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STRUCTURE OF THE GROUP OF AUTOMORPHISMS OF C*-ALGEBRAS

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1 Introduction

A non-commutative generalization of the functional representation theorem for commutative unital C*-algebras was introduced in [2]. This generalization was established via a non-commutative Gelfand transform mapping an unital C*-algebra $\mathcal{A}$ to an algebra of functions (for some non-commutative product) on the set of pure states of $\mathcal{A}$ viewed as a uniform Kähler bundle over the spectrum of $\mathcal{A}$ (See Sect. 3). The Kähler structure involved can be seen as a geometrical counterpart of Shultz' characterization [5] of the set of pure states of a unital C*-algebra.

As a consequence, any statement about C*-algebras can be translated into an equivalent statement in terms of uniform Kähler bundles. For example, the set of *-isomorphisms between two C*-algebras $\mathcal{A}$ and $\mathcal{A}'$ is in one-to-one correspondence with the set of uniform Kähler isomorphisms between the uniform Kähler bundles associated with $\mathcal{A}$ and $\mathcal{A}'$ [2].

We think that this correspondence between C*-algebras and Kähler geometry can be advantageously exploited to get new insights in some problems occurring in C*-algebras theory. Also, the non-commutative structure on the

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space of functions on the set of pure states seems to be related with deformation quantization of Poisson manifolds and a better understanding of this link might result in a fruitful interaction between these fields.

In this paper, by using non-commutative functional representation theorem, we study the structure of the group of automorphisms of C*-algebras in terms of geometry of uniform Kähler bundles.

The paper is organized as follows. In section 2, we state our main theorem. In section 3, we review the theory of uniform Kähler bundle [2]. In section 4, we introduce the orbit spectrum of a C*-algebra $\mathcal{A}$. It is the space of orbits in the spectrum of $\mathcal{A}$ of the group of automorphisms of $\mathcal{A}$. We decompose the uniform Kähler bundle associated to $\mathcal{A}$ by the orbit spectrum. In section 5, we prove the main theorem.

2 Structure of the group of automorphisms

We first state our main theorem using the following notation:
Let $\mathcal{A}$ be a unital C*-algebra. Aut$\mathcal{A}$ is the group of *-automorphisms of $\mathcal{A}$, B is the spectrum of $\mathcal{A}$ defined as the set of all the equivalence classes of irreducible representations of $\mathcal{A}$, and $\mathcal{P}$ is the set of pure states of $\mathcal{A}$. With respect to the weak* topology, $\mathcal{P}$ is a uniform space [2], [1]. Since Aut$\mathcal{A}$ acts on B naturally, we define the orbit space $\Lambda \equiv B/\mathrm{Aut}\mathcal{A}$ denoting the corresponding natural projection by $p^{'}: B \to \Lambda$.

Theorem 2.1 *There is an injective homomorphism*

$$\pi : \mathrm{Aut}\mathcal{A} \hookrightarrow PU(\mathcal{P}) \rtimes_{\delta} S(B)^{\Lambda}$$

where

$$PU(\mathcal{P}) \equiv \prod_{b \in B} PU_b,$$

$$S(B)^{\Lambda} \equiv \{\phi : B \to B : \phi \textit{ is a bijection such that } p^{'} \circ \phi = p^{'}\},$$

PU_b is the projective unitary group on the representation space of the representative element b in the spectrum B and δ is the right action of $S(B)^{\Lambda}$ on $PU(\mathcal{P})$ defined by

$$\{u_b\}\delta_{\phi} \equiv \{u_{\phi^{-1}(b)}\}$$

for $\{u_b\} \in PU(\mathcal{P})$, $u_b \in PU_b$ and $\phi \in S(B)^{\Lambda}$.

