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FREE BOUNDARY PROBLEM FOR QUASILINEAR PARABOLIC EQUATION WITH FIXED ANGLE OF CONTACT TO A BOUNDARY

北海道大学大学院理学研究科数学専攻 D2

高坂 良史 (YOSHIHITO KOHSAKA)

1. Introduction

We consider the following free boundary problem of form:

$$u_t = (a(u_x))_x, \qquad s(t) < x < 0,\ t > 0, \tag{1.1}$$

$$u_x(s(t), t) = \tan\theta_0, \quad t \geq 0, \tag{1.2}$$

$$u_x(0, t) = \tan\theta_1, \qquad t \geq 0, \tag{1.3}$$

$$u(s(t), t) = 0, \qquad t \geq 0, \tag{1.4}$$

$$u(x, 0) = u_0(x), \qquad s(0) := s_0 \leq x \leq 0, \tag{1.5}$$

where $a \in C^2(\mathbb{R})$ and $a'(\sigma) > 0$ for $\sigma \in \mathbb{R}$ ($\prime = \frac{d}{d\sigma}$), and s_0 is a given negative number, and $u_0 \in C^2[s_0, 0]$. We also assume a compatibility condition $u_{0x}(s_0) = \tan\theta_0, u_{0x}(0) = \tan\theta_1, u_0(s_0) = 0$, and assume $u_0(x) > 0$ for $x \in (s_0, 0]$. The angles $\theta_i \in (0, \frac{\pi}{2})$ for $i = 0, 1$ will be measured counter-clockwise from the x-axis.

If we set $a(\sigma) = \arctan\sigma$, the equation (1.1) is the curvature flow equation for the graph of u separating two phase. The curvature flow equation is one of the typical evolution equations which describe the motion of the phase boundary. In this case, this problem is the curvature flow problem with prescribed angle on the boundary of the second quadrant.

If we set $a(\sigma) = \sigma$, the equation (1.1) is the heat equation. In this case, this problem appears in the combustion theory.

In this note, we consider the convergence of the solution of (1.1)-(1.5) as $t \to \infty$ in the case $\theta_0 < \theta_1$. Our main goal of this paper is to show that the solution of (1.1)-(1.5) converges as $t \to \infty$ to the unique self-similar solution in the case $\theta_0 < \theta_1$.

Main Theorem. *Assume that* $\theta_0 < \theta_1$.

(I) There exists an expanding self-similar solution S_t *corresponding to the problem (1.1)-(1.5) which is unique up to translation of time. Moreover,* S_t *is convex.*

(II) Let Γ_t *be a solution of (1.1)-(1.5). Then, for each* $\delta \in (0, 1/2)$, *there is a constant* C_δ *such that*

$$d_H(\Gamma_t, S_t) \leq C_\delta t^{-\delta} \quad for \quad t \geq 1$$

where d_H *denotes the Hausdorff distance.*

To prove this theorem, we employ what is called similarity change of variables;

$$u(x,t) = \sqrt{2t+1}\, U(\xi, \tau), \quad s(t) = \sqrt{2t+1}\, p(\tau), \tag{1.6}$$

where

$$\xi = \frac{x}{\sqrt{2t+1}}, \quad \tau = \frac{1}{2}\log(2t+1). \tag{1.7}$$

Then, problem (1.1)-(1.5) becomes

$$U_\tau = (a(U_\xi))_\xi + \xi U_\xi - U, \quad p(\tau) < \xi < 0, \ \tau > 0, \tag{1.8}$$

$$U_\xi(p(\tau),\tau) = \tan\theta_0, \quad \tau \geq 0, \tag{1.9}$$

$$U_\xi(0,\tau) = \tan\theta_1, \qquad \tau \geq 0, \tag{1.10}$$

$$U(p(\tau),\tau) = 0, \qquad \tau \geq 0, \tag{1.11}$$

$$U(\xi,0) = U_0(\xi), \qquad p(0) = s_0 \leq \xi \leq 0. \tag{1.12}$$

A stationary solution to (1.8) is called a self-similar solution.

