac{\kappa}{1-\kappa}\right)\right)$ *there exists* $C^\star < \infty$ *such that*

$$|\Phi_{(t_0,x_0)}(r) - \Phi_{(t_0,x_0)}(0+)| \leq C^\star r^\Lambda \text{ for } (t_0,x_0) \in K \text{ and } r \in (0, r_0) ,$$

$$\int_{\partial B_1(0)} \left| \frac{u(t_0, x_0 + rx)}{r^{\frac{2}{1-\gamma}}} - u_0(0,x) \right| d\mathcal{H}^{n-1} \leq C^\star r^{\frac{\Lambda}{2}} \text{ for } (t_0,x_0) \in K$$

and $r \in (0, \frac{r_0}{2})$, *and* u_0 *is the unique blow-up limit of* u *at the point* (t_0, x_0) .

5 Asymptotic behaviour near regular points

The point is that the assumptions of Theorem 4.1 can be verified uniformly in an open neighborhood of a regular free boundary point, i.e. a free boundary point at which at least one blow-up limit coincides with a half-plane solution. Thus we obtain the following

COROLLARY 5.1 (differentiability) *Let u be a solution of the Cauchy problem (4) and suppose that $(t_0, x_0) \in R$. Then there exist $\delta > 0$, a function $\nu : \overline{Q_\delta(t_0, x_0)} \cap R \to \partial B_1(0)$ and $\frac{\omega(z)}{z} \to 0$ as $z \to 0$ such that*

$$\left| u(t,x) - \left(\frac{1-\gamma}{2} \max((x-\bar{x})\cdot\nu(\bar{t},\bar{x}), 0)\right)^{\frac{2}{1-\gamma}} \right| \le \omega\left(|x-\bar{x}|^{\frac{2}{1-\gamma}} + |t-\bar{t}|^{\frac{1}{1-\gamma}}\right)$$

for every $(\bar{t}, \bar{x}) \in \overline{Q_\delta(t_0, x_0)} \cap R$.

6 Regularity

The key idea to prove the relative openness and regularity of the set R is that the half-plane solutions take a lower energy value than any other homogeneous solution of degree $\frac{2}{1-\gamma}$ whose coincidence set has a non-empty interior. This proven, it is possible to derive the following regularity result:

THEOREM 6.1 *Let u be a solution of the Cauchy problem (4). Then $\partial\{u > 0\}$ is locally in an open neighborhood of the set R a $\mathbf{C}^{\frac{1}{2}, 1+\mu}$-surface. The space outer normal $\nu(t,x)$ to $\partial\{u > 0\}$ is locally in R a Hölder-continuous function.*

By the Hausdorff measure estimate in [6] this implies

COROLLARY 6.1 *Let u be a solution of the Cauchy problem (4). Then $((0,\infty) \times \mathbf{R}^n) \cap \partial\{u > 0\}$ is the disjoint union of R and Σ, $\partial\{u > 0\}$ is locally* *in an open neighborhood of R a $\mathbf{C}^{\frac{1}{2},1+\mu}$-surface and the set Σ is ignored by spatial integration by parts in $\{u>0\}$, i.e.*

$$\int_{\{u(t)>0\}} \partial_i \zeta = \int_{\partial_{\mathrm{red}}\{u(t)>0\}} \zeta\, \nu_i\, \mathcal{H}^{n-1} = \int_{R\cap\{s=t\}} \zeta\, \nu_i\, \mathcal{H}^{n-1}$$

for a.e. $t \in (0,\infty)$ and every $\zeta \in C_0^{0,1}(\mathbf{R}^n)$; here the reduced boundary $\partial_{\mathrm{red}}\{u(t)>0\}$ is defined as the set of free boundary points at which the outer normal of of H. Federer [7, 4.5.5] exists.

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