The image of $\mathrm{Aut}\mathcal{A}$ *under* π *is given in terms of a faithful action* κ *of* $PU(\mathcal{P})\rtimes_\delta S(B)^\Lambda$ *on* $\mathcal{P}$*, by*

$$\{g \in PU(\mathcal{P})\rtimes_\delta S(B)^\Lambda : \kappa_g \textit{ is acting as a uniform homeomorphism on } \mathcal{P}\}.$$

By this theorem, we characterize an element of image of $\mathrm{Aut}\mathcal{A}$ under π as an element of $PU(\mathcal{P})\rtimes_\delta S(B)^\Lambda$ which is a uniform homeomorphism on $\mathcal{P}$.

For $\alpha \in \mathrm{Aut}\mathcal{A}$, define $\alpha[\pi] \equiv [\pi \circ \alpha^{-1}]$ for $[\pi] \in B$ where $[\pi]$ is an equivalence class of irreducible representations with the representative element π.

Corollary 2.1 *In Theorem 2.1, the image of the subgroup* $\{\alpha \in \mathrm{Aut}\mathcal{A} : \alpha b = b \textit{ for any } b \in B\}$ *by* π *is*

$$PU_u(\mathcal{P}) \equiv \{v \in PU(\mathcal{P}) : \kappa_v \textit{ is a uniform homeomorphism on } \mathcal{P}\}$$

where $PU_u(\mathcal{P})$ *is identified with* $PU_u(\mathcal{P}) \times \{1\} \subset PU(\mathcal{P})\rtimes_\delta S(B)^\Lambda$*.*

Example 2.1 Let X be a compact Hausdorff space. For the commutative C*-algebra $\mathcal{A} \equiv C(X)$, $\mathrm{Aut}\mathcal{A}$ is isomorphic to the group $\mathrm{Homeo}X$ of homeomorphisms on X. So, $\Lambda \equiv X/\mathrm{Homeo}X$ depends on the topological structure of X. Since $\mathcal{P} \cong X \cong B$, $PU(\mathcal{P}$ is trivial. So, $PU(\mathcal{P})\rtimes S(B)^\Lambda \cong S(B)^\Lambda$.

Any compact Hausdorff space has uniformity and $\mathrm{Homeo}X$ is equal to the set of uniform homeomorphisms on X [1]. The image of the injection of $\mathrm{Aut}\mathcal{A}$ into $S(B_\mathcal{A})^\Lambda$ is then equal to $\mathrm{Homeo}X$.

By above argument, an element of $S(B_\mathcal{A})^\Lambda$ is considered as a " *topological* " symmetry of a general noncommutative C*-algebra $\mathcal{A}$.

Example 2.2 Let $\mathcal{H}$ be a Hilbert space with $\dim\mathcal{H} \geq 1$ and $\mathcal{A} \equiv \mathcal{L}(\mathcal{H})$. It is known that

$$\mathrm{Aut}\mathcal{A} = \{\mathrm{Ad}U : U \text{ is unitary on } \mathcal{H}\} \cong PU(\mathcal{H}) \equiv \{\text{ projective unitaries on } \mathcal{H}\}$$

by the isomorphism $\mathrm{Aut}\mathcal{A} \ni \mathrm{Ad}U \mapsto [U] \in PU(\mathcal{H})$. Any automorphism leaves unchanged the elements of the spectrum of $\mathcal{A}$ and $\Lambda \equiv B/\mathrm{Aut}\mathcal{A} \cong B$. Therefore $S(B)^\Lambda = \{id_B\}$. Therefore

$$PU(\mathcal{P})\rtimes S(B)^\Lambda = PU(\mathcal{P}) \times \{1\}.$$

In theorem 2.1, the image of π of Aut$\mathcal{A}$ is

$$\{\{[\pi_b(U)]\}_{b\in B} : U \text{ is unitary on } \mathcal{H}\} \subset PU(\mathcal{P})$$

where the irreducible representation π_b of $\mathcal{A}$ is the representative element of $b \in B$ and we denote $[U] \equiv \{e^{it}U : t \in \mathbf{R}\}$ for a unitary operator U .

