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THE EXISTENCE OF POSITIVE SOLUTIONS FOR A CLASS OF INDEFINITE WEIGHT SEMILINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS

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1. Introduction

We discuss the existence of positive classical solutions of the boundary value problems:

$$(I_\lambda^\alpha) \qquad \begin{cases} -\Delta u = \lambda g(x) f(u) & \text{in} \quad \Omega \\ (1-\alpha)\dfrac{\partial u}{\partial n} + \alpha u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where λ and α are real parameters and Ω is an open bounded region of $\mathbf{R}^N$, $N \geq 2$ with smooth boundary $\partial\Omega$. We shall suppose that $\alpha \leq 1$; thus $\alpha = 0$ corresponds to the Neumann problem, $\alpha = 1$ to the Dirichlet problem and $0 < \alpha < 1$ to the usual Robin problem. We shall assume throughout that $g : \overline{\Omega} \to \mathbf{R}$ is a smooth function which changes sign on Ω.

Equation (I_λ^α) arises in population genetics with $f(u) = u(1-u)$ (see [7]). In this setting (I_λ^α) is a reaction-diffusion equation where the real parameter $\lambda > 0$ corresponds to the reciprocal of the diffusion coefficient and the unknown function u represents a relative frequency so that there is interest only in solutions satisfying $0 \leq u \leq 1$. In this paper we shall study the structure of the set of positive solutions of (I_λ^α) in the cases where $f(u) = u(1 - |u|^p)$ and $f(u) = u(1 + |u|^p)$, $p > 0$. In order to obtain a better understanding of this structure we no longer impose the restrictions that $\lambda > 0$ or that $u \leq 1$.

We obtain new existence results by using a variational method based on the properties of eigencurves, i.e., properties of the map $\lambda \to \mu(\lambda)$ where $\mu(\lambda)$ denotes the principal eigenvalue of the linear problem

$$(1.1) \qquad \begin{cases} -\Delta u - \lambda g(x)u = \mu u & \text{in} \quad \Omega \\ (1-\alpha)\frac{\partial u}{\partial n} + \alpha u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Our method works provided that the linearized problem for (I_λ^α), viz,

$$(L^\alpha) \qquad \begin{cases} -\Delta u = \lambda g(x)u & \text{in} \quad \Omega \\ (1-\alpha)\dfrac{\partial u}{\partial n} + \alpha u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

has principal eigenvalues and it is shown in Afrouzi and Brown [1] that this occurs on an interval $[\alpha_0, 1]$ where $\alpha_0 \leq 0$. Thus we are able to obtain existence results for (I_λ^α) even in the case of nonstandard Robin boundary conditions where α is small and negative. Our method depends on using eigencurves to produce an equivalent norm on $W^{1,2}(\Omega)$; such an equivalent norm is also introduced in [4].

Solutions of (I_λ^α) also arise from the bifurcation of solutions from the zero solution in the $(\lambda, u)-$ plane. We shall investigate the nature of bifurcating solutions in the cases $f(u) = u(1-|u|^p)$ and $f(u) = u(1+|u|^p)$; in the former case we show that the solutions whose existence has been established by variational means are completely distinct from those arising from bifurcation but that in the latter case variational and bifurcation methods give existence results for precisely the same λ-ranges.

Our results illustrate the very significant role played by the indefinite weight function $g(x)$ in the existence of positive solutions of (I_λ^α). If $g(x) \equiv 1$ and $f(u) =$

$u(1-|u|^p)$ then (I_λ^α) becomes

$$\begin{cases} -\Delta u = \lambda u(1-|u|^p) & \text{in} \quad \Omega \\ (1-\alpha)\dfrac{\partial u}{\partial n} + \alpha u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Then, when $\alpha > 0$, it is well known that positive solutions must satisfy $0 < u < 1$ and are precisely those arising out of bifurcation from the zero solution; moreover the equation has no positive solutions if $\lambda < \lambda_1$ where λ_1 denotes the least eigenvalue of the Laplacian. We shall show, however, that, when g changes sign, the variational method proves the existence of a positive solution for all λ, $0 < \lambda < \lambda^+(\alpha)$, where $\lambda^+(\alpha)$ denotes the positive principal eigenvalue of (L^α) and that such solutions are not bounded above by 1.

