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TYPE *III* FACTORS ARISING AS AMALGAMATED FREE PRODUCTS

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1. Introduction.

Type *III* factors arising as free products were studied in detail by F. Rădulescu, L. Barnett, K. Dykema and D. Shlykhtenko mainly based on D. Voiculescu's powerful machine. (However, the question of type classification is not yet completed. An attempt to the question can be also found in [U3].) On the other hand, amalgamated free products in the type *III* setting had never been investigated before our study. In this note, we will report our recent study on type *III* factors arising as amalgamated free products.

2. Amalgamated free products of von Neumann algebras.

Let $A \supseteq D \subseteq B$ be (σ-finite) von Neumann algebras and $E_D^A : A \to D$, $E_D^B : B \to D$ be faithful normal conditional expectations. The amalgamated free product of A and B over D with respect to E_D^A and E_D^B is defined as a von Neumann algebra M with unital

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(normal) embeddings of A and B into M which coincide on D and a faithful normal conditional expectation $E_D^M : M \to D$ such that

(1) A and B generate M as subalgebras of M;

(2) the restrictions of E_D^M to A and B (as subalgebras of M) are just E_D^A and E_D^B respectively;

(3) A and B (as subalgebras of M) are free with respect to E_D^M.

We denote the amalgamated free product by

$$(M, E_D^M) = (A, E_D^A) *_D (B, E_D^B).$$

In analysis on type III factors, modular automorphisms have central importance. Hence, we should at first compute the modular automorphisms $\sigma_t^{\varphi\circ E_D^M}$, $t \in \mathbf{R}$, for a fixed faithful normal state φ on D. This can be done by using the freeness condition.

Theorem 1. (cf. [U2]) *We have*

$$\sigma_t^{\varphi\circ E_D^M}|_A = \sigma_t^{\varphi\circ E_D^A}, \quad \sigma_t^{\varphi\circ E_D^M}|_B = \sigma_t^{\varphi\circ E_D^B}$$

for $t \in \mathbf{R}$.

3. Amalgamated free products over Cartan subalgebras.

Amalgamated free products of factors over their common Cartan subalgebras were investigated as a first step of our attempt towards investigation on amalgamated free products in the type III setting.

Our main theorem is as follows:

Theorem 2. ([U2]) *Let A and B be factors with separable preduals having a common non-atomic (i.e., A and B are not of type I) Cartan subalgebra D. Let*

$$(M, E_D^M) = (A, E_D^A) *_D (B, E_D^B)$$

be the amalgamated free product. Then there exists a faithful normal state φ on D such that

$$(A_{\varphi \circ E_D^A})' \cap M \subseteq A.$$

Moreover, if A is of type III_λ $(\lambda \neq 0)$, the above state φ can be chosen in such a way that

$$(A_{\varphi \circ E_D^A})' \cap A = \mathbf{C}1.$$

The analogous result holds for B.

The proof depends on several results related to ergodic theory and the averaging technique, but the crucial idea is very simple. Indeed, the theorem is proved based on the essentially same simple spirit as in the proof of the classical fact that $\{a\}' \cap L(\mathbb{F}_2) = \{a\}''$, where a is one of the free generators in $L(\mathbb{F}_2)$.

Remark. (The type I case.) If A and B are type I factors having a common Cartan subalgebra D, then $A = B$, that is, $A \supseteq D \subseteq B$ is isomorphic to $A \supseteq D \subseteq A$. In this case, it can be shown that the amalgamated free product of A and B over D is isomorphic to $L(\mathbb{F}_{n-1}) \otimes A$ when A (or B) is of type I_n (possibly $n = \infty$) (and hence M is not of type III). This computation is essentially contained in [D3], and also T. Sakamoto checked the same formula in the C^*-case. Furthermore, the same formula also holds even in the level of groupoid free products (see below). It should be remarked that the relative commutant property in the theorem does not hold in the type I case.

Indeed, via the above isomorphism $M = A *_D B \cong L(\mathbb{F}_{n-1}) \otimes A$ the subalgebra A corresponds to $\mathbf{C}1 \otimes A$, and hence $A' \cap M \cong L(\mathbb{F}_{n-1})$. This shows that the non-type I case is different from the type I case.

