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GORENSTEIN LIAISON VIA DIVISORS

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1. INTRODUCTION

This expository paper is a slightly expanded version of a series of four talks that I gave at the Research Institute for Mathematical Sciences at Kyoto University on July 28–31, 1998 in the workshop "Free resolutions of the defining ideals of projective varieties," organized by Professor C. Miyazaki. The purpose of this paper is to describe some recent work, almost entirely from [12], which gives a new approach to the theory of liaison (or linkage) via Gorenstein ideals, which we will refer to as *Gorenstein liaison* or simply *G-liaison*. We will discuss the new ideas, and we will show how they fit in nicely with the theory of liaison via complete intersections, which we will call *complete intersection liaison* or simply *CI-liaison*. We will be particularly interested in seeing how the new approach fits in with Hartshorne's approach [11] to CI-liaison using generalized divisors on complete intersections. Indeed, the main common theme throughout this paper is that Gorenstein liaison is also a theory about divisors, this time on arithmetically Cohen-Macaulay schemes. A second common theme is to try to compare and contrast the theory of CI-liaison with that of G-liaison in various ways.

In Section 2 we give the necessary definitions for liaison, and many examples of Gorenstein subschemes of $\mathbb{P}^n$. We also give some background material and motivate the idea of passing to higher codimension. We see that Gorenstein liaison is a natural generalization of the well-understood codimension two case. Section 3 gives further background material. In particular, since the common theme in this paper is that of divisors on schemes which are not necessarily smooth, we discuss some of the pitfalls that must be avoided in talking about divisors. Section 4 makes the connection between divisors and Gorenstein liaison. We give a construction for arithmetically Gorenstein subschemes of $\mathbb{P}^n$ and give some important consequences for liaison. We show that linearly equivalent divisors are in the same even Gorenstein liaison class, and that adding hyperplane sections preserves the even Gorenstein liaison class.

Section 5 describes two generalizations of the codimension two case given in [12]. One is a generalization of Gaeta's theorem, and it says basically that any determinantal subscheme of $\mathbb{P}^n$ is in the Gorenstein liaison class of a complete intersection. The second is a generalization of a theorem of Rao, and says basically that geometric liaison and algebraic liaison generate the same equivalence relation.

Section 6 describes some results on curves on smooth arithmetically Cohen-Macaulay surfaces, including the fact that on a smooth rational surface in $\mathbb{P}^4$, all arithmetically Cohen-Macaulay curves are in the Gorenstein liaison class of a complete intersection. In contrast, we mention some complete intersection liaison invariants from [12], and we give a corollary which shows that adding hyperplane sections does *not* preserve the complete intersection liaison class. Finally, in Section 7 we give some further similarities and differences between CI-liaison and G-liaison, this time primarily from [18].

I would like to stress that the paper [12] in fact does a great deal more than what we describe here, treating also invariants of complete intersection liaison and unobstructedness of subschemes of projective space. Here we report almost exclusively on those aspects of the paper which deal with Gorenstein liaison. For additional background, especially on Gorenstein liaison, we refer the reader to [18] for a broad treatment of the theory, and to [21] for some more advanced results.

I would like to thank my co-authors of [12] for the enjoyable collaboration which led to the material described in this paper and in my talks: they are J. Kleppe, R.M. Miró-Roig, U. Nagel and C. Peterson. I would also like to express here my sincere thanks to C. Miyazaki for inviting me to the workshop and for giving me the opportunity to discuss this material, and also for his great hospitality during my stay. I am also very grateful to K. Yoshida for the generous financial support which made my participation in the workshop possible.

2. Preliminaries and motivation

We denote by R the polynomial ring $k[x_0, \dots, x_n]$, where k is an algebraically closed field, and we denote by $\mathbb{P}^n$ the projective space $\mathrm{Proj}(R)$. For a sheaf $\mathcal{F}$ on $\mathbb{P}^n$, we write

$$H^i_*(\mathcal{F}) := \bigoplus_{t \in \mathbb{Z}} H^i(\mathbb{P}^n, \mathcal{F}(t)).$$

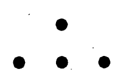
This is a graded R-module. For a subscheme V of $\mathbb{P}^n$, we denote by $\mathcal{I}_V$ its ideal sheaf and by I_V its saturated homogeneous ideal; note that $I_V = H^0_*(\mathcal{I}_V)$. We recall that a codimension c subscheme X of $\mathbb{P}^n$ is *arithmetically Gorenstein* if its saturated ideal has a minimal free resolution of the form

$$0 \to R(-t) \to F_{c-1} \to \cdots \to I_X \to 0.$$

In particular, X is arithmetically Cohen-Macaulay. The *socle degree* of X is the integer $t - c$. If the h-vector of X is $(1, c, h_2, \dots, h_{r-2}, c, 1)$ (which we recall must be symmetric in order for X to be arithmetically Gorenstein) then the socle degree is also r.

Example 2.1. We first mention some interesting examples of arithmetically Gorenstein schemes, and in particular some useful ways of constructing them.

(1) Any complete intersection, X, of forms of degree $a_1, \dots, a_c$ is arithmetically Gorenstein. In this case $t = \sum a_i$. These are easy to produce, but for many purposes they are "too big," as we shall see.
(2) A set of five points in $\mathbb{P}^3$ in linear general position is arithmetically Gorenstein. More generally, a set of $n+2$ points in $\mathbb{P}^n$ in linear general position is arithmetically Gorenstein. This follows from a result of [7], which says that a reduced zeroscheme Z is arithmetically Gorenstein if and only if its h-vector is symmetric and Z satisfies the *Cayley-Bacharach property.* (We recall that Z satisfies the Cayley-Bacharach property if all subsets of cardinality $|Z| - 1$ have the same Hilbert function. So, for instance, the simplest set of points not satisfying the Cayley-Bacharach property is the following configuration of points in $\mathbb{P}^2$:



since one subset of three points lies on a line, while the other subsets of three points do not. This set of points has symmetric h-vector but is not arithmetically Gorenstein.) The result of [7] was generalized in [13] to the non-reduced case.

(3) A *Buchsbaum-Rim sheaf* is, by definition, the reflexive sheaf $\mathcal{B}_\varphi$ obtained as the kernel of a generically surjective morphism $\varphi : \mathcal{F} \to \mathcal{G}$, where $\mathcal{F}$ and $\mathcal{G}$ are free sheaves of ranks f and g, respectively, and we assume that φ has a degeneracy locus of the expected codimension $f - g + 1$. Various properties of these sheaves were studied in [20], [14] and [19]. In particular, it was shown in [19] that if the rank, r, of $\mathcal{B}_\varphi$ is odd then the top dimensional part of the scheme defined by a regular section of $\mathcal{B}_\varphi$ is arithmetically Gorenstein of codimension r. See also the discussion on page 19.

(4) If X is arithmetically Gorenstein of dimension ≥ 1 and if F is any homogeneous polynomial not vanishing on any component of X then F cuts out a divisor on X which, viewed as a subscheme of $\mathbb{P}^n$, is again arithmetically Gorenstein. This fact can be used in conjunction with the Buchsbaum-Rim construction above to construct arithmetically Gorenstein schemes of even codimension.

(5) Below we will recall the notion of two codimension c schemes, V_1 and V_2, in $\mathbb{P}^n$ being *geometrically linked* by an arithmetically Gorenstein scheme X. This means simply that $I_{V_1} \cap I_{V_2} = I_X$. If this is the case, it follows that the sum $I_{V_1} + I_{V_2}$ defines a scheme which is arithmetically Gorenstein of codimension $c+1$. This comes from the exact sequence

$$0 \to I_X \to I_{V_1} \oplus I_{V_2} \to I_{V_1} + I_{V_2} \to 0$$

and a mapping cone.

(6) Below in Corollary 4.2 we will see a very useful construction, via linear systems, for constructing Gorenstein ideals. This was, in a sense, the initial observation which generated the work contained in [12]. □

(7) In codimension two, the converse of (1) holds: any arithmetically Gorenstein scheme is a complete intersection. In higher codimension, (2) shows that there are arithmetically Gorenstein schemes which are not complete intersections. □

We now come to the central definitions of this paper.

Definition 2.2. We say that two subschemes V_1 and V_2 of $\mathbb{P}^n$ are *Gorenstein linked*, or simply *G-linked*, by an arithmetically Gorenstein scheme X if $I_X \subset I_{V_1} \cap I_{V_2}$ and if we have $I_X : I_{V_1} = I_{V_2}$ and $I_X : I_{V_2} = I_{V_1}$. We denote this by $V_1 \overset{X}{\sim} V_2$. The equivalence relation generated by G-links is called *Gorenstein liaison*, or simply *G-liaison*. If X is a complete intersection, we say that V_1 and V_2 are *complete intersection linked*, or simply *CI-linked*. The equivalence relation generated by CI-links is called *complete intersection liaison*, or simply *CI-liaison*. A subscheme V is said to be *licci* if it is in the (complete intersection) **LI**aison **C**lass of a **C**omplete **I**ntersection. Analogously, we say that a subscheme V is *glicci* if it is in the **G**orenstein **LI**aison **C**lass of a **C**omplete **I**ntersection. If V_1 is linked to V_2 in two steps by complete intersection (resp. by arithmetically Gorenstein schemes), we say that they are *CI-bilinked* (resp. *G-bilinked*). □

The notion of using complete intersections to link varieties has been used for a long time, going back at least to work of Macaulay and Severi.