The plan of the paper is as follows. In section 2 we first recall the facts that we shall require about eigencurves and show how eigencurves can be used to generate an equivalent norm for $W^{1,2}(\Omega)$; then using this equivalent norm we prove the existence of solutions by applying variational methods. In section 3 we discuss the solutions of (I_λ^α) which arise from bifurcations and compare these with the variational solutions obtained in section 2 for the case where $\alpha \in (0,1]$, i.e., where we have Dirichlet or the standard Robin boundary condition.

2. Variational Solutions

We first recall some facts about how the method of eigencurves can be used to prove the existence of principal eigenvalues of (L^α) (see, e.g., [1]). For fixed λ we denote by $\mu(\alpha,\lambda)$ the principal eigenvalue of the Schrödinger problem (1.1). Clearly λ is a principal eigenvalue of (L^α) if and only if $\mu(\alpha,\lambda) = 0$.

It can be shown that $\mu(\alpha,\lambda)$ has the variational characterisation

$$\mu(\alpha,\lambda) = \inf\{\int_\Omega (|\nabla u|^2 - \lambda g u^2)\, dx + \frac{\alpha}{1-\alpha}\int_{\partial\Omega} u^2\, dS_x : u \in W^{1,2}(\Omega),\ \int_\Omega u^2\, dx = 1\}$$

from whence it follows that

(i) $\alpha \to \mu(\alpha, \lambda)$ is an increasing function;

(ii) $\lambda \to \mu(\alpha, \lambda)$ is a concave function with a unique maximum such that $\mu(\alpha, \lambda) \to -\infty$ as $\lambda \to \pm\infty$.

If $\alpha \in (0, 1]$, then $\mu(\alpha, 0) > 0$. In particular, $\lambda \to \mu(\alpha, \lambda)$ has exactly one negative zero $\lambda^-(\alpha)$ and one positive zero $\lambda^+(\alpha)$. Thus $\lambda^-(\alpha)$ and $\lambda^+(\alpha)$ are principal eigenvalues for (L^α).

If $\alpha = 0$, i.e., we have the Neumann problem, then $\mu(0, 0) = 0$. If $\int_\Omega g\, dx < 0$, (L^α) has principal eigenvalues $\lambda^-(0) = 0$ and $\lambda^+(0) > 0$. On the other hand, if $\int_\Omega g\, dx > 0$, there exist principal eigenvalues such that $\lambda^-(0) < \lambda^+(0) = 0$.

Suppose now that $\int_\Omega g\, dx < 0$ and that α is small and negative. Then, since $\alpha \to \mu(\alpha, \lambda)$ is increasing, there still exist principal eigenvalues $\lambda^-(\alpha) < \lambda^+(\alpha)$ of (L^α) but now both $\lambda^-(\alpha)$ and $\lambda^+(\alpha)$ are positive. It can be shown that there exists $\alpha_0 < 0$ such that the above is true for all $\alpha \in (\alpha_0, 0)$, but for $\alpha < \alpha_0$ $\mu(\alpha, \lambda) < 0$ for all λ so that principal eigenvalues no longer exist.

Similar considerations show that when $\int_\Omega g\, dx > 0$ there exists $\alpha_0 < 0$ such that there principal eigenvalues $\lambda^-(\alpha) < \lambda^+(\alpha) < 0$ for $\lambda_0 < \lambda < 0$ but when $\int_\Omega g\, dx = 0$ there are no principal eigenvalues for $\alpha < 0$.

We now show how the above eigencurves $\lambda \to \mu(\alpha, \lambda)$ may be used to produce an equivalent norm for $W^{1,2}(\Omega)$.

Theorem 2.1. *Suppose $\alpha \in (0, 1)$ or that $\int_\Omega g\, dx \neq 0$ and $\alpha \in (\alpha_0, 0]$ so that (L^α) has principal eigenvalues $\lambda^-(\alpha)$ and $\lambda^+(\alpha)$. For any $\lambda \in (\lambda^-(\alpha), \lambda^+(\alpha))$*

$$||u||_\lambda = \{\int_\Omega [|\nabla u|^2 - \lambda g u^2]\, dx + \frac{\alpha}{1-\alpha} \int_{\partial\Omega} u^2\, dS_x\}^{1/2}$$

defines a norm in $W^{1,2}(\Omega)$ which is equivalent to the usual norm for $W^{1,2}(\Omega)$.