Example 2.3 For a C*-algebra $\mathcal{A}$, $\mathcal{I}$ is a *primitive* ideal of $\mathcal{A}$ if there is an irreducible representation π of $\mathcal{A}$ such that ker$\pi = \mathcal{I}$. The *primitive spectrum* of $\mathcal{A}$ is the set of all primitive spectrums of $\mathcal{A}$. Assume $\mathcal{A}$ is a simple C*-algebra. Then the primitive spectrum of $\mathcal{A}$ consists of only one point. By definition, the Jacobson topology of the spectrum B is the trivial topology, that is, the open sets of B are the empty set and B itself [4]. So Homeo$B$ = $S(B) \equiv$ { permutation of B}. Furthermore, if Λ is 1-point (we call $\mathcal{A}$ automorphic), then $S(B)^\Lambda = S(B) =$ Homeo$B$. Thus, $PU(\mathcal{P})\rtimes S(B)^\Lambda = PU(\mathcal{P})\rtimes$Homeo$B$. Therefore Aut$\mathcal{A}$ is a subgroup of $PU(\mathcal{P})\rtimes$Homeo$B$ if $\mathcal{A}$ is simple and automorphic.

3 C*-geometry

In this section, we review the characterization of the set of pure states and the spectrum of a C*-algebra following [2].

Let (f, E, M) be a surjective map $f : E \to M$ between two sets E, M.

Definition 3.1 *(f, E, M) is a formal Kähler bundle if there is a family $\{E_m\}_{m\in M}$ of Kähler manifolds indexed by M and $E = \cup_{m\in M}E_m$ and $f(x) = m$ if $x \in E_m$.*

We simply denote (f, E, M) by E.

Assume now that E and M are topological spaces.

Definition 3.2 *(f, E, M) is called a uniform Kähler bundle if (f, E, M) is a formal Kähler bundle, f is open,continuous, the topology of E is a uniform topology and the relative topology of each fiber is equivalent to the Kähler topology of its fiber.*

For a uniform topology, see [1]. The weak*-topology on the set of pure states of C*-algebra is a uniform topology.

Definition 3.3 *Two formal Kähler bundle* (f, E, M), (f', E', M') *are isomorphic if there is a pair* (β, ϕ) *of bijections* $\beta : E \to E'$ *and* $\phi : M \to M'$, *such that* $f' \circ \beta = \phi \circ f$

$$\begin{array}{ccccc} & & \beta & & \\ & E & \cong & E' & \\ f & \downarrow & & \downarrow & f' \\ & M & \cong & M' & \\ & & \phi & & \end{array}$$

and any restriction $\beta|_{f^{-1}(m)} : f^{-1}(m) \to (f')^{-1}(\phi(m))$ *is a holomorphic Kähler isometry for any* $m \in M$. *We call* (β, ϕ) *a formal Kähler isomorphism between* (f, E, M) *and* (f', E', M').

By definition of a formal Kähler bundle isomorphism (β, ϕ) between (f, E, M) and (f', E', M'), ϕ is uniquely determined by β: For $m \in M$, the value $\phi(m)$ is given by $\phi(m) = f'(\beta(e))$ with arbitrary $e \in f^{-1}(\{m\})$.

Definition 3.4 *Two uniform Kähler bundles* (f, E, M), (f', E', M') *are isomorphic if there is a formal Kähler isomorphism* (β, ϕ) *between* (f, E, M) *and* (f', E', M') *such that* ϕ *is a homeomorphisms, and* β *is a uniform homeomorphism. We call* (β, ϕ) *a uniform Kähler isomorphism between* (f, E, M) *and* (f', E', M').

By definition, any uniform Kähler bundle is a formal Kähler bundle. For a uniform Kähler bundle E, we define:

Definition 3.5

$$\widetilde{\mathrm{Iso}(E)} \equiv \textit{ the group of formal Kähler bundle isomorphisms of } E,$$

$$\mathrm{Iso}(E) \equiv \textit{ the group of uniform Kähler bundle isomorphisms of } E.$$

By the GNS representation, there is a natural projection $p : \mathcal{P} \to B$ from the set $\mathcal{P}$ of pure states onto the spectrum B. We consider $(p, \mathcal{P}, B)$ as a map of topological spaces where $\mathcal{P}$ is endowed with weak* topology and B is endowed with the Jacobson topology.

