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Remarks on High Linear Syzygy

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In this note we explain some properties that follow from a high linear syzygy. We consider the r-th, (r-1)-st, and (r-2)-nd linear syzygies over a polynomial ring in r variables. The most interesting, and the only nontrivial, case is the (r-2)-nd linear syzygy which produces skew-symmetric matrices that are helpful in understanding certain geometric situations.

Let $S=K[x_1,\cdots,x_r]=\oplus_{d\geq 0}S_d$ be a polynomial ring over a field K with the usual $\mathbb{N}$-grading. Let $M=\oplus_{d\geq t}M_d$ be a finitely generated graded S-module. As usual, $M(n)$ denotes the same module M with its degrees shifted to the left by n units, i.e., $M(n)_d:=M_{d+n}$. Let $F_\bullet$ denote the minimal graded free resolution of M over S, i.e.,

$$F_\bullet : 0\rightarrow F_r\rightarrow F_{r-1}\rightarrow\cdots\rightarrow F_p\rightarrow\cdots\rightarrow F_0\rightarrow 0,$$

where $F_p=\oplus_{q\in\mathbb{Z}}S(-q-p)^{b_{p,q}(M)}$.

The reason for the extra degree shift of $-p$ in the p-th free module F_p is because the entries of the maps in the minimal resolution are all of positive degrees. We say that M has a q-linear p-th syzygy if the graded betti number $b_{p,q}(M)\neq 0$. When $q=0$ we drop 0- to call it a linear p-th syzygy. The most important result concerning the linear syzygy is the vanishing theorem of Green ([G, Theorem 3.a.1]) which asserts that if M has a linear p-th syzygy, then dim $M_0 \geq p$ under certain conditions, which are satisfied in geometric situations. Some progress in finding more precise algebraic conditions affecting the linear syzygies were made in [EK1] and [EK2], but much more remains a mystery.

Tor-modules of the graded modules are also graded and can be computed in the usual way using $M(n) \otimes_S N(q) \cong (M \otimes_S N)(n+q)$. Let K denote the graded S-module S/S_+, where $S_+ := \oplus_{d>0} S_d$ is the unique homogeneous maximal ideal. We note that K is a graded module concentrated in degree 0. Using $F_\bullet \otimes_S K$, we compute

$$Tor_p^S(M,K) = \oplus_{q\in\mathbb{Z}} K(-q-p)^{b_{p,q}(M)},$$

which implies that

$$b_{p,q}(M) = \dim_K Tor_p^S(M,K)_{q+p}.$$

We may also compute $Tor_p^S(M,K)$ using the Koszul resolution $G_\bullet$ of K, where

$$G_\bullet : 0 \to S(-r) \to S(-r+1)^{\binom{r}{r-1}} \to \cdots \to S(-p)^{\binom{r}{p}} \to \cdots \to S \to 0.$$

Using $M \otimes_S G_\bullet$, we again compute

$$Tor_p^S(M,K) = \text{homology}\,(M(-p-1)^{\binom{r}{p+1}} \to M(-p)^{\binom{r}{p}} \to M(-p+1)^{\binom{r}{p-1}}),$$

and hence

$$(*) \qquad Tor_p^S(M,K)_{q+p} = \text{homology}\,(M_{q-1}^{\binom{r}{p+1}} \to M_q^{\binom{r}{p}} \to M_{q+1}^{\binom{r}{p-1}}).$$

Since the differential maps in the Koszul complex is given by the natural maps between the wedge products, it is customary to write the right-hand-side of $(*)$ above as:

$$(**) \qquad \text{homology}\,(\bigwedge^{p+1} S_1 \otimes_K M_{q-1} \xrightarrow{d_{p+1}} \bigwedge^{p} S_1 \otimes_K M_q \xrightarrow{d_p} \bigwedge^{p-1} S_1 \otimes_K M_{q-1}).$$

We remark that $\mathcal{K}_{p,q}(M)$ was the notation for $Tor_p^S(M,K)_{q+p}$ Green used in [G] in his systematic study of the relationship between the graded resolution and the geometry of projective algebraic variety.