The following proposition has been known for some time, but maybe not well-known for Gorenstein links, especially (iv). It gives some idea of what is preserved when you link.

Proposition 2.3.

(a) *Assume* $V_1 \overset{X}{\sim} V_2$ *is a G-link, where*

$$0 \to R(-t) \to \cdots \to I_X \to 0.$$

(i) $\dim V_1 = \dim V_2 = \dim X = r$ *(say), all unmixed.*

(ii) *If* $r \geq 1$ *and* V_1 *is locally CM then for* $1 \leq i \leq r$ *we have*

$$(M^{r-i+1})(V_2) \cong (M^i)(V_1)^\vee(n+1-t)$$

where $(M^i)(V) := H^i_*(\mathcal{I}_V)$.

(iii) $\deg V_1 + \deg V_2 = \deg X$.

(iv) *If* $\dim X = 1$, $\deg V_i = d_i$ *and* $p_a(V_i) = g_i$ *then*

$$g_2 - g_1 = \frac{1}{2}(t-n-1)(d_2 - d_1).$$

(b) *If* V *is not necessarily unmixed,* $I_X : [I_X : I_V]$ *is the ideal of the top dimensional part of* V.

We remark that the algorithm described in (b) is very easy and fast on the standard computer programs for commutative algebra.

It is very interesting that Proposition 2.3 holds for G-liaison almost exactly as it does for the well-known case of CI-liaison. A very natural question is the following:

> Do CI-liaison and G-liaison generate the same equivalence relation on codimension c subschemes of $\mathbb{P}^n$?

In codimension two we see immediately that the answer is "yes," since complete intersections and arithmetically Gorenstein schemes coincide. In higher codimension the answer is "no." Indeed, a simple counterexample is the following. Consider a set Z of four points in $\mathbb{P}^3$ in linear general position. By adding a sufficiently general fifth point, Example 2.1 (2) shows that Z is glicci. On the other hand, it follows from a theorem of [15] that they are not licci, since they have a pure resolution but are not Gorenstein.

The study of liaison in codimension two is a remarkably complete picture, and its applications have been extremely numerous and varied. One naturally would like to carry out a program in higher codimension. We have the following picture:

$$\boxed{\begin{array}{c}\text{codimension 2 liaison}\\ \text{(with all its beautiful results)}\end{array}}$$

$$\swarrow \quad \text{higher codimension} \quad \searrow$$

$$\text{CI-liaison} \qquad\qquad\qquad\qquad \text{G-liaison}$$

The purpose of this paper is to show how results of [12] suggest that G-liaison is a very natural direction to go.

3. Remarks on divisors

Hartshorne [11] developed the theory of generalized divisors on (locally) Gorenstein schemes, and from it he rigorously derived many of the main facts about CI-liaison by thinking of subschemes of $\mathbb{P}^n$ as divisors on complete intersections. In this paper we would like to discuss how this approach gives a result about CI-liaison (the equivalence of geometric and algebraic liaison), and how an analogous approach can be applied to G-liaison. In this section, however, we begin by discussing what can and cannot be done with divisors, and where we have to be careful.

¿From now on, a *divisor* on a subscheme S of $\mathbb{P}^n$ is a pure codimension 1 subscheme of S with no embedded components. Note that we usually talk about a divisor C on a subscheme S of $\mathbb{P}^n$. The notation suggests curves on surfaces, and this is one of our main applications, but we do not intend that to be the case in general.

The most basic kind of divisor is the hypersurface section divisor:

Definition 3.1. Let $S \subset \mathbb{P}^n$ and let $F \in R$ be a homogeneous polynomial not vanishing on any component of S (i.e. $I_S : F = I_S$). Then H_F is the divisor cut out on S by F. □

Remark 3.2.

(a) Viewing H_F as a subscheme of $\mathbb{P}^n$, its ideal I_{H_F} is the saturation of $I_S + (F)$, and $\deg H_F = \deg F \cdot \deg S$.
(b) If S is arithmetically Cohen-Macaulay then $I_S + (F)$ is already saturated and H_F is also arithmetically Cohen-Macaulay.
(c) Note that we do not assume anything about S. If S is arithmetically Cohen-Macaulay (no other assumptions) and $\deg F = t$ then we denote the set of all divisors cut out by hypersurfaces of degree t by $|tH|$. □

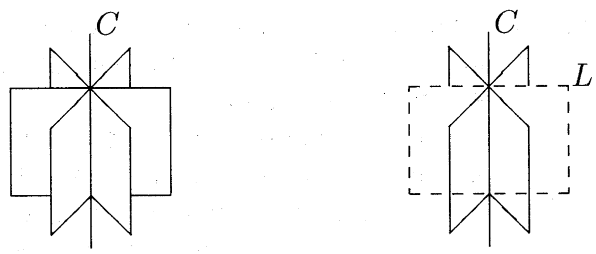
Now we turn to other divisors and to operations on divisors. Hartshorne extends the theory of divisors to schemes S having property G_1 and S_2. We always assume arithmetically Cohen-Macaulay, so in particular S_2, but we now examine the G_1 condition.

Definition 3.3. A noetherian ring A (resp. a noetherian scheme X) satisfies the condition G_r, "Gorenstein in codimension $\leq r$" if every localization $A_\mathfrak{p}$ (resp. every local ring $\mathcal{O}_x$) of dimension $\leq r$ is a Gorenstein local ring. □

Remark 3.4. Definition 3.3 means that the non locally Gorenstein locus has codimension greater than r. In particular, G_0 is "generically Gorenstein." See [11] for more details on schemes satisfying the condition G_r. □

What does this notion give us, and what doesn't it give us? We will see, for example, that if S is smooth we can talk about sums of divisors, while if S satisfies only G_1 then we cannot. However, there is much more that we can do with *differences* of divisors, especially in the case where $C \subseteq C'$ and we want to define $C' - C$ as an effective divisor of degree equal to $\deg C' - \deg C$. Here is a simple example.

Example 3.5. Let S_1 be the union of three planes in $\mathbb{P}^3$, all containing a line C. Let S_2 be the union of two planes in $\mathbb{P}^3$ containing C. Both S_1 and S_2 are hypersurfaces, hence satisfy G_1, and C is a divisor on either S_1 or S_2.



S_1 = three planes $\qquad$ S_2 = two planes

What should $C + C$ be? On S_1 there is no hope for a natural, degree-preserving sum. On S_2 one can see that C is self-linked (via the complete intersection (S_2, L)), so there is almost some hope that $C + C$ might make sense; but different choices of L give different complete intersection subschemes, so it does not quite work. *However*, $H_L - C = C$ works unambiguously on S_2, where $H_L - C$ is defined by the ideal $(S_2, L) : I_C$. $\square$

Although the above example shows that in general sums do not make sense on arithmetically Cohen-Macaulay schemes satisfying G_1, we will see shortly with "basic double G-linkage" that in some special cases a sum does exist. We now consider differences. The idea in general is to mimic the preceding example to define $C' - C$. S need not be a complete intersection, or even arithmetically Gorenstein. It is enough that *locally* C' is Gorenstein along every component:

Definition 3.6. Let $S \subset \mathbb{P}^n$ be a subscheme, and let C, C' be divisors on S. Assume that $C \subseteq C'$ as schemes, and that C' satisfies property G_0. Then $C' - C$ is the effective divisor on S, of degree equal to $\deg C' - \deg C$, whose defining ideal as a subscheme of $\mathbb{P}^n$ is $I_{C'} : I_C$. In particular, suppose S is arithmetically Cohen-Macaulay and satisfies property G_1. Let $C \subset S$ be a divisor and let $F \in I_C$ be a homogeneous polynomial such that $I_S : F = I_S$. Then $H_F - C$ is the effective divisor defined by $I_{H_F} = (I_S + (F)) : I_C$. $\square$

Example 3.7. We illustrate why we require S to satisfy property G_1 in the last part of Definition 3.6. Let S be the union of three lines in $\mathbb{P}^3$ meeting at a point, P. S satisfies G_0 but not G_1. Let L be a general linear form vanishing at P. Then the ideal $I_S + (L)$ is the square of the ideal of P in $\mathbb{P}^2$, i.e. H_L is a zeroscheme in $\mathbb{P}^2$ of degree 3 supported at P. On the other hand, $[I_S + (L)] : I_P$ gives back I_P. Hence $H_L - P = P$, and the degree is not preserved as we would like. $\square$

We now define certain linear systems on arithmetically Cohen-Macaulay schemes satisfying G_1. This notion agrees with that of Hartshorne [11]. It also suggests that there is sometimes a chance for sums of divisors, and we will make this precise shortly.