In Ref.[2], the following results are proved.

Theorem 3.1 *(Reduced atomic realization) For any unital C^*-algebra $\mathcal{A}$, $(p, \mathcal{P}, B)$ is a uniform Kähler bundle.*

For a fiber $\mathcal{P}_b \equiv p^{-1}(b)$, let $(\pi_b, \mathcal{H}_b)$ be some irreducible representation belonging to $b \in B$. To $\rho \in \mathcal{P}_b$, correspond $[x_\rho] \in \mathcal{P}(\mathcal{H}_b) \equiv (\mathcal{H}_b \setminus \{0\})/\mathbf{C}^\times$ where $\rho = \omega_{x_\rho} \circ \pi_b$ with ω_{x_ρ} denoting a vector state $\omega_{x_\rho} =< x_\rho|(\cdot)x_\rho >$. Then $\mathcal{P}_b$ has a Kähler manifold structure induced by this correspondence from projective Hilbert space $\mathcal{P}(\mathcal{H}_b)$.

Theorem 3.2 *Let $\mathcal{A}_i$ be C^*-algebras with associated uniform Kähler bundles $(p_i, \mathcal{P}_i, B_i)$, $i = 1, 2$. Then $\mathcal{A}_1$ and $\mathcal{A}_2$ are *-isomorphic if and only if $(p_1, \mathcal{P}_1, B_1)$ and $(p_2, \mathcal{P}_2, B_2)$ are isomorphic as uniform Kähler bundles.*

Corollary 3.1 *Let* Aut$\mathcal{A}$ *be the group of *-automorphisms of a C^*-algebra $\mathcal{A}$ with an associated uniform Kähler bundle $\mathcal{P} = (p, \mathcal{P}, B)$, and* Iso$\mathcal{P}$ *be the group of uniform Kähler bundle automorphisms on $\mathcal{P}$. Then there is a group isomorphism*

$$\mathrm{Aut}\mathcal{A} \cong \mathrm{Iso}\mathcal{P}.$$

For $\alpha \in \mathrm{Aut}\mathcal{A}$, let $\beta_\alpha \equiv \alpha^*|_{\mathcal{P}} : \mathcal{P} \to \mathcal{P}$, $\alpha^*(\rho) \equiv \rho \circ \alpha^{-1}$ and induced bijection $\phi_\alpha : B \to B$ defined by $\phi_\alpha([\pi]) \equiv [\pi \circ \alpha^{-1}]$. Then $(\beta_\alpha, \phi_\alpha)$ becomes a uniform Kähler bundle automorphism of $\mathcal{P}$.

We call these objects *C^*-geometry* since any C*-algebra can be reconstructed from the associated uniform Kähler bundle [2] and, therefore, any C*-algebra is determined by such a geometry.

By the above result, we can consider the structure of Aut$\mathcal{A}$ in the language of Iso$\mathcal{P}$.

4 Orbit decomposition of a Kähler bundle

We decompose the set of pure states and the spectrum of a C*-algebra $\mathcal{A}$ as a uniform Kähler bundle. By using this decomposition, we describe automorphisms of $\mathcal{A}$ in each decomposed component in the next section.

In [3](II, p 906), two pure states ρ and ρ' of a C*-algebra $\mathcal{A}$ are called *automorphic* if there is an automorphism α of $\mathcal{A}$ such that $\rho' = \rho \circ \alpha$. For example, any two pure states of a uniform matricial algebra are automorphic ([3] II, Theorem 12.3.4). In general, the set of pure states of a C*-algebra is divided into a disjoint union of automorphic component. Therefore, each automorphism induces transformations on each automorphic components. The idea on which this section is based comes from this point of view.