Let $\{x_1, \cdots, x_r\}$ be a basis of S_1. To simplify notation we write $x^*_{i_1 \ldots i_j}$ to denote the wedge product of $\{x_1, \cdots, x_r\} - \{x_{i_1}, \cdots, x_{i_j}\}$. We first consider some trivial cases.

p=r. Suppose that M has a q-linear r-th syzygy. Since $\bigwedge^{r+1} S_1 = 0$, this syzygy corresponds to a nonzero element $a \in M_q$ in the kernel of d_r. Since

$$d_r(x_1 \wedge \cdots \wedge x_r \otimes a) = \sum_{1 \leq i \leq r} x^*_i \otimes (-1)^i x_i a,$$

$x_i a = 0$, for all $1 \leq i \leq r$. Hence a is a nonzero element of degree q that is killed by S_+. The converse is equally trivial for us to state:

M has a q-linear r-th syzygy if and only if $(\text{Soc } M)_q \neq 0$.

p=r-1. Suppose now that M has a q-linear (r-1)-st syzygy. By (**) above this syzygy is determined by an element in the kernel of d_{r-1} that is not in the image of d_r. Let a_i, $1 \leq i \leq r$, be elements of M_q such that $\Sigma_{1 \leq i \leq r}\, x^*_i \otimes a_i$ is in the kernel of d_{r-1}. Using

$$d_{r-1}\left(\sum_{1 \leq i \leq r} x^*_i \otimes a_i\right) = \sum_{1 \leq i < j \leq r} x^*_{ij} \otimes \pm(x_i a_j - x_j a_i),$$

We can easily check the validity of the following statement:

M has a q-linear (r-1)-st syzygy if and only if there is a $2 \times r$ matrix

$$\begin{pmatrix} x_1 & \cdots & x_r \\ a_1 & \cdots & a_r \end{pmatrix}$$

such that

i) $a_i \in M_q$, for all $1 \leq i \leq r$,

ii) all of its 2 $\times$2 minors are 0, and

iii) there is no element $a \in M_{q-1}$ such that $a_i = (-1)^i x_i a$ for all $1 \leq i \leq r$.

We now consider the main case.

p=r-2. Let M has a q-linear (r-2)-nd syzygy. As before, we can find elements a_{ij}, $1 \leq i < j \leq r$, of M_q such that $\Sigma_{1 \leq i < j \leq r}\, x^*_{ij} \otimes a_{ij}$ is in the kernel of d_{r-2}. Since

$$d_{r-2}\left(\sum_{1 \leq i < j \leq r} x^*_{ij} \otimes a_{ij}\right) = \sum_{1 \leq i < j < k \leq r} x^*_{ijk} \otimes \pm(x_i a_{jk} - x_j a_{ik} + x_k a_{ij}),$$

$x_i a_{jk} - x_j a_{ik} + x_k a_{ij} = 0$, for all $1 \le i < j < k \le r$. Since these are nothing other than 4×4 pfaffians of Q below involving the first row and column, we have the following characterization:

M has a q-linear (r-2)-nd syzygy if and only if there is a $(r+1) \times (r+1)$ skew symmetric matrix

$$Q = \begin{pmatrix} 0 & x_1 & \cdots & \cdots & x_r \\ -x_1 & 0 & \cdots & a_{ij} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & -a_{ji} & \cdots & \ddots & \vdots \\ -x_r & \vdots & \cdots & \cdots & 0 \end{pmatrix} \qquad (1)$$

such that

i) the first row spans S_1,
ii) $a_{ij} \in M_q$ for $1 \le i < j \le r$,
iii) each 4×4 pfaffian of Q involving the first row and column is zero, and
iv) there are no elements $a_i \in M_{q-1}$ such that $a_{ij} = \pm(x_i a_j - x_j a_i)$ for all $i < j$.

We consider two geometric situations where all, not just the ones involving the first row and column, 4×4 pfaffians are zero. To consider general phaffians the products of elements in M have to be defined. The first situation deals with the homogeneous coordinate ring of a set of points in, or more generally, a 0-dimensional subscheme of, $\mathbb{P}^{r-1}$, and the second deals with the canonical image of a nonsingular projective curve. We assume that the field K is algebraically closed in the rest of this note.