Definition 3.8. Let S be arithmetically Cohen-Macaulay satisfying G_1, and let $C, Y \subset S$ be divisors. Then

(i) Y is in the linear system $|C + tH|$ if there exists a divisor D and homogeneous polynomials $F \in I_C$ and $G \in I_Y$ with $\deg G = \deg F + t$, such that $H_F - C = D = H_G - Y$.

(ii) If $t = 0$ we say that C and Y are *linearly equivalent*. $\square$

Remark 3.9. Part (i) says $(I_S + (F)) : I_C = I_D = (I_S + (G)) : I_Y$. $\square$

As we saw above, in general the sum of two divisors does not make sense. We now give one special case where it does, the so-called "basic double linkage." We give the construction from the literature and a new generalization from [12]. The reason for this terminology will be explained in the next section.

Definition 3.10. Given $C \subset \mathbb{P}^n$ of codimension r, choose $F_2, ..., F_r \in I_C$ forming a regular sequence, and consider the ideal

$$I_{C'} = F \cdot I_C + (F_2, ..., F_r)$$

where $F \in R$ is homogeneous and $(F, F_2, \dots, F_r)$ is also a regular sequence. $I_{C'}$ is a saturated ideal, defining a scheme C' which is called a *basic double CI-link* of C (originally just "basic double link"). □

Remark 3.11. Basic double CI-linkage, originally called simply "basic double linkage," was introduced in the paper [16] in the context of curves in $\mathbb{P}^3$. Those authors pointed out that it is a special case of "liaison addition" [25]. Its main use is as a key component of the structure theorem for even liaison classes introduced in [16] and generalized in [3], [17], [5], [22] and [21]. All of these papers used it in the context of codimension two. In the generality of the above definition it was introduced in [4] and put into the context of "generalized liaison addition" in [10]. □

As sets, it can be shown (see [10]) that C' is simply the union of C and the complete intersection, X, defined by $I_X = (F, F_2, ..., F_r)$. Furthermore, $\deg C' = \deg C + \deg X$. This motivates the following **Geometric Interpretation of Basic Double CI-linkage:**

(3.1) *Let $C \subset \mathbb{P}^n$ have codimension $r \geq 2$. Choose a complete intersection S of codimension $r-1$ containing C, and view C as a divisor on S. Then a basic double CI-link is just a divisor of the form $C' = C + H_F$, where $I_S : F = I_S$. The saturated ideal is given by $I_{C'} = F \cdot I_C + I_S$, and we have* $\deg C' = \deg C + \deg F \cdot \deg S$.

Note that H_F is simply the complete intersection X mentioned above. We now give a generalization, which we will return to in the next section and explain the name.

Lemma 3.12. (Basic Double G-linkage) *Let $C \subset \mathbb{P}^n$ be equidimensional (i.e. I_C is unmixed) and have codimension $r \geq 2$. Choose a scheme $S \subset \mathbb{P}^n$ such that S is arithmetically Cohen-Macaulay of codimension $r-1$ and $I_S \subset I_C$. Let $F \in I_C$ be such that $I_S : F = I_S$. Then the ideal $F \cdot I_C + I_S$ is saturated and unmixed, defining a projective subscheme on S which we will write as the divisor $C' = C + H_F$. As a subscheme of $\mathbb{P}^n$, C' has codimension r and satisfies* $\deg C' = \deg C + \deg F \cdot \deg S$.

Proof. The proof follows primarily from the exact sequence

$$0 \to I_S(-d) \to I_S \oplus I_C(-d) \to I_S + F \cdot I_C \to 0$$

where the first map is given by $A \mapsto (AF, F)$ and the second map is given by $(A, B) \mapsto A - FB$. The details can be found in [12], Lemma 4.8. □

Remark 3.13. In Section 6 we will be interested in divisors in the linear system $|C - H|$, where C is effective and H is a hyperplane section. In that section we will assume that S is smooth, since that is all that we will need. Here we comment on the more general situation. In view of Definition 3.6, one might expect that C would have to satisfy at least property G_0 in order for $C - H$ to make sense. However, it seems that this is not the case. More generally, assume that

- S is arithmetically Cohen-Macaulay (no further assumption),
- C is a divisor on S (no further assumption), and
- there exists a hypersurface section H_F of S such that $H_F \subset C$ as schemes.

Then $C - H_F$ seems to make sense in a degree-preserving way. Since the proof is incomplete, we state it as a conjecture:

If F is a homogeneous polynomial defining the hypersurface section H_F, then $C - H_F$ is the scheme with defining ideal $I_C : (I_S + (F))$, and $\deg(C - H_F) = \deg C - (\deg S)(\deg F)$. In fact, C is a basic double G-link of $C - H_F$. That is,

$$I_C = F \cdot [I_C : (I_S + (F))] + I_S.$$

The idea is to use Lemma 3.12. In view of Example 3.7, it is surprising that C can fail to satisfy G_0, and yet the very fact that it lies on S and contains H_F allows $C - H_F$ to make sense. If the conjecture is correct, and if S satisfies at least G_0, then $C - H_F$ is G-bilinked to C, by Proposition 4.6 below. □

4. Gorenstein ideals and liaison

In this section we develop the theory of Gorenstein liaison via divisors. We first give a construction, introduced in [12], which allows us to construct (arithmetically) Gorenstein divisors on suitable arithmetically Cohen-Macaulay schemes. We state it first in algebraic language, in which case it was essentially known.

Lemma 4.1 ([6]). *Let $S \subset \mathbb{P}^n$ be an arithmetically Cohen-Macaulay subscheme satisfying G_0. Then there is a homogeneous Gorenstein ideal J of R, with $\operatorname{codim} J = \operatorname{codim} S + 1$, such that J contains I_S and $K_S \cong J/I_S(t)$ for some $t \in \mathbb{Z}$.*

Again, we have a geometric interpretation:

Corollary 4.2. *Let $S \subset \mathbb{P}^n$ be an arithmetically Cohen-Macaulay subscheme satisfying G_1. Let K be an effective subcanonical divisor on S. Note that*

K is an effective subcanonical divisor	$\Leftrightarrow$	*K is the scheme associated to a global section of $\omega_S(t)$ for some t*
	$\Leftrightarrow$	*K is an element of the linear system $\lvert\bar{K} + tH\rvert$, where $\bar{K}$ is a canonical divisor.*

Let $F \in (I_K)_d$ such that $I_S : F = I_S$. Then the effective divisor $H_F - K$, viewed as a subscheme of $\mathbb{P}^n$, is arithmetically Gorenstein. In fact, any divisor in the linear system $|H_F - K|$ is arithmetically Gorenstein.

Proof (sketch). Let $Y \in |H_F - K|$ and let $d = \deg F$. Then we have

$$\mathcal{I}_{Y|S}(d) \cong \mathcal{O}_S(dH - Y) \cong \mathcal{O}_S(K) \cong \omega_S(t)$$

(cf. [11]). This leads to the following exact sequence, after taking cohomology:

$$0 \to I_S \to I_Y \to H^0_*(\omega_S)(t - d) \to 0. \tag{4.1}$$

Since S is arithmetically Cohen-Macaulay, the minimal free resolution of R/I_S is dual to that of the canonical module $H^0_*(\omega_S)$ (up to twist). Then an application of the Horseshoe Lemma ([26] 2.2.8, p. 37) gives the result. □

Note that the exact sequence (4.1) gives the connection between Corollary 4.2 and Lemma 4.1.

Remark 4.3. It is rather amazing that *any* element of this linear system is arithmetically Gorenstein. The graded Betti numbers are upper-semicontinuous, and there are examples of linear systems of the form $|H_F - K|$ where the other graded Betti numbers jump, but not the last one!! It is also possible to have linear systems (not of the form $|H_F - K|$!) where the general element is arithmetically Gorenstein but special elements are not, i.e. this property is special to this kind of linear system, not to arithmetically Gorenstein divisors in general. See [12] Example 5.8. □

With this preparation, we will now see how CI-liaison can be viewed as a theory of divisors on complete intersections, and how it has been generalized to G-liaison in [12]. The idea (from [11]) is as follows. If S is a complete intersection then so is H_F. Let $C \subset \mathbb{P}^n$ be a subscheme and let S be any complete intersection containing C as a divisor. Let $F \in I_C$ such that $I_S : F = I_S$. Then the divisor $H_F - C$ on S, viewed as a subscheme of $\mathbb{P}^n$, is the scheme residual to C in the complete intersection H_F. In this way Hartshorne re-derives many of the basic facts about CI-liaison as a theory of divisors. In section 5 we will see that this sort of consideration can be used to generalize some other standard results from the codimension two case. For now we give a more elementary result, which is an immediate consequence of Definition 3.8.