2. Existence and uniqueness of self-similar solution

In this section we show that the self-similar solution corresponding to the problem (1.1)-(1.5) exists uniquely. We consider the following ordinary differential equation of form (P):

$$(a(U_\xi))_\xi + \lambda\xi U_\xi - \lambda U = 0, \quad \xi \in (q,0), \tag{2.1}$$

$$U_\xi(q) = \tan\ \theta_0, \tag{2.2}$$

$$U_\xi(0) = \tan\ \theta_1, \tag{2.3}$$

$$U(q) = 0. \tag{2.4}$$

This is the stationary problem of (1.8)-(1.12) for $\lambda = 1$. Here, the function U and the number λ is unknown and we shall discuss the existence of solutions.

Theorem 2.1. *(Existence and uniqueness) Let q, θ_0, θ_1 be given constants. Assume that*

$$q < 0,\ 0 < \theta_0 \leq \theta_1 < \frac{\pi}{2}.$$

Then there exists a unique solution $(\lambda, U) \in [0,\infty) \times C^2[q,0]$ to (P). Moreover, $\lambda = 0$ is if and only if $\theta_0 = \theta_1$.

Remark 2.1. **(Relation between λ and q)** We set $\lambda = \lambda(q)$. Then, $\lambda(\zeta q) = \lambda(q)/\zeta^2$ holds for $\zeta \in (0, \infty)$. Here, we set $q = -1$ and replace $-\zeta$ by q. Then,

$$\lambda(q) = \frac{\lambda(-1)}{q^2} \tag{2.5}$$

where $\lambda(-1)$ is a constant satisfying $\lambda(-1) = 0$ if $\theta_0 = \theta_1$ and $\lambda(-1) > 0$ if $\theta_0 < \theta_1$.

In Theorem 2.1, we determined (λ, U) by giving q, θ_0, θ_1. But since (2.5) holds, we can determine (q, U) by giving $\lambda, \theta_0, \theta_1$.

To prove Theorem 2.1, we shall employ the shooting method.

For given $\lambda \in [0, \infty)$, let (P_λ) be the initial-value problem (2.1), (2.2), (2.4). We define the set J as

$$J := \{\lambda \in [0, \infty) \mid \text{there exists a } U \in C^2[q, 0] \text{ satisfying } (P_\lambda) \\ \text{for the interval } [q, 0]\}.$$

Clearly, J includes $\lambda = 0$. Thus, $\lambda \neq \emptyset$.

Then, we obtain sevral lemmas with respect to J.

Lemma 2.1. *(Openness of J)* *Assume that $\lambda_0 \in J$. Then there is a small $\hat{\delta} > 0$ so that the set $(\lambda_0 - \hat{\delta},\ \lambda_0 + \hat{\delta}) \cap [0, \infty)$ is including in the set J.*

Lemma 2.2. *(Connectedness)* *Assume that $\lambda_0,\ \lambda_1 \in J$ and $\lambda_0 < \lambda_1$. If $\lambda_0 \leq \lambda \leq \lambda_1$, then λ is included in the set J.*

Moreover, we study qualitative properties of solution.

Lemma 2.3. *Assume that $\lambda \in [\alpha, \beta]$, with constants α, β satisfying $0 \leq \alpha < \beta$.*

and that $U \in C^2[q,\gamma]$ with constants q, γ satisfying $q < \gamma \leq 0$ fulfills

$$(a(U_\xi))_\xi + \lambda \xi U_\xi - \lambda U = 0, \ \xi \in [q,\gamma],$$

$$U_\xi(q) = \tan \theta_0,$$

$$U(q) = 0.$$

Then, the following estimates are valid;

(i) $U_{\xi\xi}(\xi \ ; \ \lambda) > 0$ *for* $\xi \in [q,\gamma], \ \lambda > 0,$

(ii) $\dot{U}_\xi(\xi \ ; \ \lambda) > 0$ *for* $\xi \in [q,\gamma], \ \lambda \in [\alpha,\beta],$

(iii) $U_\xi(\xi \ ; \ \lambda) > 0$ *for* $\xi \in [q,\gamma], \ \lambda \geq 0,$

$\dot{U}(\xi \ ; \ \lambda) > 0$ *for* $\xi \in [q,\gamma], \ \lambda \in [\alpha,\beta],$

where $\cdot$ is the differential with respect to λ.