X is a set of points. Let X be a 0-dimensional subscheme of $\mathbb{P}^{r-1}$ in "general" position. Our discussion of this case is not rigorous because we use "general" to mean the argument below works. Let S be the homogeneous coordinate ring of $\mathbb{P}^{r-1}$, and I the saturated ideal defining X. Suppose that S/I has a 1-linear (r-2)-nd syzygy. Then we may view Q in (1) above as a matrix of linear forms of S. The following trick expresses any 4×4 pfaffian of Q in terms of those involving the first row and column: for $1 \le i < j < k < l \le r$,

$$x_i(a_{ij}a_{kl} - a_{ik}a_{jl} + a_{il}a_{jk})$$

$$= a_{ij}(x_l a_{ik} - x_k a_{il} + x_i a_{kl}) - a_{ik}(x_l a_{ij} - x_j a_{il} + x_i a_{jl})$$
$$+ a_{il}(x_k a_{ij} - x_j a_{ik} + x_i a_{jk}) \in I. \quad (2)$$

Since S/I is a 1-dimensional Cohen-Macaulay ring, we may assume that each x_i is a nonzero divisor on S/I, and hence the 4×4 pfaffian $a_{ij}a_{kl} - a_{ik}a_{jl} + a_{il}a_{jk}$ determined by $i < j < k < l$ is in I.

Since the vector space spanned by the entries of Q is of dimension r, the following result forces Q to have a generalized zero, i.e., one can produce a 0 off the diagonal after performing suitable (symmetric) row and column operations on Q.

Lemma ([KS, Lemma 1.5]). Let T be a $v \times v$ skew symmetric matrix of linear forms. If $\dim T < 2v - 3$, then T has a generalized zero.

We may, after a suitable row and column operations, put Q in the form

$$\begin{pmatrix} \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & | & A \\ - & & - \\ -A^t & | & * \end{pmatrix},$$

where A is a $2 \times (r-1)$ matrix of linear forms. We assume that the points in X are in "genera"l position so that if A is not 1-generic, i.e., one can produce a 0 after performing suitable row and column operations, then the whole column of A containing zero is zero. Since the 2×2 minors of A are 4×4 pfaffians of Q, this assumption is satisfied when I doesn't contain too many rank 2 quadrics, e.g., when X contains at least $2r - 1$ reduced points in linearly general position because I can't contain a product of linear forms in this case. Under this assumption we may put Q in the form

$$\begin{pmatrix} 0 & | & A \\ - & & - \\ -A^t & | & * \end{pmatrix},$$

where A is a $m \times n$ 1-generic matrix. Since $m + n = r + 1$ and $\dim A = r$, a result of Eisenbud ([E,Theorem 5.1]) implies that 2×2 minors of A define a rational normal curve. This argument provides a reason for one, more involved, direction of the following result of Green ([G, Theorem 3.c.6]).

Theorem (Strong Castelnuovo Lemma). Let X be a set of points in $\mathbb{P}^{r-1}$ in general position. Then X lies on a rational normal curve if and only if S/I has a 1-linear $(r-2)$-nd syzygy.

We remark here that Yanagawa used the same result of Eisenbud in proving his Generalized Castelnuovo's Lemma ([Y, Theorem 2.1]).

X **is a nonsingular projective curve.** We sketch the argument given in [KS] to prove a result of Green and Lazarsfeld ([GL]) on normal generation of line bundles. Let X be a nonsingular projective curve in $\mathbb{P}^{r-1}$. Let $\mathcal{L}$ be a very ample line bundle on X. Write $r = h^0(\mathcal{L})$, the dimension of $H^0(X, \mathcal{L})$, and $S =$ Sym $H^0(X, \mathcal{L})$, the symmetric algebra. For a line bundle $\mathcal{F}$ on X, let $M(\mathcal{F})$ denote the graded S-module $\oplus_{n\in\mathbb{Z}} H^0(X, \mathcal{F}\mathcal{L}^n)$. There is a natural map $\varphi : S \to M(\mathcal{O})$ whose kernel is the ideal I of the image of the morphism f defined by $\mathcal{L}$. $\mathcal{L}$ is said to be normally generated if $f(X)$ is a normal subvariety of $\mathbb{P}^{r-1}$, or equivalently, the map φ is onto. In terms of the graded betti numbers, this condition is equivalent to $b_{0,q}(M(\mathcal{O})) = 0$, for all $q > 0$. (In fact, for all $q \geq 2$ because φ is onto in degree 1.) To obtain a $(r$-$2)$-nd syzygy we apply the following result of Green.