Proof. Since $|| \, ||_\lambda$ corresponds to the bilinear form

$$< u, v >_\lambda = \int_\Omega (\nabla u \, . \, \nabla v - \lambda g u v) \, dx + \frac{\alpha}{1-\alpha} \int_{\partial\Omega} u v \, dS_x$$

in order to prove that $|| \, ||_\lambda$ is a norm it suffices to prove that $< u, u >_\lambda > 0$ for all $u \in W^{1,2}(\Omega) - \{0\}$. By the variational characterisation of $\mu(\alpha, \lambda)$ we have

$$(2.1) \qquad < u, u >_\lambda = \int_\Omega [|\nabla u|^2 - \lambda g u^2] \, dx + \frac{\alpha}{1-\alpha} \int_{\partial\Omega} u^2 \, dS_x \geq \mu(\alpha, \lambda) \int_\Omega u^2 \, dx.$$

Hence, if $\lambda^-(\alpha) < \lambda < \lambda^+(\alpha)$, $\mu(\alpha, \lambda) > 0$ and so $< u, u >_\lambda > 0$ whenever $u \neq 0$. Thus $|| \, ||_\lambda$ is a norm.

We now prove the equivalence of the norms. It is easy to see that there exists a constant $K > 0$ such that $||u||_\lambda \leq K ||u||_{W^{1,2}(\Omega)}$. Suppose that there exists a sequence $\{u_n\} \subseteq W^{1,2}(\Omega)$ such that $||u_n||_{W^{1,2}(\Omega)} = 1$ and $||u_n||_\lambda \to 0$ as $n \to \infty$. Since $\{u_n\}$ is bounded in $W^{1,2}(\Omega)$, there exists a subsequence, which for convenience we again denote by $\{u_n\}$, such that $u_n \rightharpoonup v$ weakly in $W^{1,2}(\Omega)$. Since $W^{1,2}(\Omega)$ may be compactly embedded in $L^2(\Omega)$ and in $L^2(\partial\Omega)$, we have $u_n \to v$ in $L^2(\Omega)$ and $u_n \to v$ in $L^2(\partial\Omega)$. Since $||u_n||_\lambda \to 0$, it follows from equation (2.1) that $u_n \to 0$ in $L^2(\Omega)$, i.e., $v = 0$. Thus $u_n \to 0$ in $L^2(\Omega)$ and $u_n \to 0$ in $L^2(\partial\Omega)$ and so, since $\lim_{n\to\infty} [\int_\Omega [|\nabla u_n|^2 - \lambda g u_n^2] \, dx + \frac{\alpha}{1-\alpha} \int_{\partial\Omega} u_n^2 \, dS_x] = 0$, we must have that $\lim_{n\to\infty} \int_\Omega |\nabla u_n|^2 \, dx = 0$. This is impossible, however, as $||u_n||_{W^{1,2}(\Omega)} = 1$ for all n and so we have a contradiction. It follows that $||u||_\lambda$ and $||u||_{W^{1,2}(\Omega)}$ are equivalent norms.

Using a similar argument it can be proved that

Corollary 2.2. *If $\lambda \in (\lambda^-(1), \lambda^+(1))$ where $\lambda^-(1)$ and $\lambda^+(1)$ denote the principal eigenvalues of (L^α) in the case of Dirichlet boundary conditions, then*

$$||u||_\lambda = \{ \int_\Omega [|\nabla u|^2 - \lambda g u^2] \, dx \}^{1/2}$$

defines a norm on $W_0^{1,2}(\Omega)$ which is equivalent to the usual norm for $W_0^{1,2}(\Omega)$.

We can now prove the existence of solutions to nonlinear equations by using variational methods. We first consider the case where $f(u) = u(1 - |u|^p)$.