After the work [U2] was completed, H. Kosaki began to study amalgamated free products over Cartan subalgebras from the view-point of measured groupoids (see [K]). He discussed, among other things, the construction (in a rigorous measurable fashion) of "free products" of discrete measured equivalence relations as well as some detailed analyses in the type I setting. His approach might be useful to clarify the above-mentioned difference between the type I case and the non-type I case.

Theorem 2, in particular, shows that the resulting amalgamated free product M is a factor. Moreover, based on Theorem 2 we can obtain the following type classification results:

(1) If M is of type III_0, then both of A and B must be of type III_0;

(2) If either A or B is of type II_1 and M is semi-finite, then M is of type II_1;

(3) If either A or B is of type III_1, then M is of type III_1;

(4) If A is of type III_λ and B is of type III_μ such that $\log\lambda$ and $\log\mu$ are rationally independent, then M is of type III_1.

The assertions (3), (4) follow from a simple computation of the T-set $T(M)$ based on the relative commutant property of Theorem 2 together with the assertion (1).

It should be here mentioned that an example of AFD type II_1 (or II_∞) factors having a common Cartan subalgebra whose amalgamated free product is of type III_λ, $0 < \lambda \leq 1$, can be constructed.

To get more detailed information on type classification we should compute the flow of weights of the resulting amalgamated free product M in terms of those of A and B. This can be done based on the techniques used in the proof of Theorem 2 together with a fact, suggested by H. Kosaki, on the continuous cores (or the Takesaki duals) of amalgamated free products. Indeed, we can prove the following:

Theorem 3. ([U2]) *Let $A \supseteq D \subseteq B$ and $M = A *_D B$ be as in Theorem 2. The flow of weights (X_M, F_t^M) is determined as the (unique) maximal common factor flow of the flows of weights (X_A, F_t^A), (X_B, F_t^B) and (X_D, F_t^D).*

Here, $X_D = Y \times \mathbf{R}$ with $D = L^\infty(Y)$ and $F_t^D(y,s) = (y, s+t)$. Since D is a common Cartan subalgebra of A and B, the flows of weights (X_A, F_t^A) and (X_B, F_t^B) are factor ones of (X_D, F_t^D). Hence, the above (unique) maximal common factor flow exists.

The above formulation in terms of ergodic theory was suggested by T. Hamachi, while the author's original formulation is written in terms of the continuous cores $\widetilde{A}$ and $\widetilde{B}$ of A and B as follows: The center $\mathcal{Z}(\widetilde{M})$ of the continuous core of M is just the intersection of the centers $\mathcal{Z}(\widetilde{A})$ and $\mathcal{Z}(\widetilde{B})$.

Theorem 3, in particular, shows that in the case that $A = B$, the resulting flow of weights (X_M, F_t^M) is just (X_A, F_t^A) (or (X_B, F_t^B)).

Remark. The above theorem (Theorem 3) can be proved under a weaker condition.

The cores $\widetilde{M} := M \rtimes_{\sigma^{\varphi\circ E_D^M}} \mathbf{R}$, $\widetilde{A} := A \rtimes_{\sigma^{\varphi\circ E_D^A}} \mathbf{R}$, $\widetilde{B} := B \rtimes_{\sigma^{\varphi\circ E_D^B}} \mathbf{R}$ and $\widetilde{D} := D \rtimes_{\sigma^\varphi} \mathbf{R} = D \otimes \lambda(\mathbf{R})''$ have the natural inclusion relations:

$$\widetilde{M} = \widetilde{A} \vee \widetilde{B} \supseteq \widetilde{A},\ \widetilde{B} \supseteq \widetilde{D}.$$