Duality Theorem ([G] or [EKS]). Let ω denote the canonical bundle on X. For any line bundle $\mathcal{F}$ on X,

$$b_{p,q}(M(\mathcal{F})) = b_{r-2-p,r-q}(M(\mathcal{F}^{-1}\omega)).$$

Suppose that $\mathcal{L}$ is not normally generated. Since $b_{0,q}(M(\mathcal{O})) \neq 0$ for some $q \geq 2$, $b_{r-2,r-q}(M(\omega)) \neq 0$ by the Duality Theorem. We now assume that (Cliff(X) will be defined below.)

$$\deg \mathcal{L} \geq 2g + 1 - \text{Cliff}(X). \tag{3}$$

This implies that $H^0(X, \mathcal{L}^n\omega) = 0$ for all $n \leq -2$ and $h^1(\mathcal{L})$:= dim $H^1(X, \mathcal{L}) \leq 1$ (see [GL] or [KS]). Hence $b_{r-2,r-2}(M(\omega))$ is the only nonzero graded betti numbers for $q \geq 2$, and $M(\omega)$ has a (0-)linear (r-2)-nd syzygy. As in the previous case we get a skew symmetric matrix Q in (1), where a_{ij} are sections of the canonical bundle. Since X is irreducible, the similar argument as in (2) shows that all 4×4 pfaffians of Q are zero when viewed as elements either in $H^0(\mathcal{L}\omega)$ or $H^0(\omega^2)$. If $h^1(\mathcal{L}) = 0$, we take B to be the $r \times r$ skew symmetric submatrix of Q without the first row and the first column. If $h^1(\mathcal{L}) = 1$, we let $B = Q$. When $h^1(\mathcal{L}) = 1$, we may choose a nonzero section of $H^0(X, \mathcal{L}^{-1}\omega) \cong H^1(\mathcal{L})$ to define an injection $H^0(X, \mathcal{L}) \to H^0(X, \omega)$ so that each x_i can be viewed as a section of ω. Thus B is a $(h^0(\mathcal{L}) + h^1(\mathcal{L})) \times (h^0(\mathcal{L}) + h^1(\mathcal{L}))$ skew symmetric matrix with entries in $H^0(\omega)$ such that all of its 4×4 pfaffians are in the ideal of the canonical curve. Since dim $B \leq g$, where g is the genus, the degree bound in (3) and the earlier lemma imply that B has a generalized zero. Since X is irreducible, the ideal of the canonical curve can't have a rank 2 quadric. Hence we may, after suitable row and column operations, transform B to

$$\begin{pmatrix} 0 & | & A \\ - & & - \\ -A^t & | & * \end{pmatrix},$$

where A is 1-generic.

It is not hard to check that if A is of size $m \times n$, then $m + n = h^0(\mathcal{L}) + h^1(\mathcal{L})$ and $m, n \geq 2$. Let $\mathcal{F} := Im(A : \mathcal{O}^m \to \omega^n)$. Since all 2×2 minors vanish on the canonical image of X, $\mathcal{F}$ is a rank one subsheaf of ω^n, and hence a line bundle because X is nonsingular. Since the rows of a 1-generic matrix is linearly independent $h^0(\mathcal{F}) \geq m$. It can further be shown ([KS, Claim 2]) that $h^1(\mathcal{F}) \geq n$. We now recall the definition of the Clifford index of X:

$$\text{Cliff}(X) := \inf \{g + 1 - (h^0(\mathcal{G}) + h^1(\mathcal{G})) : \mathcal{G} \text{ is a line bundle with } h^0(\mathcal{G}), h^1(\mathcal{G}) \geq 2\}.$$

Our discussion on $\mathcal{F}$ above shows that

$$\text{Cliff}(X) \leq g + 1 - (h^0(\mathcal{F}) + h^1(\mathcal{F})) \leq g + 1 - (h^0(\mathcal{L}) + h^1(\mathcal{L})).$$

Applying Riemann-Roch Theorem, $h^0(\mathcal{L}) = \deg \mathcal{L} - g + 1 + h^1(\mathcal{L})$, we get

$$\deg \mathcal{L} \leq 2g - 2h^1(\mathcal{L}) - \text{Cliff}(X),$$

which contradicts the assumption on the degree of $\mathcal{L}$ in (3). We have thus proved the following result of Green and Lazarsfeld ([GL]):

Theorem. Let $\mathcal{L}$ be a very ample line bundle on a nonsingular projective curve X of genus g. If deg $\mathcal{L} \geq 2g + 1 - 2h^1(\mathcal{L})$ - Cliff(X), then $\mathcal{L}$ is normally generated.

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