Proposition 4.4. *Let S be a complete intersection and let C and Y be divisors on S. If $Y \in |C + tH|$ then Y is CI-bilinked to C.*

Remark 4.5. As a result of the last result, it follows from (3.1), or from a direct computation using Definition 3.10, that if C' is a basic double CI-link of C then C' is CI-bilinked to C. This was the reason that Lazarsfeld and Rao [16] originally chose the name "basic double linkage." □

We now give the connection between basic double G-linkage (Lemma 3.12) and G-liaison.

Proposition 4.6. *Let C, S, F and C' be as in Lemma 3.12; i.e. assume that*

(i) *$C \subset \mathbb{P}^n$ is equidimensional (i.e. I_C is unmixed) of codimension $r \geq 2$;*
(ii) *$S \subset \mathbb{P}^n$ is arithmetically Cohen-Macaulay of codimension $r-1$ with $I_S \subset I_C$;*
(iii) *$F \in I_C$ is such that $I_S : F = I_S$;*
(iv) *$I_{C'} = F \cdot I_C + I_S$.*

Assume further that S satisfies property G_0. Then C' is G-bilinked to C.

Proof. In the paper [12] the links are given explicitly, and we refer the reader to that paper for the details. We stress that S is assumed to satisfy property G_0, which is stronger than what was needed for Lemma 3.12, but weaker than the condition G_1 (or more) which we need for many of our results. □

We can extend Proposition 4.6, by changing the hypothesis G_0 to G_1 and allowing linear equivalence. We first state the result in algebraic language:

Proposition 4.7. *Let $S \subset \mathbb{P}^n$ be an arithmetically Cohen-Macaulay subscheme of codimension c satisfying property G_1. Let C, Y be generalized divisors on S such that there exist homogeneous polynomials $F \in I_C$ and $G \in I_Y$ with $I_S : F = I_S$, $I_S : G = I_S$ and $(I_S + (F)) : I_C = (I_S + (G)) : I_Y = I_D$ (say). Then C and Y are G-bilinked.*

The geometric version of this result is as follows:

Corollary 4.8. *Let $C, Y \subset S$ be divisors, where S is arithmetically Cohen-Macaulay satisfying G_1. If $Y \in |C + tH|$ then C and Y are G-bilinked.*

The analogous statement for CI-liaison, Proposition 4.4, was very easy because in that case the ideals of the form $I_S + (F)$ are complete intersections and the ideal quotients are links. In our current situation, if S is not Gorenstein then neither is $I_S + (F)$ in general, so there is something to prove here. The ideal quotients in Proposition 4.7 are *not* links.

We first give a simple proof of Corollary 4.8 for the case where S is smooth, and we will then see how the proof needs to be modified to allow smoothness to be weakened to G_1.

Proof of Corollary 4.8 if S is smooth. Let $Y \in |C + tH|$ and let K be subcanonical. Let $F \in I_K$ not vanishing on any component of S (i.e. $I_S : F = I_S$). By Definition 3.8 there exist forms B and B' with $\deg B' - \deg B = t$, and an effective divisor D, such that $H_B - C = D = H_{B'} - Y$. By Corollary 4.2, $H_{FB} - K$ and $H_{FB'} - K$ are Gorenstein, and clearly they contain C and C', respectively, as subschemes.

In fact, $H_{FB} - K - C = (H_F - K) + (H_B - C) = (H_F - K) + D$. This says that C is G-linked to D with a Gorenstein "tail" attached. Similarly, $H_{FB'} - K - Y = (H_F - K) + D$. This says that C' is G-linked to the same residual. Therefore C and Y are both G-linked to $(H_F - K) + D$, as a subscheme of $\mathbb{P}^n$. □

What happens in the non-smooth case? In the proof above, we were forced to use sums of divisors. We have to be more careful in the more general case.

Proof of Proposition 4.7 (sketch).

Step I: One checks that $\deg D = \deg F \cdot \deg S - \deg C = \deg G \cdot \deg S - \deg Y$.

Step II: Consider a minimal free resolution of I_S:

$$0 \to \mathbb{F}_c \xrightarrow{A} \mathbb{F}_{c-1} \to \cdots \to I_S \to 0.$$

Suppose A is a $(t + r) \times t$ matrix. A^t is a presentation matrix for $K_S(n + 1)$. Let B be a sufficiently general $t \times 1$ matrix such that the concatenation A' of A^t and B is homogeneous. Let I be the ideal of the annihilator of the module $M_{A'}$, where $M_{A'}$ is the cokernel of the map represented by A'. Then the scheme K defined by I is a subcanonical divisor of S.

Step III: The submaximal minors of A define the non locally Gorenstein locus of S. Using this fact, the G_1 property and Step II, one shows that a sufficiently general choice of B in Step II in fact gives a subcanonical divisor K which has no component in common with D.

Step IV: The ideal I of Step II is not necessarily saturated. Let I_K be the saturation. Let $A \in I_K$ be such that $I_S : A = I_S$ and $I_D : A = I_D$. The ideal $I_X := (I_S + (A)) : I_K$ is Gorenstein with no component in common with D, and so

$$\deg[I_D \cap I_X] = \deg D + \deg X.$$

Step V: One checks that $I_S + F \cdot I_X$ and $I_S + G \cdot I_X$ are Gorenstein of degrees

$$\begin{gathered}\deg F \cdot \deg S + \deg X \\ \deg G \cdot \deg S + \deg X\end{gathered}$$

respectively.

Step VI: Let

$$\begin{gathered}\mathfrak{a} = [I_S + F \cdot I_X] : I_C \\ \mathfrak{b} = [I_S + G \cdot I_X] : I_Y\end{gathered}$$

These are G-links. Show both of these are contained in $I_D \cap I_X$, and compute that they have the same degree. All ideals in this proof are unmixed. Therefore $C \sim D \cup X \sim Y$ and we are done. □

Remark 4.9.

(i) Note that the scheme X above plays the role of $H_F - K$ in the geometric proof.

(ii) One may ask why Proposition 4.7 required S to satisfy property G_1, while Proposition 4.6 required only G_0. The main point, of course, is that Proposition 4.7 requires us to let the divisors move in a linear system, while Proposition 4.6 refers to a specific divisor whose linkage and degree properties can be checked directly. In particular, the divisor $H_F - C$, with ideal $[I_S + (F)] : I_C$, makes sense (e.g. you get the degree you expect) for G_1 and arithmetically Cohen-Macaulay, but not G_0. See also Example 3.7.

(iii) If S is a complete intersection, one can check quickly that the links given in Proposition 4.7 are CI-links, so the result coincides with Proposition 4.4.

(iv) If CI-liaison is a theory about divisors on complete intersections, one might expect G-liaison to be a theory about divisors on arithmetically Gorenstein schemes. It is interesting that the "right" generalization is to arithmetically Cohen-Macaulay schemes satisfying G_1.

(v) If S is not a complete intersection, usually V and Y are not in the same CI-liaison class. See the results at the end of Section 6 for further details. This is a very important difference between CI-liaison and G-liaison. □

Proposition 4.7 has the following immediate corollary.

Corollary 4.10. *For divisors on an arithmetically Cohen-Macaulay subscheme, S, of $\mathbb{P}^n$ satisfying G_1, the properties of being arithmetically Cohen-Macaulay or arithmetically Buchsbaum are preserved under linear equivalence and adding hyperplane sections of S.*

Example 4.11. It is also natural to ask if the arithmetically Cohen-Macaulay condition can be removed in Proposition 4.7. In fact, the projection of the Veronese in $\mathbb{P}^5$ to $\mathbb{P}^4$, and the linear system $|\mathcal{O}_S(1)|$ provides a counterexample. The general element of the linear system can be shown to be arithmetically Cohen-Macaulay, while the particular elements which are actually hyperplane sections are arithmetically Buchsbaum but not arithmetically Cohen-Macaulay. See [12] for details. □

5. Generalizing Two Standard Codimension Two Results

In this section we discuss two well-known results in codimension two which have been generalized in [12]. The first is often referred to as Gaeta's theorem, because of the paper [9]. However, a less complete result along the same direction was given by Apery [1],

[2] and a modern proof was given by Peskine and Szpiro [23]. The result says that in codimension two, a subscheme of $\mathbb{P}^n$ is licci if and only if it is arithmetically Cohen-Macaulay. However, we will phrase it slightly differently so that our generalization can be better understood. The second result that we will generalize is due to Rao, and it says that under very reasonable conditions, algebraic liaison and geometric liaison form the same equivalence relation in codimension two.

5.1. ***Gaeta's theorem.*** We begin with some background.