By Lemma 2.1, J is the open set included in the interval $[0,\infty)$. Moreover, we define $\Lambda_0 \in (0,\infty]$ as the supremum of λ such that there exists a solution of (P_λ) in $[q,0]$. Then, by Lemma 2.2, that J is an interval $[0,\Lambda_0)$.

We now define the mapping

$$\Phi : [0, \ \Lambda_0) \ni \lambda \mapsto U_\xi(0 \ ; \ \lambda).$$

Then, Lemma 2.3 (ii) implies

$$\frac{\partial \Phi}{\partial \lambda} > 0.$$

Thus, Φ is a monotone increasing function, which is a bijection;

$$\Phi : [0, \ \Lambda_0) \to [\tan \theta, \ \lim_{\lambda \uparrow \Lambda_0} \Phi(\lambda)).$$

Here, we obtain the following lemma.

Lemma 2.4. *(i)Assume that* $\Lambda_0 < \infty$. *Then* $\lim_{\lambda \uparrow \Lambda_0} \ \Phi(\lambda) = \infty$.

(ii)Assume that $\Lambda_0 = \infty$. *Then* $\lim_{\lambda \uparrow \infty} \ \Phi(\lambda) = \infty$.

Remark 2.2. (i) If a is bounded from the above, i.e. there exists a constant M such that $a(\sigma) < M$ for $\sigma \in \mathbb{R}$, then

$$\Lambda_0 \leq \frac{M - a(\tan\theta_0)}{q^2 \tan\theta_0} < \infty.$$

In fact, by means of simple computation, it follows that

$$a(U_\xi(0\ ;\ \lambda)) \geq a(\tan\theta_0) + \lambda\ q^2\ \tan\theta_0.$$

Thus, for $\lambda \in (0,\ \Lambda_0)$

$$a(\tan\theta_0) + \lambda q^2\ \tan\theta_0 \leq a(U_\xi(0\ ;\ \lambda)) < M.$$

Then, for any $\varepsilon > 0$

$$a(\tan\theta_0) + (\Lambda_0 - \varepsilon)q^2 \tan\theta_0 < M.$$

Hence,

$$\Lambda_0 < \frac{M - a(\tan\theta_0)}{q^2 \tan\theta_0} + \varepsilon.$$

Since ε is arbitrary,

$$\Lambda_0 \leq \frac{M - a(\tan\theta_0)}{q^2 \tan\theta_0}.$$

(ii) We now rewrite the initial-value problem (P_λ) by introducing a new dependent variable $y(\xi)$. We set

$$a(U_\xi(\xi)) := y(\xi).$$

Since $a' > 0$, there exists a C^2 inverse function a^{-1} of the function a to get

$$U_\xi(\xi) = a^{-1}(y(\xi)).$$

The equation (2.1) becomes

$$y_\xi + \lambda\xi a^{-1}(y) - \lambda U = 0.$$

It is easy to see that (P_λ) is rewritten in the form of a system

$$\frac{d}{d\xi}\begin{pmatrix} y \\ U \end{pmatrix} = \begin{pmatrix} -\lambda\xi a^{-1}(y) + \lambda U \\ a^{-1}(y) \end{pmatrix}$$
$$\begin{pmatrix} y(q) \\ U(q) \end{pmatrix} = \begin{pmatrix} a(\tan\theta) \\ 0 \end{pmatrix}.$$

For later notation, we set

$$F(\xi,\ y,\ U,\ \lambda) := \begin{pmatrix} -\lambda\xi a^{-1}(y) + \lambda U \\ a^{-1}(y) \end{pmatrix}.$$

If the initial-value problem (P_λ) is solvable for any $\lambda \in [0, \infty)$, i.e. $\sup\limits_{\mathbb{R}} \frac{d}{dy} a^{-1}(y) < \infty$, $\Lambda_0 = \infty$. Because, if $\sup\limits_{\mathbb{R}} \frac{d}{dy} a^{-1}(y) < \infty$, F is Lipschitz continuous with respect to y, U for any $\lambda \in [0, \infty)$.