Theorem 2.3. *Suppose $\alpha \in (0,1)$ or that $\int_\Omega g\, dx \neq 0$ and $\alpha \in (\alpha_0, 0]$. Then, if $0 < p < \frac{4}{n-2}$,*

$$\begin{cases} -\Delta u = \lambda g(x) u(1 - |u|^p) & \textit{in} \quad \Omega \\ (1-\alpha)\dfrac{\partial u}{\partial n} + \alpha u = 0 & \textit{on} \quad \partial\Omega, \end{cases} \tag{2.2}$$

has a positive solution for all $\lambda \in (\lambda^-(\alpha), \lambda^+(\alpha))$, provided that $\lambda \neq 0$.

Proof. Let $M = \{u \in W^{1,2}(\Omega) : \lambda \int_\Omega g|u|^{p+2}\, dx = -1\}$. Since $g < 0$ on an open subset of Ω, M is nonempty. Moreover, as $L^{p+2}(\Omega)$ may be embedded compactly in $W^{1,2}(\Omega)$, M is weakly closed in $W^{1,2}(\Omega)$.

Since the natural energy functional associated with equation (2.2), viz.,

$$u \to \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - \frac{1}{2}\lambda g u^2 + \frac{\lambda}{p+2} g|u|^{p+2}\right) dx + \frac{\alpha}{2(1-\alpha)} \int_{\partial\Omega} u^2\, dS_x$$

is bounded neither above nor below, we are led to consider the constrained problem of minimizing the functional

$$J_\lambda(u) = \int_\Omega (|\nabla u|^2 - \lambda g u^2)\, dx + \frac{\alpha}{1-\alpha} \int_{\partial\Omega} u^2\, dS_x = ||u||_\lambda^2$$

restricted to M.

It is easy to see that J_λ is sequentially weakly lower semicontinuous and Theorem 2.1 shows that J_λ is coercive. It follows (see Struwe [9], Theorem 1.2) that J_λ is bounded from below on M and attains its infimum on M.

Suppose that J_λ assumes its infimum at $u_\lambda \in M$. Then $|u_\lambda| \in M$ and $J_\lambda(u_\lambda) = J_\lambda(|u_\lambda|)$. Thus we may assume that $u_\lambda \geq 0$ on Ω.

By the Lagrange multiplier rule there exists a parameter $\kappa \in \mathbf{R}$ such that

$$\int_\Omega \nabla u_\lambda \cdot \nabla \phi \, dx - \lambda \int_\Omega g u_\lambda \phi \, dx + \frac{\alpha}{1-\alpha} \int_{\partial\Omega} u_\lambda \phi \, dS_x + \kappa\lambda \int_\Omega g u_\lambda |u_\lambda|^p \phi \, dx = 0$$

for all $\phi \in W^{1,2}(\Omega)$. Setting $\phi = u_\lambda$ above gives

$$||u_\lambda||_\lambda^2 = -\kappa\lambda \int_\Omega g|u_\lambda|^{p+2} = \kappa.$$

Since $u_\lambda \in M$ cannot vanish identically, $||u_\lambda||_\lambda > 0$ and so $\kappa > 0$.

Let $u = \kappa^{\frac{1}{p}} u_\lambda \in W^{1,2}(\Omega)$. Then u is a weak solution of equation (I_λ^α) in the sense that

$$\int_\Omega (\nabla u \nabla \phi - \lambda g u \phi + \lambda g u |u|^p \phi) \, dx + \frac{\alpha}{1-\alpha} \int_{\partial\Omega} u\phi \, dS_x = 0$$

for all $\phi \in W^{1,2}(\Omega)$. It follows from standard regularity arguments that $u \in C^2(\Omega)$ is a classical solution satisfying the appropriate boundary condition.

Since $u \geq 0$ on Ω, it is easy to deduce from the maximum principle that $u > 0$ on Ω.

Corollary 2.4. *If $0 < p < \frac{4}{n-2}$, then the equation*

$$\begin{cases} -\Delta u = \lambda g(x) u(1 - |u|^p) & \text{in} \quad \Omega \\ u = 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

has a positive solution for $\lambda \in (\lambda^-(1), \lambda^+(1))$, provided that $\lambda \neq 0$.

Proof. The result follows as in the proof of Theorem 2.3 but considering the functional

$$u \to \int_\Omega (|\nabla u|^2 - \lambda g u^2) \, dx$$

for $u \in W_0^{1,2}(\Omega)$.