Let $G \equiv \mathrm{Aut}\mathcal{A}$. G is naturally acting on the spectrum B by $g[\pi] \equiv [\pi \circ g^{-1}]$ for $g \in G$ and $[\pi] \in B$. So, we define the space Λ of orbits of G in B,

$$\Lambda \equiv B/G,$$

and call it the *orbit spectrum.* G acts naturally also on $\mathcal{P}$ by $g\rho \equiv \rho \circ g^{-1}$.

Lemma 4.1 *The orbit of G in $\mathcal{P}$ and B are in one-to-one correspondence.*

Proof. Let $G\rho$ be an orbit of G through $\rho \in \mathcal{P}$. We define $\Psi(G\rho) \equiv G[\pi_\rho] \in \Lambda$ where $[\pi_\rho]$ is the unitary equivalence class of irreducible representations of $\mathcal{A}$ with a representative element π_ρ, given by the GNS representation of ρ. For $g\rho \in G\rho$, $\pi_{g\rho}$ is unitarily equivalent to $g\pi_\rho \equiv \pi_\rho \circ g^{-1} \in G[\pi_\rho]$ by uniqueness of the GNS representation. Then the map $\Psi : \mathcal{P}/G \to \Lambda$ is well defined. By definition, Ψ maps orbits in $\mathcal{P}$ to orbits in B. And $\Psi(G\rho) = Gp(\rho)$. Hence, Ψ is onto.

If $\Psi(G\rho) = \Psi(G\rho^{'})$ and, $(\pi_\rho, \mathcal{H}_\rho, x_\rho)$ and $(\pi_{\rho'}, \mathcal{H}_{\rho'}, x_{\rho'})$ are GNS representations of ρ, $\rho^{'} \in \mathcal{P}$ respectively, then, $G[\pi_\rho] = G[\pi_{\rho'}]$. Since two automorphic pure states have GNS representation spaces with the same dimension, there are $g \in G$, a representative element $\pi^{'} \in [\pi_{\rho'}]$ which acts on $\mathcal{H}_\rho$, $\rho^{'} = \omega_{x_\rho} \circ \pi^{'}$ and a unitary operator U on $\mathcal{H}_\rho$ such that $\pi^{'} = \mathrm{Ad}U \circ g\pi_\rho$. By irreducibility of π_ρ, we can choose a unitary element V in $\mathcal{A}$ such that

$$\rho^{'} = (g\rho) \circ \mathrm{Ad}V = (\mathrm{Ad}V^{*} \circ g)\rho \in G\rho$$

(see [3], II, 10.2.6.). Therefore $G\rho = G\rho^{'}$. Ψ is an injection.

$$\begin{array}{ccccc} \mathcal{P} & \stackrel{p}{\to} & B & \stackrel{p^{'}}{\to} & B/G = \Lambda \\ \downarrow & & & \nearrow_{\Psi} & \\ \mathcal{P}/G & & & & \end{array}$$

∎

We decompose $\mathcal{P}$ by Λ into the family of uniform Kähler bundles.

Let $p' : B \to \Lambda = B/\mathrm{Aut}\mathcal{A}$ be the natural projection with fibers given by $B^\lambda \equiv (p')^{-1}(\lambda)$, $\lambda \in \Lambda$. Let $\mathcal{P}^\lambda \equiv \cup_{b\in B^\lambda}\mathcal{P}_b$. By lemma 4.1, B^λ and $\mathcal{P}^\lambda$ are orbits of G in B and $\mathcal{P}$, respectively. Let $p^\lambda \equiv p|_{\mathcal{P}^\lambda}$ for $\lambda \in \Lambda$. Then $(p^\lambda, \mathcal{P}^\lambda, B^\lambda)$ for each $\lambda \in \Lambda$ becomes a uniform Kähler bundle with the relative topology such that its total space $\mathcal{P}^\lambda$ is automorphic, that is, any two elements of $\mathcal{P}^\lambda$ are transformed by some automorphisms of $\mathcal{A}$ each other. We obtain a decomposition

$$(p, \mathcal{P}, B) = \bigcup_{\lambda\in\Lambda} (p^\lambda, \mathcal{P}^\lambda, B^\lambda)$$

of a uniform Kähler bundle.