Definition 5.1. A codimension $c+1$ subscheme, V, of $\mathbb{P}^n$ is *standard determinantal* if I_V is the ideal of maximal minors of a homogeneous $t \times (t+c)$ matrix. ("Standard" means the ideal of maximal minors has the expected codimension.) Note that a very special case is when V is a complete intersection ($t = 1$). □

We note that if V is standard determinantal then the minimal free resolution for I_V is given by the Eagon-Northcott complex. In particular, V is arithmetically Cohen-Macaulay. What is particularly important here is that in codimension two, the converse holds: If V is arithmetically Cohen-Macaulay of codimension two then it is standard determinantal. In fact, if I_V has a minimal free resolution

$$0 \to \mathbb{F}_1 \xrightarrow{A} \mathbb{F}_0 \to I_V \to 0$$

then I_V is the ideal of maximal minors of A, which is a $(t+1) \times t$ matrix. This is the Hilbert-Burch theorem (cf. [8]), and A is often called the *Hilbert-Burch matrix* of V.

Returning to liaison, it is clear that if V is licci then it is arithmetically Cohen-Macaulay, and hence if we also have $\operatorname{codim} V = 2$, then V is standard determinantal. The real content of Gaeta's theorem, then, is the converse:

Theorem 5.2 (Gaeta, Apery, Peskine-Szpiro). *In codimension two, any standard determinantal subscheme V of $\mathbb{P}^n$ is licci.*

Proof (sketch). Let V be standard determinantal of codimension two, so I_V is the ideal of maximal minors of some $t \times (t+1)$ homogeneous matrix, A. Link $V \overset{X}{\sim} W$ using as the generators of I_X two minimal generators of I_V. Of course W is again standard determinantal, since it is arithmetically Cohen-Macaulay of codimension two. What Gaeta showed is that the matrix, A', defining I_W is obtained from A by deleting two columns and transposing:

$$A = \begin{bmatrix} \times & \times & \times & \times & \times & \square & \square \\ \times & \times & \times & \times & \times & \square & \square \\ \times & \times & \times & \times & \times & \square & \square \\ \times & \times & \times & \times & \times & \square & \square \\ \times & \times & \times & \times & \times & \square & \square \\ \times & \times & \times & \times & \times & \square & \square \end{bmatrix} \qquad A' = \begin{bmatrix} \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \end{bmatrix}$$

(In this case, we can write $I_X = (F, G)$ where F is the determinant of the matrix obtained by removing the last column and G is the determinant of the matrix obtained by removing the penultimate column.) Continuing in this manner, one reaches in a finite number of steps a 1×2 matrix, i.e. a complete intersection. □

Example 5.3. In codimension two, let $I = (F, G)$ be a complete intersection. Then I^k is again arithmetically Cohen-Macaulay, given by the maximal minors of

$$k\left\{\underbrace{\begin{bmatrix} F & G & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & F & G & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & F & G & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & F & G \end{bmatrix}}_{k+1}\right.$$

One can check that the complete intersection (F^k, G^k) CI-links I^k to I^{k-1} by removing the first and last columns and transposing along the minor diagonal. $\square$

Theorem 5.2 (and its converse as well) is certainly false in higher codimension. For instance, a general set, Z, of four points in $\mathbb{P}^3$ can be shown to be standard determinantal, but we saw on page 4 that Z is not licci. On the other hand, we also saw that Z is glicci, since the addition of a general fifth point forms a Gorenstein zeroscheme. It was a very pleasant surprise to discover that this phenomenon is true in a much broader setting:

Theorem 5.4. *In arbitrary codimension, any standard determinantal subscheme, V, of $\mathbb{P}^n$ is glicci.*

Proof (sketch). This is one of the main results of [12]. The proof is very technical, but we will describe the main ideas.

We use the following notation: given a homogeneous matrix A, we denote by $I(A)$ the ideal of maximal minors of A. Assume that $\operatorname{codim} V = c+1$ and that $I_V = I(A)$ for some homogeneous $t \times (t+c)$ matrix A. We make the following definitions:

- Let B be the matrix obtained by deleting a *suitable* column of A. ("Suitable" means that $\operatorname{codim} I(B) = c$. First take general linear combinations of the rows and columns, if necessary.) Define S by $I_S = I(B)$.
- Let A' be the matrix obtained by deleting a *suitable* row of B. ("Suitable" means that $\operatorname{codim} I(A') = c+1$. First take general linear combinations of the rows and columns, if necessary.) Define V' by $I_{V'} = I(A')$.

We have the following picture:

$$A \ = \ \underbrace{\begin{bmatrix} \times & \times & \times & \times & \times & \times & \times & \square \\ \times & \times & \times & \times & \times & \times & \times & \square \\ \times & \times & \times & \times & \times & \times & \times & \square \\ \times & \times & \times & \times & \times & \times & \times & \square \\ \circ & \circ & \circ & \circ & \circ & \circ & \circ & \square \end{bmatrix}}_{t+c} \left.\vphantom{\begin{bmatrix}a\\a\\a\\a\\a\end{bmatrix}}\right\} t \qquad I(A) = I_V, \ \ \operatorname{codim} V = c+1$$

$$B \ = \ \underbrace{\begin{bmatrix} \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times \\ \circ & \circ & \circ & \circ & \circ & \circ & \circ \end{bmatrix}}_{t+c-1} \left.\vphantom{\begin{bmatrix}a\\a\\a\\a\\a\end{bmatrix}}\right\} t \qquad I(B) = I_S, \ \ \operatorname{codim} S = c$$

$$A' = \underbrace{\begin{bmatrix} \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times \end{bmatrix}}_{t+c-1} \left.\vphantom{\begin{bmatrix} \times \\ \times \\ \times \\ \times \end{bmatrix}}\right\} t-1 \qquad I(A') = I_{V'}, \quad \operatorname{codim} V' = c+1.$$

The goal will be to show that one can go from $I(A)$ to $I(A')$ using two G-links. Hence in $2t-2$ steps one arrives at a complete intersection.

Here is an intuitive (geometric) idea of what is going on:

> View V, V' as divisors on S. S does not necessarily satisfy G_1, so we have to be very careful. In any case, essentially we have that $V' \in |V + tH|$ for some t. Then Corollary 4.8 suggests that V and V' are G-bilinked.

Note that again, the liaison result is a result about divisors. Of course the failure to guarantee that S satisfies G_1 means that we have to work a little harder. In fact, we show that V is G-bilinked to V' by explicitly giving the links. We need some more notation.

- Let A_1 be the submatrix of A consisting of the first $t-1$ columns.
- Let D be the determinant of the matrix consisting of the first $t-1$ and the last column of A.

We have the following picture:

$$A = \left[\begin{matrix} \times & \times & \times & \times & \times & \times & \times & \square \\ \times & \times & \times & \times & \times & \times & \times & \square \\ \times & \times & \times & \times & \times & \times & \times & \square \\ \times & \times & \times & \times & \times & \times & \times & \square \\ \underbrace{\circ \quad \circ \quad \circ \quad \circ}_{\substack{A_1 \\ \| \\ t \times (t-1)}} & & & & \circ & \circ & \circ & \square \end{matrix}\right] \qquad D = \left|\begin{matrix} \times & \times & \times & \times & \square \\ \times & \times & \times & \times & \square \\ \times & \times & \times & \times & \square \\ \times & \times & \times & \times & \square \\ \underbrace{\circ \quad \circ \quad \circ \quad \circ}_{A_1} & & & & \square \end{matrix}\right|$$

The main steps of the proof are as follows:

(i) Let $J = I(A_1)$. For $i = 1, \ldots, c-1$ we have that $I_S + J^i$ is perfect of codimension $c+1$.

(ii) $I_S + J^{c-1}$ is Gorenstein.

(iii) $I_S + D \cdot J^{c-1}$ is also Gorenstein of codimension $c+1$, contained in I_V.

(iv) For $i = 0, \ldots, c$ we have

$$\deg(I_S + J^i) = i \cdot [\deg D \cdot \deg S - \deg V].$$

(v) Now we G-link I_V by $I_S + D \cdot J^{c-1}$. The residual is $I_S + J^c$. (One shows one inclusion and computes degrees.)