Conclusions identical to those of Theorem 2.3 and Corollary 2.4 can also be obtained for the case where $f(u) = u(1 + |u|^p)$ by considering the same functional

J_λ constrained to the set $\{u \in W^{1,2}(\Omega) : \lambda \int_\Omega g|u|^{p+2}\,dx = 1\}$; in this case the Lagrange multiplier $\kappa < 0$ and the change of variable $u = (-\kappa)^{\frac{1}{p}} u_\lambda$ is required.

Finally in this section we remark that since the function J_λ is even, using the Krasnoselskii genus and minimax principles (see [9]), it can be shown that the above equations have infinitely many distinct pairs of solutions for all $\alpha \in (\alpha_0, 1]$.

3. Solutions arising from bifurcation

The following lemma is central in proving that bifurcation occurs and in determining the direction of bifurcation.

Lemma 3.1. *Let $\alpha \in [0,1]$ and suppose that $\lambda \neq 0$ is a principal eigenvalue of (L^α) with corresponding positive principal eigenfunction ϕ. Then $\lambda \int_\Omega g\phi^{p+1}\,dx > 0$ for all $p \geq 1$.*

Proof. Suppose $0 < \alpha < 1$. Multiplying (L^α) by ϕ^p we obtain $-\Delta\phi\,\phi^p = \lambda g \phi^{p+1}$ on Ω and so

$$-\int_{\partial\Omega} \frac{\partial \phi}{\partial n} \phi^p \, dS_x + p \int_\Omega \phi^{p-1} |\nabla \phi|^2 \, dx = \int_\Omega \lambda g \phi^{p+1} \, dx. \tag{3.2}$$

Hence

$$\lambda \int_\Omega g\phi^{p+1}\,dx = \frac{\alpha}{1-\alpha} \int_{\partial\Omega} \phi^{p+1}\,dS_x + p \int_\Omega \phi^{p-1} |\nabla\phi|^2\,dx$$

and so the required result holds.

If $\alpha = 0$ or $\alpha = 1$, the surface integral term in (3.2) vanishes and the result follows easily.

We now show that bifurcation occurs at our principal eigenvalues by using the Crandall and Rabinowitz theorem on bifurcation from simple eigenvalues [6].

Suppose that $f : \mathbf{R} \to \mathbf{R}$ is any smooth function such that $f(0) = 0$ and $f'(0) = 1$. Consider $F : \mathbf{R} \times C_B^{2+\tau}(\Omega) \to C^\tau(\Omega)$ defined by

$$F(\lambda, u) = -\Delta u - \lambda g f(u)$$

where $C_B^{2+\tau}(\Omega) = \{u \in C^{2+\tau}(\Omega) : (1-\alpha)\frac{\partial u}{\partial n} + \alpha u = 0 \text{ on } \partial\Omega\}$. Then F is a smooth map with Fréchet derivative F_u such that

$$F_u(\lambda, 0)u = -\Delta u - \lambda g u.$$

Thus, if λ_0 denotes a principal eigenvalue of (L^α) and ϕ_0 a corresponding positive eigenfunction, then $N(F_u(\lambda, 0)) = [\phi_0]$ and $R(F_u(\lambda_0, 0)) = [\phi_0]^\perp = \{u \in C^\alpha(\Omega) : \int_\Omega u\phi_0 \, dx = 0\}$. Moreover $F_{\lambda u}(\lambda_0, 0)\phi_0 = -g\phi_0$ and since, by Lemma 3.1, $\lambda \int_\Omega g\phi^2 \, dx > 0$, it follows that $F_{\lambda u}(\lambda_0, 0)\phi_0 \notin R(F_u(\lambda_0, 0))$. Thus by the Crandall and Rabinowitz theorem there exists a curve of nontrivial solutions of the form $s \to (\lambda(s), s(\phi_0 + \psi(s))$ bifurcating from $(\lambda_0, 0)$ where $\lambda(0) = \lambda_0$, $\psi(0) = 0$ and $\psi(s) \in C_B^{2+\alpha}(\Omega) \cap [\phi_0]^\perp$.