Proof of Theorem 1.1. By Lemma 2.4,

$$\Phi([0,\ \Lambda_0)) = [\tan\theta_0,\ \infty).$$

Moreover, since $\partial\Phi/\partial\lambda > 0$ by Lemma 2.3 (ii), Φ is one-to-one. Thus, Φ is a bijection. Consequently, for any $\alpha := \tan\theta_1 \in [\tan\theta_0,\ \infty)$, there exist a unique $(\lambda, U) \in [0,\ \Lambda_0) \times C^2[q, 0]$ satisfying the initial-value problem (P_λ) and $U_\xi(0) = \tan\theta_1$. $\square$

3. Convergence of a solution for $\theta_0 < \theta_1$

We consider the convergence of the solution of (1.1)-(1.5). Here, we shall discuss the convergence of a solution for problem (1.8)-(1.12).

Theorem 3.1. *Assume that $u_0 \in C^2[s_0, 0]$ satisfying $u_{0\xi}(s_0) = \tan\theta_0$, $u_{0\xi}(0) = \tan\theta_1$, $u_0(s_0) = 0$, and $u_0(\xi) > 0$ for $\xi \in (s_0, 0]$. Moreover, assume that $(U(\xi,\tau), p(\tau))$ is a smooth solution for problem (1.8)-(1.12). Then $(U(\xi,\tau), p(\tau))$ converge as $\tau \to \infty$ to $(U^*(\xi), p^*)$ satisfying*

$$(a(U^*_\xi))_\xi + \xi U^*_\xi - U^* = 0, \quad p^* < \xi < 0, \tag{3.1}$$

$$U^*_\xi(p^*) = \tan\theta_0, \tag{3.2}$$

$$U^*_\xi(0) = \tan\theta_1, \tag{3.3}$$

$$U^*(p^*) = 0. \tag{3.4}$$

Moreover, this convergence is exponential:

$$\hat{d}_H(\hat{\Gamma}_\tau, \hat{S}) \le Ce^{-\delta_0 \tau}$$

for each $\delta_0 \in (0, 2)$ and $\tau \ge 0$ where $\hat{d}_H$ denotes the Hausdorff distance, $\hat{\Gamma}_\tau$ is the solution of (1.8)-(1.12), $\hat{S}$ is the solution of (3.1)-(3.4), and C is a constant and is independent of τ.

For the proof of Theorem 3.1, we construct a subsolution and a supersolution for the problem (1.8)-(1.12), which converges as $\tau \to \infty$ to U^* satisfying (3.1)-(3.4), and use the strong maximum principle.

3.1 Structure of a subsolution

We first define $v_0(\xi)$ as the following. We set

$$K := \min\left\{\tan\theta_0,\ \inf_{\xi\in(s_0,0)} \left(\frac{u_0(\xi)}{\xi - s_0}\right)\right\}.$$

Here we choose a constant ℓ satisfying

$$\frac{s_0 K}{\tan\theta_1} < \ell < 0. \tag{3.5}$$

(cf. Figure 3.1). Then, by Theorem 2.1, there exist a unique $(\lambda_\ell, v_0) \in (0,\infty) \times C^2[\ell, 0]$ satisfying

$$(a(v_{0\xi}))_\xi + \lambda_\ell \, \xi v_{0\xi} - \lambda_\ell \, v_0 = 0, \quad \ell < \xi < 0, \tag{3.6}$$

$$v_{0\xi}(\ell) = \tan\theta_0, \tag{3.7}$$

$$v_{0\xi}(0) = \tan\theta_1, \tag{3.8}$$

$$v_0(\ell) = 0. \tag{3.9}$$

By Remark 2.1, if necessary, we choose λ_ℓ such that $\lambda_\ell > 1$. Then, we get the following relation between u_0 and v_0.

Lemma 3.1. *Assume that v_0 satisfies (3.6)-(3.9). Then the following estimate is valid;*

$$u_0(\xi) > v_0(\xi) \quad \text{for} \quad \xi \in [\ell, 0].$$

Moreover, we get the following relation between U^* and v_0.