Any two elements b, b' in the same orbit B^λ have representative representation spaces with the same dimensions. For an orbit $\lambda \in \Lambda$, let $\mathcal{H}_\lambda$ be a Hilbert space corresponding to a representative element of some point in an orbit B^λ. We can choose a representative element belonging to B^λ which acts on the same Hilbert space $\mathcal{H}_\lambda$.

Let $\mathcal{P}^o \equiv \cup_{\lambda\in\Lambda}\mathcal{P}(\mathcal{H}_\lambda) \times B^\lambda$ and $p^o : \mathcal{P}^o \to B$ defined by $p^o(\xi, b) = b$ for $\xi \in \mathcal{P}(\mathcal{H}_\lambda)$ and $b \in B^\lambda$. $(p^o, \mathcal{P}^o, B)$ becomes a formal Kähler bundle with fiber $\mathcal{P}(\mathcal{H}_\lambda) \times \{b\}$ for $b \in B^\lambda \subset B$.

Theorem 4.1 *$(p, \mathcal{P}, B)$ and $(p^o, \mathcal{P}^o, B)$ are isomorphic as formal Kähler bundles.*

Proof. Fix a family of representative elements $\{\pi_b\}_{b\in B}$ of B such that π_b acts on $\mathcal{H}_\lambda$ if $b \in B^\lambda$. Define $\beta : \mathcal{P} \to \mathcal{P}^0$ by $\beta(\rho) \equiv ([x_\rho], b) \in \mathcal{P}(\mathcal{H}_\lambda) \times B^\lambda$ if $[\pi_\rho] = b \in B^\lambda$, where $\rho = \omega_{x_\rho} \circ \pi_b$ and $x_\rho \in \mathcal{H}_\lambda$. Let $\phi : B \to B$ be the identity map on B. Then $\beta(\mathcal{P}_b) = \mathcal{P}(\mathcal{H}_\lambda) \times \{b\}$ for $b \in B^\lambda$. And β is fiber-wise holomorphic isometry. Then, (β, ϕ) becomes a formal Kähler isomorphism between $(p, \mathcal{P}, B)$ and $(p^o, \mathcal{P}^o, B)$. ∎

Corollary 4.1 *(Orbit decomposition) Let $\mathcal{P} = (p, \mathcal{P}, B)$ be a uniform Kähler bundle associated with a C^*-algebra. Then there is a uniform Kähler bundle $\mathcal{P}^o = (p^o, \mathcal{P}^o, B)$ with $\mathcal{P}^o = \cup_{\lambda\in\Lambda}\mathcal{P}(\mathcal{H}_\lambda) \times B^\lambda$ such that $\mathcal{P} \cong \mathcal{P}^o$.*

Proof. In the previous theorem, let the topology of $\mathcal{P}^o$ be the induced topology from $\mathcal{P}$. Then $(p^o, \mathcal{P}^o, B)$ becomes a uniform Kähler bundle which is isomorphic to $(p, \mathcal{P}, B)$. ∎

5 Proof of the main theorem

By Corollary 4.1, we identify the uniform Kähler bundle $\mathcal{P}=(p,\mathcal{P},B)$ associated with a C*-algebra $\mathcal{A}$ and its orbit decomposition $\mathcal{P}^o$ corresponding to the orbit spectrum Λ. Let Aut$\mathcal{A}$ be the group of *-automorphisms of a C*-algebra $\mathcal{A}$ with associated uniform Kähler bundle $\mathcal{P}=(p,\mathcal{P},B)$ and the orbit spectrum Λ. By corollary 4.1, we identify $\mathcal{P}$ with its orbit decomposition $\mathcal{P}^o$.

Recall that $PU(\mathcal{P})$, $S(B)^\Lambda$ and $\widetilde{\mathrm{Iso}\mathcal{P}}$ are defined in Theorem 2.1 and Definition 3.5.