Now suppose that $f(u) = u(1 - |u|^p)$ where $p > 0$. We shall determine the direction of bifurcation. For sufficiently small s we have

$$-\Delta\phi_0 - \Delta\psi(s) = \lambda(s)g[\phi_0 + \psi(s)]\,[1 - |u(s)|^p]$$

and so

$$-\Delta\phi_0 - \Delta\psi(s) = \lambda_0 g[\phi_0 + \psi(s)]\,[1 - |u(s)|^p] + (\lambda(s) - \lambda_0)g[\phi_0 + \psi(s)]\,[1 - |u(s)|^p].$$

Hence

$$-\Delta\psi(s) - \lambda_0 g\psi(s) = -\lambda_0 g[\phi_0 + \psi(s)]|u(s)|^p + (\lambda(s) - \lambda_0)g[\phi_0 + \psi(s)]\,[1 - |u(s)|^p]$$

and so, since $R(-\Delta - \lambda_0 g) = [\phi_0]^\perp$, we must have

$$\lambda_0 \int_\Omega g[\phi_0 + \psi(s)]|u(s)|^p \phi_0 \, dx = (\lambda(s) - \lambda_0) \int_\Omega g[\phi_0 + \psi(s)]\,[1 - |u(s)|^p]\phi_0 \, dx.$$

Thus, dividing by s^p and letting $s \to 0$, we obtain

$$\lim_{s\to 0} \frac{\lambda(s) - \lambda_0}{s^p} = \lambda_0 \frac{\int_\Omega g\phi_0^{p+2}\, dx}{\int_\Omega g\phi_0^2 \, dx}$$

The formula above determines the direction of bifurcation of the branch of positive solutions. In particular we have

Theorem 3.2. *Let $\alpha \in [0, 1]$ and suppose that $\lambda_0 \neq 0$ is a principal eigenvalue of (L^α). Then a curve of positive solutions for equation (2.2) bifurcates from the line of zero solutions at $(\lambda_0, 0)$; bifurcation is to the right (left) if $\lambda_0 > 0 (< 0)$.*

We now investigate in more detail the curve of positive solutions bifurcating from $(\lambda_0, 0)$ where $\lambda_0 > 0$.

It is straightforward to show that, when $\alpha \in (0, 1]$, equation (2.2) is equivalent to the operator equation

$$u = \lambda K_B N u \tag{3.2}$$

where $K_B : C(\Omega) \to C(\Omega)$ is the compact integral operator with kernel the Green's function associated with $-\Delta$ and the corresponding boundary condition and $N : C(\Omega) \to C(\Omega)$ is the Nemytskii operator $N(u)(x) = g(x)u(x)[1 - |u(x)|^p]$. It is also easy to show that the Rabinowitz global bifurcation theorem (see [8]) can be applied to equation (3.2) to give the existence of a continuum $\mathcal{C}$ of positive solutions of (2.2) joining $(\lambda_0, 0)$ to ∞ in $\mathbf{R} \times C(\Omega)$.

We now show that the variational solutions whose existence was proved in the previous section cannot lie on $\mathcal{C}$.

Theorem 3.3. *Suppose $0 < \alpha < 1$. If $(\lambda, u) \in \mathcal{C}$, then $u(x) < 1$ for $x \in \overline{\Omega}$.*

Proof. Close to the bifurcation point $(\lambda_0, 0)$ the continuum $\mathcal{C}$ must coincide with the curve of positive solutions given by the Crandall and Rabinowitz theorem and so, if $(\lambda, u) \in \mathcal{C}$ lies close to the bifurcation point, we must have that $u(x) < 1$ for all $x \in \overline{\Omega}$.

Suppose that there exists $(\lambda, u) \in \mathcal{C}$ such that $u(x_0) \geq 1$ for $x_0 \in \overline{\Omega}$. Then there must exist $(\lambda^*, u^*) \in \mathcal{C}$ such that $0 \leq u^*(x) \leq 1$ for all $x \in \overline{\Omega}$ and $u^*(x^*) = 1$ for some $x^* \in \overline{\Omega}$. Let $v = 1 - u^*$. Then v satisfies

$$-\Delta v = \lambda(-g)\frac{f(1-v)}{v}v \quad \text{in} \quad \Omega; \qquad (1-\alpha)\frac{\partial v}{\partial n} + \alpha v = \alpha \quad \text{in} \quad \partial\Omega$$

where $f(u) = u(1 - |u|^p)$. Thus $v(x) \geq 0$ for $x \in \overline{\Omega}$, $v(x^*) = 0$ and $-\Delta v + q(x)v = 0$ on Ω for some smooth function q. It follows from the maximum principle that, if $x^* \in \Omega$ then $v(x) \equiv 0$ in Ω which is impossible. But, if $x^* \in \partial\Omega$, then $(1-\alpha)\frac{\partial v}{\partial n} = \alpha$ and so $\frac{\partial v}{\partial n}(x^*) > 0$ which is also impossible as v attains its minimum value at x^*. Hence $u(x) < 1$ for all $x \in \overline{\Omega}$ whenever $(\lambda, u) \in \mathcal{C}$.