Lemma 3.2. *Assume that U^* satisfies (3.1)-(3.4) and v_0 satisfies (3.6)- (3.9). Then U^* is represented by v_0 as the following;*

$$U^*(\xi) = \sqrt{\lambda_\ell}\; v_0\left(\frac{\xi}{\sqrt{\lambda_\ell}}\right).$$

Moreover, this representation is unique.

Then, applying Lemma3.1, Lemma3.2 and that U^* satisfies (3.1)-(3.4), we obtain the following proposition.

Proposition 3.1. *For any $\delta_1 \in (0, 2]$, we define*

$$V(\eta,\tau) := \varphi(\tau) U^*\left(\frac{\eta}{\varphi(\tau)}\right) \tag{3.10}$$

where $\varphi(\tau) = 1 + \left(\frac{1}{\sqrt{\lambda_\ell}} - 1\right) e^{-\delta_1 \tau}$. *Then, V is a subsolution of (1.8)-(1.12).*

3.2 Structure of a supersolution

We first define $w_0(\xi)$ as the following. Now, we choose a constant L satisfying $L < s_0$ and

$$0 < \sup_{\xi \in [s_0, 0]} \left(\frac{u_0(\xi)}{\xi - L} \right) \leq \tan \theta_0. \tag{3.11}$$

(cf. Figure 3.2). Then, by Theorem 2.1, there exist a unique $(\lambda_L, w_0) \in (0, \infty) \times C^2[L, 0]$ satisfying

$$(a(w_{0\xi}))_\xi + \lambda_L \xi w_0 = 0, \; L < \xi < 0, \tag{3.12}$$

$$w_{0\xi}(L) = \tan \theta_0, \tag{3.13}$$

$$w_{0\xi}(0) = \tan \theta_1, \tag{3.14}$$

$$w_0(L) = 0. \tag{3.15}$$

By Remark 2.1, if necessary, we choose λ_L such that $0 < \lambda_L < 1$. Then, we get the following relation between u_0 and w_0.

Lemma 3.3. *Assume that w_0 satisfies (3.12)-(3.15). Then the following estimate is valid;*

$$u_0(\xi) < w_0(\xi) \quad for \quad \xi \in [s_0, 0].$$

Moreover, we get the following relation between U^* and w_0.

Lemma 3.4. *Assume that U^* satisfies (3.1)-(3.4) and w_0 satisfies (3.12)-(3.15). Then U^* is represented by w_0 as the following;*

$$U^*(\xi) = \sqrt{\lambda_L}\; w_0\left(\frac{\xi}{\sqrt{\lambda_L}}\right).$$

Moreover, this representation is unique.

Then, applying Lemma3.3, Lemma3.4 and that U^* satisfies (3.1)-(3.4), we obtain the following proposition.

Proposition 3.2. *For any $\delta_2 \in (0, \sqrt{\lambda_L}+1)$, we define*

$$W(\rho,\tau) := \psi(\tau)U^*\left(\frac{\rho}{\psi(\tau)}\right) \tag{3.16}$$

where $\psi(\tau) = 1 + \left(\frac{1}{\sqrt{\lambda_L}} - 1\right)e^{-\delta_2\tau}$. Then, W is a supersolution of (1.8)-(1.12).

3.3 Proof of Theorem 3.1

We now set

$$d(\tau) := \inf\{[(\xi-\eta)^2 + (U(\xi,\tau) - V(\eta,\tau))^2]^{1/2}$$
$$\mid p(\tau) \le \xi \le 0,\ \varphi(\tau)p^* \le \eta \le 0\}.$$

Then, we get the following.

Lemma 3.5. *For $\tau \ge 0$, $d(\tau) > 0$.*

This lemmma is proved by using the strong maximaum principle.