We define actions t,s of $PU(\mathcal{P})$, $S(B)^\Lambda$ respectively on the set $\mathcal{P}$ of pure states by

$$\begin{array}{rcl} t_u(\xi,b) & \equiv & (\ u_b\xi\ ,\ b\ ),\\ s_\phi(\xi,b) & \equiv & (\ \xi\ ,\ \phi(b)\ ),\end{array}$$

for $u=\{u_b\}_{b\in B}\in PU(\mathcal{P})$, $\phi\in S(B)^\Lambda$ and $(\xi,b)\in\mathcal{P}^o$. With these actions, we define injective homomorphisms τ and σ of $PU(\mathcal{P})$ and of $S(B)^\Lambda$ into $\widetilde{\mathrm{Iso}\mathcal{P}}$, by

$$\begin{array}{rcl} \tau_u & \equiv & (\ t_u\ ,\ id_B\ ),\\ \sigma_\phi & \equiv & (\ s_\phi\ ,\ \phi\ ),\end{array}$$

respectively, for $u\in PU(\mathcal{P})$, $\phi\in S(B)^\Lambda$.

Lemma 5.1

$$\tau(PU(\mathcal{P}))=\{(\beta,\phi)\in\widetilde{\mathrm{Iso}\mathcal{P}}:\phi=id_B\}\equiv G_3.$$

Proof. By definition of τ, $\tau(PU(\mathcal{P}))$ is contained in G_3. On the other hand, for any $g=(\beta,id_B)\in G_3$, β becomes a holomorphic Kähler isometry on each fiber by definition of G_3. So it becomes a projective unitary on each fiber by Wigner's theorem. Thus, there is a family of projective unitaries corresponding to g. ∎

For $u\in PU(\mathcal{P})$ and $\phi\in S(B)^\Lambda$, we obtain $s_\phi t_u s_{\phi^{-1}}=t_{u\delta_\phi}$, where δ is the right action of $S(B)^\Lambda$ on $PU(\mathcal{P})$ defined in Theorem 2.1. From this follows the relation

$$\sigma_\phi\tau_u\sigma_{\phi^{-1}}=\tau_{u\delta_\phi}. \qquad \text{(Eq.5.1)}$$

Consider the action $\tilde{\delta}_a=\mathrm{Ad}_a$ of $a\in\sigma(S(B)^\Lambda)$ on $\tau(PU(\mathcal{P}))$. Let G_2 be the group generated by $\tau(PU(\mathcal{P}))$ and $\sigma(S(B)^\Lambda)$. We obtain the following isomorphism between semi-direct products of groups.

Lemma 5.2

$$PU(\mathcal{P})\rtimes_{\delta}S(B)^{\Lambda} \cong \tau(PU(\mathcal{P}))\rtimes_{\tilde{\delta}}\sigma(S(B)^{\Lambda}) = G_2.$$

Proof. By Eq.5.1 and the definition of semi-direct product, the lemma follows.

∎

Proof of Theorem 2.1(main Theorem). By lemma 5.2, $PU(\mathcal{P})\rtimes_{\delta}S(B)^{\Lambda}$ is embedded into $\widetilde{\mathrm{Iso}\mathcal{P}}$ as a subgroup.

Let

$$\kappa : PU(\mathcal{P})\rtimes_{\delta}S(B)^{\Lambda} \hookrightarrow \widetilde{\mathrm{Iso}\mathcal{P}} \tag{Eq.5.2}$$

be defined by $\kappa(u,\phi) \equiv \tau_u\sigma_\phi = (t_u s_\phi, \phi)$ for $(u,\phi) \in PU(\mathcal{P})\rtimes_{\delta}S(B)^{\Lambda}$.