(vi) Let D' be the determinant of the matrix consisting of the first $t-1$ columns of A'. Repeat the steps using $I_{V'}$ instead of I_V and D' instead of D, and show that the residual is again $I_S + J^c$. □

Example 5.5. We give the analog to Example 5.3 for G-liaison in arbitrary codimension. In codimension $c+1$, let $I = (F_1, \ldots, F_{c+1})$ be a complete intersection. I^k is again standard determinantal, being the ideal of maximal minors of the matrix

$$
k\left\{\begin{bmatrix}
F_1 & F_2 & F_3 & \cdots & F_{c+1} & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & F_1 & F_2 & \cdots & F_c & F_{c+1} & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & F_1 & F_2 & F_3 & \cdots & F_c & F_{c+1} & 0 \\
0 & 0 & 0 & \cdots & 0 & F_1 & F_2 & \cdots & F_{c-1} & F_c & F_{c+1}
\end{bmatrix}\right.
$$

$$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}_{k+c}$$

Then I^k is G-bilinked to I^{k-1}: simply remove the last row and column. (Note that in Example 5.3 the powers were directly linked, while here they are bilinked.) This is false for CI-links. For example, consider $I = (x, y, z)$, the ideal of a single point in $\mathbb{P}^3$. Then I and I^2 are not in the same CI-liaison class. □

Remark 5.6. As mentioned above, the way that Gaeta's theorem is usually stated is that if V has codimension two, then V is arithmetically Cohen-Macaulay if and only if V is licci. In higher codimension, we know that

$$\text{standard determinantal} \Rightarrow \text{glicci} \Rightarrow \text{arithmetically Cohen-Macaulay.}$$

The first converse is certainly false. The second is almost certainly false, except possibly in codimension 3. See also Remark 7.7. □

5.2. ***Rao's theorem on algebraic and geometric liaison.*** Throughout this subsection, what we called *CI-links* in Definition 2.2 will be referred to as *algebraic CI-links.* More precisely, we have

Definition 5.7.

(i) Two schemes V_1 and V_2 are *algebraically CI-linked* by a complete intersection X, denoted $V_1 \overset{X}{\sim} V_2$, if

$$I_X \subset I_{V_1} \cap I_{V_2}, \quad I_X : I_{V_1} = I_{V_2} \quad \text{and} \quad I_X : I_{V_2} = I_{V_1}.$$

(ii) Assume that $V_1 \overset{X}{\sim} V_2$, where X is a complete intersection, and assume that V_1 and V_2 have no common component. Then $V_1 \cup V_2 = X$ as schemes, and we say that V_1 and V_2 are *geometrically CI-linked.* □

Notice that if V_1 and V_2 are geometrically CI-linked then both are generic complete intersections, since they have no component in common and their union is a complete intersection. This leads to the following:

Question 5.8. *Among generic complete intersections of codimension c, do geometric CI-links and algebraic CI-links generate the same equivalence relation?*

Unfortunately, this question is not well posed. The reason for this is that the expression "among generic complete intersections" is not well-defined. More precisely, let V_1 and V_2 be generic complete intersections. Suppose

$$V_1 \sim W_1 \sim \cdots \sim W_k \sim V_2$$

are algebraic links. We want to know if there exist *geometric* links

$$V_1 \sim Y_1 \sim \cdots \sim Y_\ell \sim V_2.$$

The heart of the matter is the following:

Problem: *Do we require the W_i to all be generic complete intersections?*

Conceivably there could be fewer equivalence classes if we do not make this restriction. To give the known answers to Question 5.8, we introduce the following notation, adapted from [24].

Definition 5.9.

(i) $H(c,n)$ is the set of subschemes of $\mathbb{P}^n$ which are of pure codimension c (i.e. the defining ideal is unmixed) and generic complete intersections.

(ii) $\tilde{H}(c,n)$ is the set of elements of $H(c,n)$ which are in addition locally Cohen-Macaulay.

Remark 5.10. Note that $H(n-1,n) = \tilde{H}(n-1,n)$ (i.e. for curves they are the same). Also, note that the property of being locally Cohen-Macaulay is preserved under liaison. □

There are no known situations where Question 5.8 has a negative answer. We now give the known affirmative answers to the question. First we have a result of Rao [24], who assumed codimension two, locally Cohen-Macaulay and all W_i are generic complete intersections:

Theorem 5.11 ([24]). *If $c = 2$ and if all W_i are generic complete intersections then algebraic and geometric CI-liaison generate the same equivalence relation on $\tilde{H}(2,n)$*

Next, Schwartau [25] removed the assumption that the W_i are all generic complete intersections from Rao's theorem, but only for curves in $\mathbb{P}^3$ (and so, by virtue of Remark 5.10, he also did not need to worry about the locally Cohen-Macaulay assumption).

Theorem 5.12 ([25]). *If $c = 2$ and $n = 3$ and we do* not *assume that the W_i are generic complete intersections, then algebraic and geometric CI-liaison generate the same equivalence relation on $H(2,3) = \tilde{H}(2,3)$.*

Theorem 5.11 was generalized in [12] to arbitrary codimension, and the assumption of locally Cohen-Macaulay was removed:

Theorem 5.13. *For any c, n, if all W_i are generic complete intersections then algebraic and geometric CI-liaison generate the same equivalence relations on $H(c,n)$.*

Theorem 5.11 and Theorem 5.12 were also generalized in [12] by passing to codimension two in any $\mathbb{P}^n$ and removing the locally Cohen-Macaulay assumption:

Theorem 5.14. *If $c = 2$ and if we do* not *assume that the W_i are generic complete intersections, then algebraic and geometric CI-liaison generate the same equivalence relation on $H(2,n)$.*

Remark 5.15.

(i) The analog of the above results for Gorenstein liaison is open. Similarly, the analog of Theorem 5.14 for CI-liaison in higher codimension is open.

(ii) Theorem 5.13, for arbitrary codimension, is largely based on Hartshorne's approach with divisors. If $V_1 \overset{X}{\sim} V_2$ where $I_X = (F_1, \ldots, F_c)$, we let S be the complete intersection defined by $I_S = (F_2, \ldots, F_c)$ and view everything as divisors on S.

(iii) Theorem 5.14, for codimension 2, uses ideas of Rao, Nagel and Nollet to reduce to the case where each step in the sequence of links is a generic complete intersection, and then we apply the first theorem. □

6. Curves on a Smooth arithmetically Cohen-Macaulay Surface

In this section we continue the theme of studying liaison via divisors, and we focus on the case of curves on a smooth arithmetically Cohen-Macaulay surface. Recall first the following two general facts.

1. (Corollary 4.8) The even G-liaison class of a divisor C on an arithmetically Cohen-Macaulay subscheme S satisfying G_1 is preserved under linear equivalence and adding hyperplane sections. Hence any element of the linear system $|C+tH|$ is in the same even G-liaison class as C, where H is the hyperplane section divisor. Note that t can be negative, as long as S is smooth; see Remark 3.13.
2. (Proposition 2.3) The deficiency modules are preserved under even G-liaison (up to shift).

Fact 1, in particular, motivates the following definition.

Definition 6.1. Let S be a smooth subscheme of $\mathbb{P}^n$. We will say that an effective divisor C on S is *minimal* if there is no effective divisor in the linear system $|C-H|$. □

Proposition 6.2. *Let S be a smooth, arithmetically Cohen-Macaulay surface in $\mathbb{P}^n$ and let M be a graded R-module of finite length. Then there exist only finitely many G-liaison classes of curves $C \subset S$ with deficiency module $M(C) \cong M$ (up to twist).*

Proof (sketch). One can check that $\operatorname{Pic} S$ is finitely generated, since $H^1(\mathcal{O}_S) = 0$. Say $\operatorname{Pic} S = \langle L_1, \dots, L_r\rangle$. We make the following definitions:

- Let $H \in \operatorname{Pic} S$ be the class of the hyperplane sections, and write $H = \sum_{i=1}^r h_i L_i$;
- Let $K \in \operatorname{Pic} S$ be the class of canonical divisors, and write$K = \sum_{i=1}^r k_i L_i$;
- Let $C \in \operatorname{Pic} S$ be the class of a minimal curve on S with $M(C) = M$ (up to shift), and write $C = \nu H + \sum_{i=1}^r c_i L_i$;
- Let $n_t = \dim M(C)_t$.

The idea of the proof is to show that there exist upper and lower bounds for c_i, $i = 1, \dots, r$, in terms of the h_i, the k_i, the n_t and various invariants of the L_i. □

One of the original motivating questions of [12] was whether something special happens with G-liaison in codimension three, analogous to what happens in codimension two with complete intersections. For instance, we wondered if it is true that in codimension three, any arithmetically Cohen-Macaulay subscheme of $\mathbb{P}^n$ is glicci (see also Remark 5.6 and Question 4.2). The following result is a partial answer to this question.

Theorem 6.3. *Let S be a smooth, rational arithmetically Cohen-Macaulay surface in $\mathbb{P}^4$. Then all arithmetically Cohen-Macaulay curves on S are glicci.*

Proof (sketch). We first classify the smooth rational arithmetically Cohen-Macaulay surfaces in $\mathbb{P}^4$; there are four of them: the cubic scroll, the Del Pezzo surface, the Castelnuovo surface and the Bordiga surface. For each smooth rational surface S, we classify the minimal arithmetically Cohen-Macaulay curves on S, since we completely know $\operatorname{Pic} S$. Finally, we show that each minimal arithmetically Cohen-Macaulay curve is glicci by direct examination. □

In contrast, in the case of CI-liaison we have a quite different story. A number of quite deep results are given in [12], both to describe some invariants of CI-liaison and to apply them to very general situations. Rather than describe these results in detail, we give a simple example of their application.