Consequently, by means of Lemma 3.1 and Lemma 3.5, we get

$$p(\tau) < \varphi(\tau)p^*,\ U(\eta,\tau) > V(\eta,\tau) \quad \text{for} \quad \varphi(\tau)p^* \le \eta \le 0,\ \tau \ge 0. \tag{3.17}$$

In the same way, we get

$$\psi(\tau)p^* < p(\tau),\ W(\rho,\tau) > U(\rho,\tau) \quad \text{for} \quad p(\tau) \le \rho \le 0,\ \tau \ge 0. \tag{3.18}$$

Here, we assume $\xi_0 \in [p^*, 0]$. Then, by the definition of V and W (see (3.10), (3.16)), the intersection points of the straight line $\{(\xi, r) \mid U^*(\xi_0)\xi - \xi_0\, r = 0\}$ and the graphs $\{(\xi, r) \mid r = V(\xi, \tau),\ \varphi(\tau)p^* \leq \xi \leq 0\}$, $\{(\xi, r) \mid r = W(\xi, \tau),\ \psi(\tau)p^* \leq \xi \leq 0\}$ are represented as the following;

$$(\varphi(\tau)\xi_0,\ \varphi(\tau)U^*(\xi_0)),\ (\psi(\tau)\xi_0,\ \psi(\tau)U^*(\xi_0)).$$

(cf. Figure 3.3). We set

$$\begin{aligned}\mathcal{D}(\xi_0, \tau) := \{\xi \mid \xi &- \text{coordinate of intersection points of the straight} \\ &\text{line } \{(\xi, r) \mid U^*(\xi_0)\xi - \xi_0 r = 0\} \text{ and the solution } \hat{\Gamma}_\tau\}.\end{aligned}$$

where $\hat{\Gamma}_\tau$ is the solution of (1.8)-(1.12) (cf. Figure 3.4). Since $U(\xi, \tau)$ is a smooth function in the set $\{(\xi, \tau) | \ p(\tau) \leq \xi \leq 0,\ \tau \geq 0\}$, we get $\mathcal{D}(\xi_0, \tau) \neq \emptyset$.

Then, by means of (3.17) and (3.18), we obtain for $\xi \in \mathcal{D}(\xi_0, \tau)$

$$\begin{aligned}(\varphi(\tau)\xi_0)^2 + (\varphi(\tau)U^*(\xi_0))^2 &\leq \xi^2 + (U(\xi, \tau))^2 \\ &\leq (\psi(\tau)\xi_0)^2 + (\psi(\tau)U^*(\xi_0))^2. \qquad (3.19)\end{aligned}$$

Here, we see

$$\begin{aligned}&[(\varphi(\tau)\xi_0)^2 + (\varphi(\tau)U^*(\xi_0))^2]^{1/2} - [\xi_0^2 + (U^*(\xi_0))^2]^{1/2} \\ &\qquad = \left(\frac{1}{\sqrt{\lambda_\ell}} - 1\right) e^{-\delta_1 \tau}[\xi_0^2 + (U^*(\xi_0))^2]^{1/2}, \qquad (3.20)\\ &[(\psi(\tau)\xi_0)^2 + (\psi(\tau)U^*(\xi_0))^2]^{1/2} - [\xi_0^2 + (U^*(\xi_0))^2]^{1/2} \\ &\qquad = \left(\frac{1}{\sqrt{\lambda_L}} - 1\right) e^{-\delta_2 \tau}[\xi_0^2 + (U^*(\xi_0))^2]^{1/2}. \qquad (3.21)\end{aligned}$$

Thus, by (3.19)-(3.21) and $\lambda_\ell > 1$ and $0 < \lambda_L < 1$, we get for $\xi \in \mathcal{D}(\xi_0, \tau)$

$$\begin{aligned}C\left(\frac{1}{\sqrt{\lambda_\ell}} - 1\right) e^{-\delta_1 \tau} &< [\xi^2 + U(\xi, \tau))^2]^{1/2} - [\xi_0^2 + (U^*(\xi_0))^2]^{1/2} \\ &< C\left(\frac{1}{\sqrt{\lambda_L}} - 1\right) e^{-\delta_2 \tau}\end{aligned}$$

where $C = \sup_{\xi_0 \in [p^*,0]} [\xi_0^2 + (U^*(\xi_0))^2]^{1/2}$.