As is well-known in the subfactor theory, the conditional expectations E_D^M, E_D^A and E_D^B can be lifted as $\widetilde{E_D^M} : \widetilde{M} \to \widetilde{D}$, $\widetilde{E_D^A} : \widetilde{A} \to \widetilde{D}$ and $\widetilde{E_D^B} : \widetilde{B} \to \widetilde{D}$ by the quite natural way. In the current setting, we have known that $\mathcal{Z}(\widetilde{M}) = \mathcal{Z}(\widetilde{A}) \cap \mathcal{Z}(\widetilde{B}) \subseteq \widetilde{D}$, and hence we have

$$\widetilde{M} = \int_{X_M}^{\oplus} \widetilde{M}(\omega) d\mu(\omega) \supseteq$$

$$\widetilde{A} = \int_{X_M}^{\oplus} \widetilde{A}(\omega) d\mu(\omega), \quad \widetilde{B} = \int_{X_M}^{\oplus} \widetilde{B}(\omega) d\mu(\omega)$$

$$\supseteq \widetilde{D} = \int_{X_M}^{\oplus} \widetilde{D}(\omega) d\mu(\omega)$$

under the identification $\mathcal{Z}(\widetilde{M}) = L^\infty(X_M, \mu)$. Based on Voiculescu's striking result ([V2]) we can show the following:

Proposition 4. *Let $A \supseteq D \subseteq B$ and $M = A *_D B$ be as in Theorem 2. Suppose that M is of type III. For a.e. ω, the type II_∞ factor $\widetilde{M}(\omega)$ is not of the form $L(\mathbb{F}_r) \otimes B(\mathcal{H})$ for $r > 1$. Therefore, if M is of type III_λ $(0 < \lambda < 1)$, then the type II_∞ factor appearing in its discrete decomposition is also not isomorphic to $L(\mathbb{F}_r) \otimes B(\mathcal{H})$ for any $r > 1$.*

This proposition suggests us that type III factors arising as our amalgamated free products are different from those arising as free products. Indeed, K. Dykema showed, in [D2], that the discrete cores of many type III_λ factors arising as free products of AFD von Neumann algebras are of the form $L(\mathbb{F}_\infty) \otimes B(\mathcal{H})$.

Remark. (The case that M is of type II_1.) The factor M is not isomorphic to any free group factor even in the case that M is of type II_1. This also follows from D. Voiculescu's striking result ([V2]).

In closing this section, it is mentioned that the (non-)amenability of our amalgamated free product M is unknown. (Of course, if M is of type II_1, then M is always not amenable thanks to the presence of a copy of the free group factor $L(\mathbb{F}_2)$ in M.)

4. The crossed-product by a minimal action of $SU_q(2)$.

In [U1], the author introduced (and established) the concept of free products of actions of quantum groups, and by using it a minimal action of the quantum group $SU_q(n)$ was constructed. However, the question of type classification of the fixed-point algebra (or equivalently the crossed-product thanks to the Takesaki duality) of the minimal action has been unsolved. The crossed product can be written as a certain amalgamated free product, and hence this question can be regarded as that for an amalgamated free product. The author recently solved it as follows:

Theorem 5. ([U4]) *The crossed-product by the minimal action of $SU_q(2)$ constructed in* [U1] *is of type III_{q^2} (and hence the fixed-point algebra is also of type III_{q^2}).*

The main part of the proof is as follows: If a real number t is in the T-set of the crossed-product, then the following equations hold:

$$x \otimes (q^{2it}x) + u \otimes (q^{2it}v) = x \otimes (wxw^*) + u \otimes (wvw^*);$$

$$v \otimes x + y \otimes v = v \otimes (wxw^*) + y \otimes (wvw^*)$$

for some unitary w on the standard Hilbert space $L^2(SU_q(2))$. Here x, u, v, y be the standard generators of $SU_q(2)$ (see [M^{2}N^{2}U] for example). By the orthogonality of the generators with respect to the Haar state h (thanks to the Peter-Weyl type theorem for $SU_q(2)$) we obtain

$$q^{2it}x = wxw^* = x, \quad q^{2it}v = wvw^* = v.$$

Hence the number t is in $(\frac{-2\pi}{\log q^2})\mathbb{Z}$.

Furthermore, it can be shown that the ambient factor and the fixed-point algebra under the above minimal action of $SU_q(2)$ have the identical flow of weights.

The details together with further investigations (related to subfactor theory) will be presented elsewhere.

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