The existence of positive solutions to

$$(3.3) \qquad -\Delta u = \lambda g(x) f(u) \text{ in } \Omega; \qquad (1-\alpha)\frac{\partial u}{\partial n} + \alpha u = 0 \text{ in } \partial\Omega$$

where $\alpha > 0$, $f : [0,1] \to \mathbf{R}^+$, $f(0) = f(1) = 0$, $f'(0) = 1$, $f''(u) < 0$ for $u \in (0,1)$ is studied in [5] where it is shown that (3.3) has only the zero solution for $0 < \lambda < \lambda^+(\alpha)$. Clearly solutions of equation (2.2) satisfying $0 < u < 1$ are also solutions of equation (3.3) with $f(u) = u(1 - |u|^p)$. But, under the hypotheses of Theorem 3.3, if $(\lambda, u) \in \mathcal{C}$, we must have that $0 < u(x) < 1$ for $x \in \Omega$ and so $\lambda \geq \lambda^+(\alpha)$.

Thus, if $0 < \alpha < 1$, $\mathcal{C}$ lies entirely in $[\lambda^+(\alpha), \infty) \times \{u \in C(\Omega) : |u(x)| < 1 \text{ for } x \in \Omega\}$ and so none of the variational solutions whose existence we established for $\lambda < \lambda^+(\alpha)$ lie on $\mathcal{C}$. Moreover, since by [5] zero is the unique nonnegative solution of (3.3) lying between 0 and 1 for $\lambda < \lambda^+(\alpha)$, it follows that if u is a variational solution of (2.2) then $u(x_0) > 1$ for some $x_0 \in \Omega$.

It is easy to adapt the above argument to deal with the case where $\alpha = 1$ (Dirichlet boundary conditions) and again show that $\mathcal{C}$ lies entirely in $[\lambda^+(\alpha), \infty) \times \{u \in C(\Omega) : u(x) < 1 \text{ for } x \in \Omega\}$ so that the bifurcation and variational solutions are completely disjoint from each other.

If $\alpha_0 < \alpha < 0$ and $\int_\Omega g\, dx < 0$ so that both $\lambda^-(\alpha)$ and $\lambda^-(\alpha)$ are positive with corresponding principal eigenfunctions ϕ_- and ϕ_+, straightforward continuity arguments show that $\int_\Omega g\phi_-^{p+1}\, dx < 0$ and $\int_\Omega g\phi_+^{p+1}\, dx > 0$ provided that α is sufficiently close to zero. It follows from (3.1) that the bifurcation of positive solutions occurs to the left at $\lambda^-(\alpha)$ and to the right at $\lambda^+(\alpha)$. When $\alpha < 0$ the argument used in the proof of Theorem 3.3 to show the boundedness of continua emanating from principal eigenvalues no longer holds and the global nature of the continua bifurcating from $\lambda^-(\alpha)$ and $\lambda^-(\alpha)$ is an interesting open problem; it is unclear which of the alternatives in the Rabinowitz theorem hold, i.e., whether the two continua join up with each other or become unbounded.

We now consider the case when $f(u) = u(1 + |u|^p)$. Formula (3.1) now becomes

$$(3.4) \qquad \lim_{s \to 0} \frac{\lambda(s) - \lambda_0}{\alpha^s} = -\lambda_0 \frac{\int_\Omega g\phi_0^{p+2}\, dx}{\int_\Omega g\phi_0^2\, dx}.$$

Suppose $0 < \alpha \le 1$. It follows easily from (3.4) that a curve of positive solutions bifurcates to the left at $(\lambda^+(\alpha), 0)$. The Rabinowitz global bifurcation theorem can again be applied in this case to give the existence of a continuum of positive solutions $\mathcal{C}$ joining $(\lambda^+(\alpha), 0)$ to ∞ in $\mathbf{R} \times C(\Omega)$. The next lemma shows that $\mathcal{C}$ intersects $\lambda = \lambda^+(\alpha)$ only at $(\lambda^+(\alpha), 0)$.