Consequently, if we choose $\delta_0 \in (0, \sqrt{\lambda_L} + 1)$, we obtain for $\tau \geq 0$

$$\hat{d}_0(\hat{\Gamma}_\tau, \hat{S}) \leq \hat{C}\ e^{-\delta_0 \tau}$$

where $\hat{C} = \max\left\{ -C\left(\frac{1}{\sqrt{\lambda_\ell}} - 1\right),\ C\left(\frac{1}{\sqrt{\lambda_L}} - 1\right)\right\}$, and

$$\hat{d}_0(\hat{\Gamma}_\tau, \hat{S}) := \sup_{\xi_0 \in [p^*,0]} \sup_{\xi \in \mathcal{D}(\xi_0,\tau)} \mid [\xi^2 + (U(\xi,\tau))^2]^{1/2} - [\xi_0^2 + (U^*(\xi_0))^2]^{1/2} \mid.$$

Then, we note that $\hat{d}_0$ is equivalent to the Hausdorff distance $\hat{d}_H$. Thus, the proof of Theorem 3.1 is completed.

3.4 Proof of (II) of Main Theorem

We define

$$d_0(\Gamma_t, S_t) := \sup_{X_0 \in S_t} \sup_{Y \in Q} \mid d(O, X_0) - d(O, Y) \mid$$

where

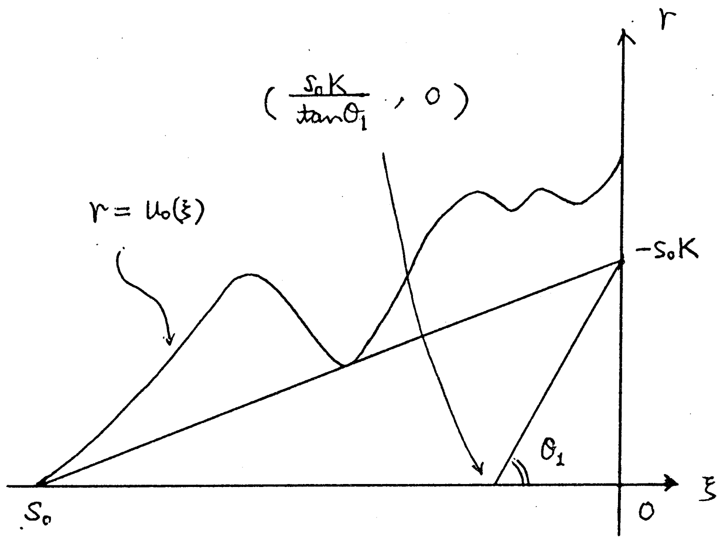
$$Q := \{Y \in \Gamma_t \mid \text{the intersection points between } \Gamma_t \text{ and the straight line passing the origin O and } X_0 (\in S_t)\}.$$

Then, we note that d_0 is equivalent to the Hausdorff distance d_H.

Consequently, if we choose $\hat{\delta} \in (1, 2)$, by means of Theorem 3.1,

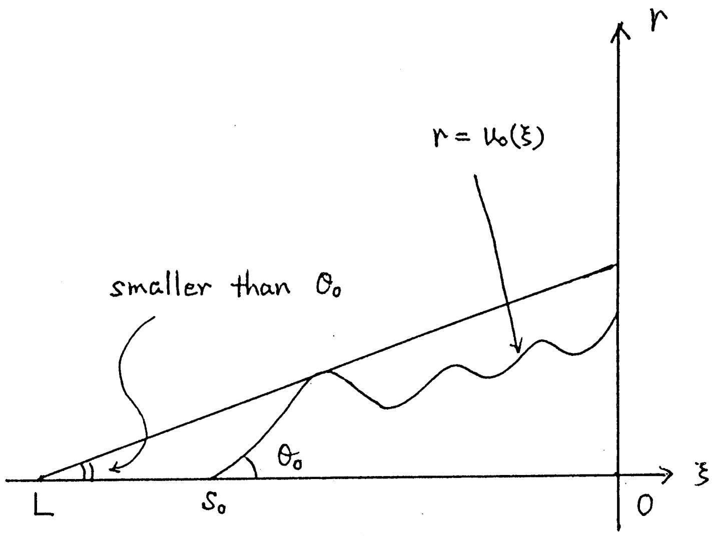
$$d_0(\Gamma_t, S_t) \leq C(2t+1)^{-(\hat{\delta}-1)/2} \leq \tilde{C} t^{-(\hat{\delta}-1)/2}$$