On the other hand, by Corollary 3.1, there is an isomorphism

$$\pi_1 : \mathrm{Aut}\mathcal{A} \cong \mathrm{Iso}\mathcal{P} \subset \widetilde{\mathrm{Iso}\mathcal{P}}. \tag{Eq.5.3}$$

We denote $\pi_1(\alpha) \equiv (\beta_\alpha, \phi_\alpha)$ for $\alpha \in \mathrm{Aut}\mathcal{A}$. Then, $(\beta_\alpha \circ s_{\phi_\alpha^{-1}}, id_B) \in G_3$. By lemma 5.1, there is $u^\alpha \in PU(\mathcal{P})$ such that $\beta_\alpha \circ s_{\phi_\alpha^{-1}} = t_{u^\alpha}$. So, $\beta_\alpha = t_{u^\alpha} s_{\phi_\alpha}$. Therefore,

$$\pi_1(\alpha) = (\beta_\alpha, \phi_\alpha) = (t_{u^\alpha} s_{\phi_\alpha}, \phi_\alpha) = \tau_{u^\alpha}\sigma_{\phi_\alpha}.$$

By this calculation and Eq.5.2, $\pi_1(\mathrm{Aut}\mathcal{A}) \subset G_2 = \kappa(PU(\mathcal{P})\rtimes_{\delta}S(B)^{\Lambda})$.

Let $\pi \equiv \kappa^{-1}|_{G_2} \circ \pi_1$. We denote an element of $PU(\mathcal{P})\rtimes_{\delta}S(B)^{\Lambda}$ by $u \cdot \delta_\phi$ which satisfies the product law

$$(u \cdot \delta_\phi)(u' \cdot \delta_{\phi'}) = \{u(u'\delta_\phi)\} \cdot \delta_{\phi\phi'}$$

for $u\cdot\delta_\phi,\ u'\cdot\delta_{\phi'} \in PU(\mathcal{P})\rtimes_{\delta}S(B)^{\Lambda})$, $u, u' \in PU(\mathcal{P})$ and $\phi, \phi' \in S(B)^{\Lambda}$. Then

$$\pi(\alpha) = u^\alpha \cdot \delta_{\phi^\alpha} \in PU(\mathcal{P})\rtimes_{\delta}S(B)^{\Lambda}$$

for $\alpha \in \mathrm{Aut}\mathcal{A}$. Then we obtain the injective homomorphism for which we have been looking

$$\begin{tikzcd}
\mathrm{Aut}\mathcal{A} \arrow[rr, hook, "\pi"] \arrow[dr, "\pi_1"'] & & PU(\mathcal{P})\rtimes_{\delta}S(B)^{\Lambda}. \\
& G_2 \arrow[ur, "\kappa^{-1}|_{G_2}"'] &
\end{tikzcd}$$

By Corollary 3.1, we obtain

$$\pi(\mathrm{Aut}\mathcal{A}) = (\kappa^{-1}|_{G_2} \circ \pi_1)(\mathrm{Aut}\mathcal{A}) = \kappa^{-1}|_{G_2}(\mathrm{Iso}\mathcal{P}) \text{ (by equation Eq.5.3)}$$
$$= \kappa^{-1}|_{G_2}(\{(\beta,\phi) \in \widetilde{\mathrm{Iso}\mathcal{P}} : \beta \text{ is a uniform homeomorphism on } \mathcal{P}\})$$
$$= \{g \in PU(\mathcal{P}) \rtimes_\delta S(B)^\Lambda : \kappa_g \text{ acts on } \mathcal{P} \text{ as a uniform homeomorphism}\},$$

from which the statement of Theorem 2.1 immediately follows. ∎

6 Conclusion

In this paper, we have obtained the orbit decomposition of the uniform Kähler bundles and the group of automorphisms.

The next step would be to consider the orbit decomposition of the algebra itself. In this context, the meaning of decomposition has to be cleared. It might be a decomposition like by crossed products, free products of C*-algebras.

We are studying geometrical objects corresponding to modules, crossed product, subalgebra, *-homomorphism and etc are currently under study for non-commutative C*-algebras. They are realization of non-commutative geometry by " *real* " geometry defined as the set of points and its function space [2]. So, they must be direct generalization of the geometry of commutative case leading to new geometrical structures for which we would like to give a better understanding.

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