Example 6.4. Let S be the Castelnuovo surface in $\mathbb{P}^4$, realized as the blow-up of $\mathbb{P}^2$ at 8 general points. Note that $\operatorname{Pic} S \cong \mathbb{Z}^{\oplus 9}$, $H = (4; 2, 1^7)$ (i.e. S is embedded via the linear system of quartics double at one of the points and passing through the remaining 7 points) and $-K = (3; 1^8)$. Let C be a curve in the linear system $(1; 1, 0^7)$ (note that C is a conic). Let $C_v \in |C+vH|$. Then C_v and $C_{v'}$ are not licci and belong to different CI-liaison classes provided $v > v' \geq 2$. More generally, if $C \subset S$ is arithmetically Cohen-Macaulay then so is $C + H$, but "usually" they are in different CI-liaison classes. □

The idea behind this example is to study certain invariants of CI-liaison, and show that for different values of v the invariants change. Some of the invariants are described as follows, but again we refer the reader to [12] for the details.

Theorem 6.5. *Let $X, X' \subset \mathbb{P}^N$ be arithmetically Cohen-Macaulay schemes algebraically linked by a complete intersection $Y \subset \mathbb{P}^N$ of dimension n. If $n > 0$ then we have an R-module isomorphism*

$$H^i_*(\mathcal{N}_X) \cong H^i_*(\mathcal{N}_{X'}) \quad \text{for} \quad 1 \leq i \leq n-1$$

where $\mathcal{N}_X = \mathcal{H}om_{\mathcal{O}_\mathbb{P}}(\mathcal{I}_X, \mathcal{O}_X)$ is the normal sheaf.

Theorem 6.6. *Let $X, X' \subset \mathbb{P}^N$ be arithmetically Cohen-Macaulay local complete intersections of dimension n belonging to the same CI-liaison class, let $A = R/I_X$ and $A' = R/I_{X'}$ be the coordinate rings, and let K_A and $K_{A'}$ be the corresponding canonical modules. Then we have an isomorphism of graded R-modules*

$$H^j_\mathfrak{m}(K_A \otimes_R I_X) \cong H^j_\mathfrak{m}(K_{A'} \otimes_R I_{X'})$$

provided $0 < j \leq n$. For $j = 0$ this is an isomorphism as graded k-modules.

7. Comparisons between CI-liaison and G-liaison

In this section we would like to compare and contrast the two theories. In contrast to the preceding material (mostly from [12]), in this section the material is from [18], except as quoted.

7.1. ***Finding "good" links.*** One of the main applications of liaison is to construct subschemes of $\mathbb{P}^n$ with desired properties. To do this, one starts with a known scheme, and tries to find links which yield the desired scheme. A basic problem, then, is the following:

Problem 7.1. *Given a scheme V, how do you find "good" G-links; i.e. how do you find a "good" Gorenstein ideal $I_X \subset I_V$ of the same height? Here "good" often means "small."*

We will discuss some approaches, and illustrate them using the following example.

Example 7.2. Let Z be a set of four points in $\mathbb{P}^3$ in linear general position. We saw on page 4 that Z is G-linked to a single point, hence is glicci. This is the smallest possible G-link for Z. We also saw that Z is not licci. □

Here are some approaches to the above problem.

1. *Complete intersections.* Given V, it is easy to find a regular sequence in I_V, for example on a computer. The problem is that they tend to be too big. For instance, with regard to Example 7.2, the smallest complete intersection containing Z is generated by three quadrics, and hence has degree 8. The residual is again a set of four

points. This reflects the fact that Z is not licci, hence Z has the least degree of any scheme in its CI-liaison class.

2. *Buchsbaum-Rim sheaves of odd rank.* These are certain reflexive sheaves, $\mathcal{B}$, on $\mathbb{P}^n$ of rank $\leq n$. The definition can be found in Example 2.1 (3), and they were studied extensively in the papers [20], [14] and [19]. They are locally free if and only if the rank is $= n$. As mentioned in in Example 2.1 (3), the top dimensional part of a regular section of such a sheaf is arithmetically Gorenstein. Using this, it is easy to get non complete intersection Gorenstein ideals $I_X \subset I_V$ on a computer (cf. [19]), and we even know the resolutions. In general, it is hard to control these sections to get Gorenstein ideals with prescribed properties. Nevertheless, considering again Example 7.2, we can take $\mathcal{B} = \Omega_{\mathbb{P}^3}(3)$ to be the twisted cotangent bundle on $\mathbb{P}^3$, and we can find a section which defines a Gorenstein ideal $I_X \subset I_Z$ of degree 5, as desired (cf. [20]). The problem, in general, is to find the "right" Buchsbaum-Rim sheaf (given V) to take sections of. If you choose a "general" one, usually the sections are still too big. It would be nice to find an algorithm for finding the "right" Buchsbaum-Rim sheaf from the syzygy matrix of V.
3. *A linear systems approach.* This is the approach of [12] described in this paper. We first look for an arithmetically Cohen-Macaulay scheme S satisfying G_1 and containing V as a divisor. Then we look at the linear systems $|V \pm tH|$. With respect to Example 7.2, let S be a twisted cubic curve containing Z. S is smooth and rational, and its canonical divisor has degree -2. From Riemann-Roch and Corollary 4.2, any divisor of degree $3t+2$ is Gorenstein, so in particular we can G-link Z down to one point ($t = 1$). Of course the problem with this approach in general is to find a suitable S.

7.2. ***The behavior of liaison under hyperplane sections.*** An important tool in algebraic geometry is the process of taking hyperplane sections. One can study a scheme V by knowing properties of its general hyperplane section, or one can deduce properties of the hyperplane section from what one knows about V. It is natural to ask how liaison behaves with respect to this process. First, it is well-known that both CI-links and G-links are preserved under hyperplane sections:

Proposition 7.3. *Let $V_1 \overset{X}{\sim} V_2$ in $\mathbb{P}^n$, where $2 \leq \operatorname{codim} X \leq n$ and X is Gorenstein. Let H be a general hyperplane. Then $V_1 \cap H \overset{X \cap H}{\sim} V_2 \cap H$ as subschemes of $\mathbb{P}^n$ (or of H).*

In fact this works equally well for hypersurface sections.

What about the converse? It turns out that CI-links lift, under a very weak hypothesis (e.g. arithmetically Cohen-Macaulay subschemes satisfy it):

Proposition 7.4. *Let $V \subset \mathbb{P}^n$ be a subscheme of codimension ≥ 2 satisfying $H^1_*(\mathcal{I}_V) = 0$. Let H be a general hyperplane. Let $\bar{V} = V \cap H$. Let $\bar{X}$ be a complete intersection in $H = \mathbb{P}^{n-1}$ linking $\bar{V}$ to some subscheme $\bar{W} \subset \mathbb{P}^{n-1}$. Then there exists a complete intersection $X \subset \mathbb{P}^n$ containing V and a subscheme $W \subset \mathbb{P}^n$ such that X links V to W, $\bar{X} = X \cap H$ and $\bar{W} = W \cap H$.*

Remark 7.5. Without the condition $H^1_*(\mathcal{I}_V) = 0$ (or something similar), Proposition 7.4 is false. For example, let V be a set of two skew lines in $\mathbb{P}^3$, so $\bar{V}$ is a set of two points in $\mathbb{P}^2$. Take $\bar{X}$ to be a complete intersection of type $(1,3)$ linking $\bar{V}$ to a single point. Then clearly $\bar{X}$ does not lift to V, since V does not lie on any plane. $\square$

In contrast, it is rather unfortunate that G-links *do not* lift in general. We have the following example.

Example 7.6. Let $V \subset \mathbb{P}^4$ be a rational normal curve and let H be a general hyperplane. Then $V \cap H$ is a set of 4 points in linear general position in $\mathbb{P}^3$. It turns out that "most" arithmetically Gorenstein zeroschemes of degree 5 containing $V \cap H$ (i.e. "most" choices of a fifth point, P) do not lift to a Gorenstein curve C of degree 5 containing V. The reason is that if such a curve C exists, it must be the union of V and a line which is a secant line to V. This line meets H in the point P. Hence P lies on the secant variety to V. But this secant variety is 3-dimensional, so "most" choices of P do not lie on it. $\square$

7.3. ***Extending the codimension two theory.*** One of the inspirations for studying liaison in higher codimension (CI or G) is the beauty and importance of the theory in codimension two, and the hope that those results have analogs in higher codimension. In this subsection we give some remarks about the prospect of extending some of the main results in codimension two.

Remark 7.7. We saw in Proposition 2.3 that we have the invariance of the deficiency modules in any codimension (assuming dimension ≥ 1), for both CI- and G-liaison. In higher codimension we have several invariants for CI-liaison, especially among arithmetically Cohen-Macaulay subschemes. We certainly expect fewer for G-liaison. We have briefly discussed this here, and we refer the reader to [12] for more details.