Lemma 3.4. *There does not exist a positive solution of the equation*

$$\begin{cases} -\Delta u = \lambda^+(\alpha) g(x) u(1 + |u|^p) & \text{in} \quad \Omega \\ (1 - \alpha)\dfrac{\partial u}{\partial n} + \alpha u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{3.4}$$

Proof. Suppose that u is a positive solution of (3.4) and let ϕ be a positive principal eigenfunction of (L^α) corresponding to $\lambda^+(\alpha)$. Multiplying (3.4) by $u^{-(p+1)}\phi^{p+2}$ and (L^α) by $u^{-p}\phi^{p+1}$, subtracting and integrating we obtain

$$\int_\Omega [\left(\frac{\phi}{u}\right)^{p+1} (u\Delta\phi - \phi\Delta u)]\, dx = \lambda^+(\alpha) \int_\Omega g(x)\phi^{p+2}\, dx. \tag{3.5}$$

But by Picone's identity (see [3] and the references therein)

$$\operatorname{div}[\xi\left(\frac{\phi}{u}\right)(u\nabla\phi - \phi\nabla u)] = \xi\left(\frac{\phi}{u}\right)(u\Delta\phi - \phi\Delta u) + \xi'\left(\frac{\phi}{u}\right)u^2\left|\nabla\left(\frac{\phi}{u}\right)\right|^2$$

which holds for any $\xi \in C^1(\mathbf{R})$, u, $\phi \in C^2$, $u > 0$. Choosing $\xi(t) = t^{p+1}$ and using integration by parts in (3.5) gives

$$\int_{\partial\Omega} \left(\frac{\phi}{u}\right)^{p+1} (u\frac{\partial\phi}{\partial n} - \phi\frac{\partial u}{\partial n})\, dS_x - (p+1)\int_\Omega \left(\frac{\phi}{u}\right)^p u^2 \left|\nabla\left(\frac{\phi}{u}\right)\right|^2 dx = \lambda \int_\Omega g\phi^{p+2}\, dx > 0$$

and so we have a contradiction.

Hence $\mathcal{C}$ bifurcates to the left at $(\lambda^+(\alpha), 0)$ and has no other intersection point with the line $\lambda = \lambda^+(\alpha)$. Since there are no positive solutions when $\lambda = 0$, $\mathcal{C} - \{(\lambda^+(\alpha), 0)\}$ must lie strictly between $\lambda = 0$ and $\lambda = \lambda^+(\alpha)$ and so must approach ∞ in such a way that $||u|| \to \infty$ in this region.

We can derive further information about $\mathcal{C}$ by making use of a priori bounds obtained by Berestycki, Capuzzo-Dolcetta and Nirenberg in [2] under some additional assumptions on g, Ω and p.

Lemma 3.5. *Suppose* $\Omega^+ = \{x \in \Omega : g(x) > 0\}$, $\Omega^- = \{x \in \Omega : g(x) < 0\}$ *and* $\Gamma = \Omega^+ \cap \Omega^-$. *If* $\Gamma \subseteq \Omega$, $\nabla g(x) \neq 0$ *for all* $x \in \Gamma$ *and* $p < \frac{3}{N-1}$, *then, for all* $\lambda \neq 0$, *there exists* $C > 0$ *such that* $u(x) \leq C$ *for all* $x \in \Omega$ *for any positive solution* u *of equation (3.4).*

Thus under the hypotheses of Lemma 3.5 $\mathcal{C}$ cannot approach ∞ at any nonzero value of λ and so must approach ∞ in such a way that $||u|| \to \infty$ as $\lambda \to 0$. It follows by a simple connectedness argument that there must exist $(\lambda, u) \in \mathcal{C}$ for every $\lambda \in (0, \lambda^+(\alpha))$. Thus in this case the variational solutions discussed in the previous section may coincide with the solutions arising from bifurcation.

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