Thus, the proof of Main Theorem is completed.



r
$\left(\frac{s_0 K}{\tan\theta_1}, 0\right)$
$r = u_0(\xi)$
$-s_0 K$
θ_1
s_0
ξ
0

Fig 3.1



r
$r = u_0(\xi)$
smaller than θ_0
θ_0
L
s_0
ξ
0

Fig 3.2

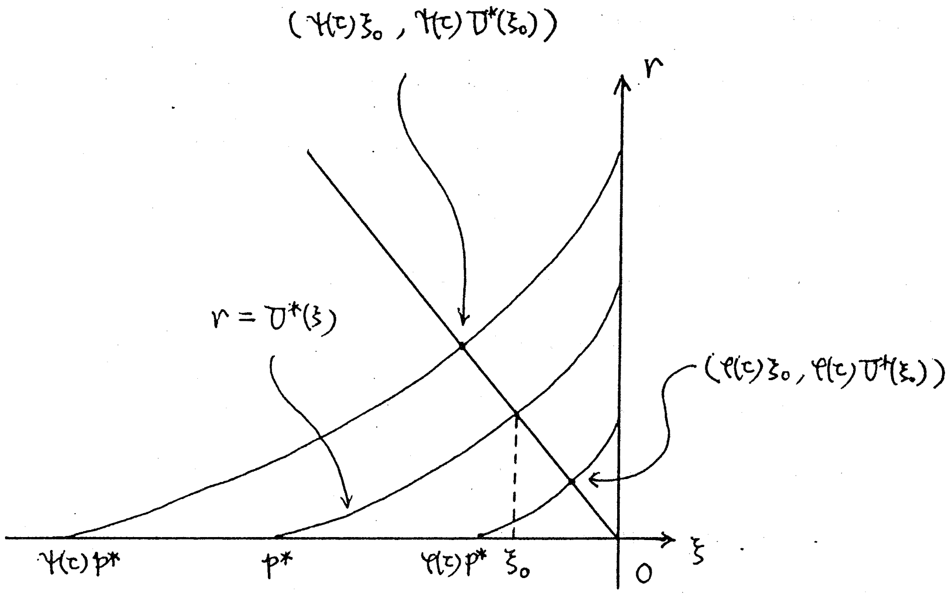


Fig 3.3 The intersection points

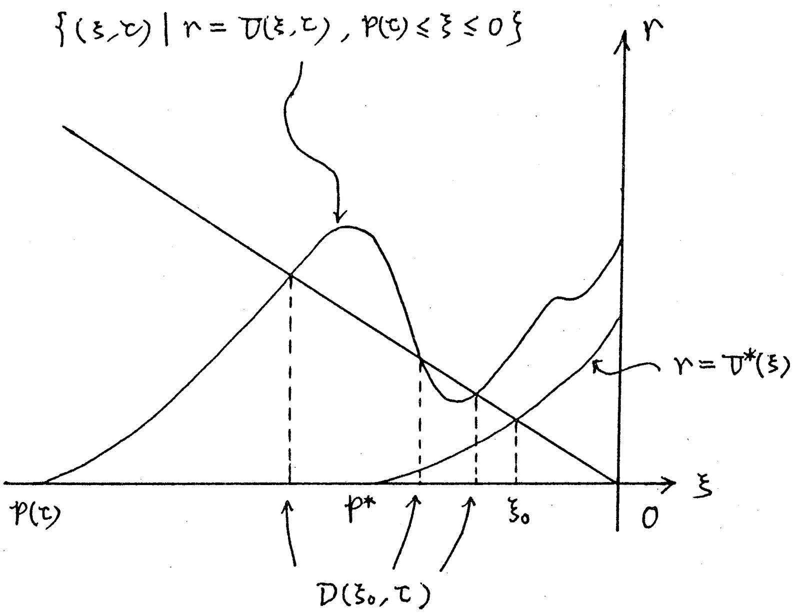


Fig 3.4 An example of $\mathcal{D}(\xi_0, \tau)$

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Address: Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan

E-mail: y-kosaka@math.sci.hokudai.ac.jp