Rao's theorems give necessary and sufficient conditions for two codimension 2 schemes to be in the same (even) liaison class, in terms of the known invariants (in particular, stable equivalence classes of vector bundles). This is quite open in higher codimension. We need to find more invariants and show that they are sufficient.

It is very likely that G-liaison behaves rather nicely in codimension three. Here is a first step:

> *Is it true that a codimension three subscheme V of $\mathbb{P}^n$ is glicci if and only if it is arithmetically Cohen-Macaulay?*

Note that the hard direction is $\Leftarrow$. We proved it for the case where V is "standard determinantal" instead of arithmetically Cohen-Macaulay (Theorem 5.4), but in any codimension. We also proved it in the special case of curves on a smooth rational surface in $\mathbb{P}^4$ (Theorem 6.3). No counterexample is known in higher codimension, but it seems less likely in codimension ≥ 4. $\square$

Remark 7.8. Another important codimension two result is the Lazarsfeld-Rao (LR) Property. This gives a common structure to all even liaison classes in codimension two (see also Remark 3.11 for some history of this problem). It says that an even liaison class $\mathcal{L}$ always possesses a set of "minimal" elements, and that

(i) The minimal elements are contained in a flat family of subschemes of $\mathbb{P}^n$, hence in particular all have the same degree and arithmetic genus (and in fact Hilbert function). In the locally Cohen-Macaulay case, their deficiency modules have the same shift. Among all elements of $\mathcal{L}$, the minimal elements are characterized as those elements with the smallest degree and arithmetic genus, and in the locally Cohen-Macaulay case they are also characterized by having the shift of the deficiency module in the leftmost possible position.

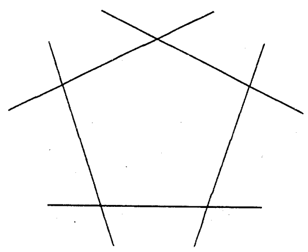
(ii) Any element of $\mathcal{L}$ can be obtained from a minimal one by a sequence of basic double links (cf. Definition 3.10 and the Geometric Interpretation (3.1)) followed by a deformation.

Is there a chance that there is an LR Property in higher codimension? My guess is that ultimately there will be an LR Property for CI-liaison, but not for G-liaison (at least not nearly as strong a property). Here are some thoughts and observations.

First, a key ingredient in the LR Property (for the locally Cohen-Macaulay case) is to partition $\mathcal{L}$ according to the shift of the deficiency modules, and to take the minimal elements to be those corresponding to the leftmost shift. In codimension two, this forces the conclusion that the minimal elements also have the same degree and arithmetic genus, which is minimal among elements of $\mathcal{L}$. This idea of partitioning and considering the leftmost shift still works for both CI-liaison and G-liaison in arbitrary codimension $< n$. For CI-liaison, the question of whether the degrees and arithmetic genera of the elements corresponding to the leftmost shift are uniquely determined and minimal is still open. For G-liaison it is false (see below).

Now, given a subscheme $V \subset \mathbb{P}^n$, the LR Property requires us to be able to apply basic double linkage to V. This is no problem for CI-liaison in arbitrary codimension. We have a version for G-liaison (Lemma 3.12 and Proposition 4.6), but it is not clear if it will be enough.

Perhaps the main reasons for skepticism can be seen in the following example. Let X be the curve in $\mathbb{P}^4$ given in the following picture:



(The lines are chosen generally, subject to the condition that they form the configuration shown.) One checks that X is Gorenstein, so it G-links two skew lines to the disjoint union of a line and a plane curve of degree 2. One also checks that both of these curves have a deficiency module which is 1-dimensional, occurring in degree 0. What should the minimal elements of the corresponding even G-liaison class be? At the least, we will lose the fact that the minimal elements lie in a flat family and hence have the same degree and arithmetic genus. (One could object that these two curves are not evenly linked, but in fact two skew lines are directly linked to a different set of two skew lines.) □

Remark 7.9. Liaison has been very useful in producing examples of smooth subschemes of $\mathbb{P}^n$ with desired properties. Peskine and Szpiro [23] proved a result which gave conditions guaranteeing that the residual scheme in a given CI-link will be smooth. In order for G-liaison to be equally useful, it will be necessary to find similar results for G-liaison. □

Remark 7.10. As mentioned above, it is possible that there may be stronger (or easier) results for G-liaison in codimension 3. This is worth exploring.

It should also be observed that unobstructedness is more often preserved under CI-liaison than G-liaison, so CI-liaison may be more useful for studying questions of unobstructedness. We refer the reader to [12] for details. □

References

[1] R. Apéry, *Sur certain caractères numeriques d'un idéal sans composant impropre*, C.R.A.S. 220 (1945), 234–236.

[2] R. Apéry, *Sur les courbes de premiére espèce de l'espace à trois dimensions*, C.R.A.S. 220 (1945), 271–272.

[3] E. Ballico, G. Bolondi and J. Migliore, *The Lazarsfeld-Rao Problem for Liaison Classes of Two-Codimensional Subschemes of* $\mathbb{P}^n$, Amer. J. of Math. **113** (1991), 117–128.

[4] G. Bolondi and J. Migliore, *The Structure of an Even Liaison Class*, Trans. Amer. Math. Soc. **316** (1989), 1–37.

[5] G. Bolondi and J. Migliore, *The Lazarsfeld-Rao property on an arithmetically Gorenstein variety*, manuscripta math. 78 (1993), 347–368.

[6] W. Bruns and J. Herzog, "Cohen-Macaulay Rings," Cambridge studies in advanced mathematics, Cambridge University Press, 1993.

[7] E. Davis, A.V. Geramita, F. Orecchia, *Gorenstein Algebras and the Cayley-Bacharach Theorem*, Proc. Amer. Math. Soc. **93** (1985), 593–597.

[8] D. Eisenbud, "Commutative Algebra with a view toward Algebraic Geometry," Springer-Verlag, Graduate Texts in Mathematics 150 (1995).

[9] F. Gaeta, *Sulle curve sghembe algebriche di residuale finito*, Annali di Matematica s. IV, t. XXVII (1948), 177–241.

[10] A.V. Geramita and J. Migliore, *A Generalized Liaison Addition*, J. Alg. 163 (1994), 139–164.

[11] R. Hartshorne, *Generalized Divisors on Gorenstein Schemes*, K-Theory **8** (1994), 287–339.

[12] J. Kleppe, J. Migliore, R.M. Miró-Roig, U. Nagel and C. Peterson, *Gorenstein Liaison, Complete Intersection Liaison Invariants and Unobstructedness*, preprint.

[13] M. Kreuzer, *On 0-dimensional complete intersections*, Mathematische Annalen **292** (1992), 43-58.

[14] M. Kreuzer, J. Migliore, U. Nagel and C. Peterson, *Determinantal Schemes and Buchsbaum-Rim Sheaves*, preprint.

[15] A. Kustin, M. Miller and B. Ulrich, *Linkage Theory for Algebras with Pure Resolutions*, J. Algebra **102** (1986), 199–228.

[16] R. Lazarsfeld and P. Rao, *Linkage of General Curves of Large Degree*, in "Algebraic Geometry–Open Problems (Ravello, 1982)," Lecture Notes in Mathematics, vol. 997, Springer–Verlag (1983), 267–289.

[17] M. Martin-Deschamps and D. Perrin, *Sur la Classification des Courbes Gauches*, Astérisque 184–185, Soc. Math. de France (1990).

[18] J. Migliore, "Introduction to Liaison Theory and Deficiency Modules," Birkhäuser, Progress in Mathematics 165 (1998).

[19] J. Migliore, U. Nagel and C. Peterson, *Buchsbaum-Rim Sheaves and their Multiple Sections*, preprint.

[20] J. Migliore and C. Peterson, *A construction of codimension three arithmetically Gorenstein subschemes of projective space*, Trans. Amer. Math. Soc. **349** (1997), 3803-3821.

[21] U. Nagel, *Even Liaison Classes Generated by Gorenstein Linkage*, to appear, J. Algebra.

[22] S. Nollet, *Even Linkage Classes*, Trans. Amer. Math. Soc. **348** (1996), no. 3, 1137–1162.

[23] C. Peskine and L. Szpiro, *Liaison des variétés algébriques. I*, Inv. Math. **26** (1974), 271–302.

[24] P. Rao, *Liaison among Curves in* $\mathbb{P}^3$, Invent. Math. **50** (1979), 205–217.

[25] P. Schwartau, *Liaison Addition and Monomial Ideals*, Ph.D. thesis, Brandeis University (1982).

[26] C. Weibel, "An introduction to homological algebra," Cambridge University Press, Cambridge studies in advanced mathematics 38 (1994).

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